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Connection for AP® Courses
class="introduction"

Galaxies are
as immense
as atoms are
small. Yet the
same laws of
physics
describe
both, and all
the rest of
nature—an
indication of
the
underlying
unity in the
universe. The
laws of
physics are
surprisingly
few in
number,
implying an
underlying
simplicity to
nature's
apparent
complexity.
(credit:
NASA, JPL-
Caltech, P.
Barmby,
Harvard-
Smithsonian
Center for

Astrophysics)



What is your first reaction when you hear the word “physics”? Did you imagine working through difficult equations or memorizing formulas that seem to have no real use in life outside the physics classroom? Many people come to the subject of physics with a bit of fear. But as you begin your exploration of this broad-ranging subject, you may soon come to realize that physics plays a much larger role in your life than you first thought, no matter your life goals or career choice.

For example, take a look at the image above. This image is of the Andromeda Galaxy, which contains billions of individual stars, huge clouds of gas, and dust. Two smaller galaxies are also visible as bright blue spots in the background. At a staggering 2.5 million light years from Earth, this galaxy is the nearest one to our own galaxy (which is called the Milky Way). The stars and planets that make up Andromeda might seem to be the furthest thing from most people's regular, everyday lives. But Andromeda is a great starting point to think about the forces that hold together the universe. The forces that cause Andromeda to act as it does are the same forces we contend with here on Earth, whether we are planning to send a rocket into space or simply raise the walls for a new home. The same gravity that causes the stars of Andromeda to rotate and revolve also causes water to flow over hydroelectric dams here on Earth. Tonight, take a moment to look up at the stars. The forces out there are the same as the ones here on Earth. Through a study of physics, you may gain a greater

understanding of the interconnectedness of everything we can see and know in this universe.

Think now about all of the technological devices that you use on a regular basis. Computers, smart phones, GPS systems, MP3 players, and satellite radio might come to mind. Next, think about the most exciting modern technologies that you have heard about in the news, such as trains that levitate above tracks, “invisibility cloaks” that bend light around them, and microscopic robots that fight cancer cells in our bodies. All of these groundbreaking advancements, commonplace or unbelievable, rely on the principles of physics. Aside from playing a significant role in technology, professionals such as engineers, pilots, physicians, physical therapists, electricians, and computer programmers apply physics concepts in their daily work. For example, a pilot must understand how wind forces affect a flight path and a physical therapist must understand how the muscles in the body experience forces as they move and bend. As you will learn in this text, physics principles are propelling new, exciting technologies, and these principles are applied in a wide range of careers.

In this text, you will begin to explore the history of the formal study of physics, beginning with natural philosophy and the ancient Greeks, and leading up through a review of Sir Isaac Newton and the laws of physics that bear his name. You will also be introduced to the standards scientists use when they study physical quantities and the interrelated system of measurements most of the scientific community uses to communicate in a single mathematical language. Finally, you will study the limits of our ability to be accurate and precise, and the reasons scientists go to painstaking lengths to be as clear as possible regarding their own limitations.

Chapter 1 introduces many fundamental skills and understandings needed for success with the AP® Learning Objectives. While this chapter does not directly address any Big Ideas, its content will allow for a more meaningful understanding when these Big Ideas are addressed in future chapters. For instance, the discussion of models, theories, and laws will assist you in understanding the concept of fields as addressed in Big Idea 2, and the

section titled 'The Evolution of Natural Philosophy into Modern Physics' will help prepare you for the statistical topics addressed in Big Idea 7.

This chapter will also prepare you to understand the Science Practices. In explicitly addressing the role of models in representing and communicating scientific phenomena, Section 1.1 supports Science Practice 1. Additionally, anecdotes about historical investigations and the inset on the scientific method will help you to engage in the scientific questioning referenced in Science Practice 3. The appropriate use of mathematics, as called for in Science Practice 2, is a major focus throughout sections 1.2, 1.3, and 1.4.

Physics: An Introduction



The flight formations of migratory birds such as Canada geese are governed by the laws of physics.
(credit: David Merrett)

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain the difference between a principle and a law.
- Explain the difference between a model and a theory.

The physical universe is enormously complex in its detail. Every day, each of us observes a great variety of objects and phenomena. Over the centuries, the curiosity of the human race has led us collectively to explore and catalog a tremendous wealth of information. From the flight of birds to the colors of flowers, from lightning to gravity, from quarks to clusters of

galaxies, from the flow of time to the mystery of the creation of the universe, we have asked questions and assembled huge arrays of facts. In the face of all these details, we have discovered that a surprisingly small and unified set of physical laws can explain what we observe. As humans, we make generalizations and seek order. We have found that nature is remarkably cooperative—it exhibits the *underlying order and simplicity* we so value.

It is the underlying order of nature that makes science in general, and physics in particular, so enjoyable to study. For example, what do a bag of chips and a car battery have in common? Both contain energy that can be converted to other forms. The law of conservation of energy (which says that energy can change form but is never lost) ties together such topics as food calories, batteries, heat, light, and watch springs. Understanding this law makes it easier to learn about the various forms energy takes and how they relate to one another. Apparently unrelated topics are connected through broadly applicable physical laws, permitting an understanding beyond just the memorization of lists of facts.

The unifying aspect of physical laws and the basic simplicity of nature form the underlying themes of this text. In learning to apply these laws, you will, of course, study the most important topics in physics. More importantly, you will gain analytical abilities that will enable you to apply these laws far beyond the scope of what can be included in a single book. These analytical skills will help you to excel academically, and they will also help you to think critically in any professional career you choose to pursue. This module discusses the realm of physics (to define what physics is), some applications of physics (to illustrate its relevance to other disciplines), and more precisely what constitutes a physical law (to illuminate the importance of experimentation to theory).

Science and the Realm of Physics

Science consists of the theories and laws that are the general truths of nature as well as the body of knowledge they encompass. Scientists are continually trying to expand this body of knowledge and to perfect the expression of the laws that describe it. **Physics** is concerned with describing the interactions

of energy, matter, space, and time, and it is especially interested in what fundamental mechanisms underlie every phenomenon. The concern for describing the basic phenomena in nature essentially defines the *realm of physics*.

Physics aims to describe the function of everything around us, from the movement of tiny charged particles to the motion of people, cars, and spaceships. In fact, almost everything around you can be described quite accurately by the laws of physics. Consider a smart phone ([\[link\]](#)). Physics describes how electricity interacts with the various circuits inside the device. This knowledge helps engineers select the appropriate materials and circuit layout when building the smart phone. Next, consider a GPS system. Physics describes the relationship between the speed of an object, the distance over which it travels, and the time it takes to travel that distance. When you use a GPS device in a vehicle, it utilizes these physics equations to determine the travel time from one location to another.



The Apple
“iPhone” is a
common
smart phone
with a GPS
function.
Physics

describes the
way that
electricity
flows through
the circuits of
this device.

Engineers use
their
knowledge of
physics to
construct an
iPhone with
features that
consumers
will enjoy.

One specific
feature of an
iPhone is the

GPS
function.

GPS uses
physics
equations to
determine the
driving time
between two
locations on a
map. (credit:
@gletham
GIS, Social,
Mobile Tech
Images)

Applications of Physics

You need not be a scientist to use physics. On the contrary, knowledge of physics is useful in everyday situations as well as in nonscientific professions. It can help you understand how microwave ovens work, why metals should not be put into them, and why they might affect pacemakers. (See [\[link\]](#) and [\[link\]](#).) Physics allows you to understand the hazards of radiation and rationally evaluate these hazards more easily. Physics also explains the reason why a black car radiator helps remove heat in a car engine, and it explains why a white roof helps keep the inside of a house cool. Similarly, the operation of a car's ignition system as well as the transmission of electrical signals through our body's nervous system are much easier to understand when you think about them in terms of basic physics.

Physics is the foundation of many important disciplines and contributes directly to others. Chemistry, for example—since it deals with the interactions of atoms and molecules—is rooted in atomic and molecular physics. Most branches of engineering are applied physics. In architecture, physics is at the heart of structural stability, and is involved in the acoustics, heating, lighting, and cooling of buildings. Parts of geology rely heavily on physics, such as radioactive dating of rocks, earthquake analysis, and heat transfer in the Earth. Some disciplines, such as biophysics and geophysics, are hybrids of physics and other disciplines.

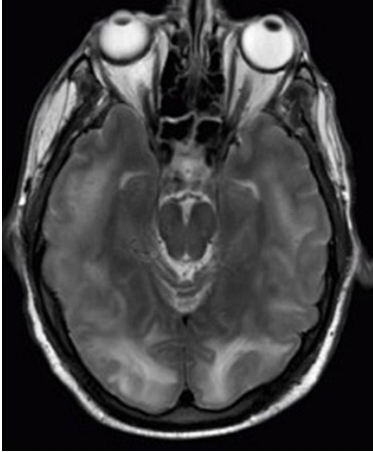
Physics has many applications in the biological sciences. On the microscopic level, it helps describe the properties of cell walls and cell membranes ([\[link\]](#) and [\[link\]](#)). On the macroscopic level, it can explain the heat, work, and power associated with the human body. Physics is involved in medical diagnostics, such as x-rays, magnetic resonance imaging (MRI), and ultrasonic blood flow measurements. Medical therapy sometimes directly involves physics; for example, cancer radiotherapy uses ionizing radiation. Physics can also explain sensory phenomena, such as how musical instruments make sound, how the eye detects color, and how lasers can transmit information.

It is not necessary to formally study all applications of physics. What is most useful is knowledge of the basic laws of physics and a skill in the analytical methods for applying them. The study of physics also can

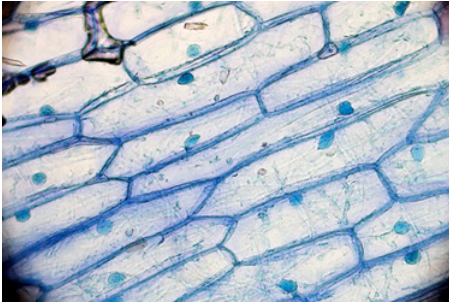
improve your problem-solving skills. Furthermore, physics has retained the most basic aspects of science, so it is used by all of the sciences, and the study of physics makes other sciences easier to understand.



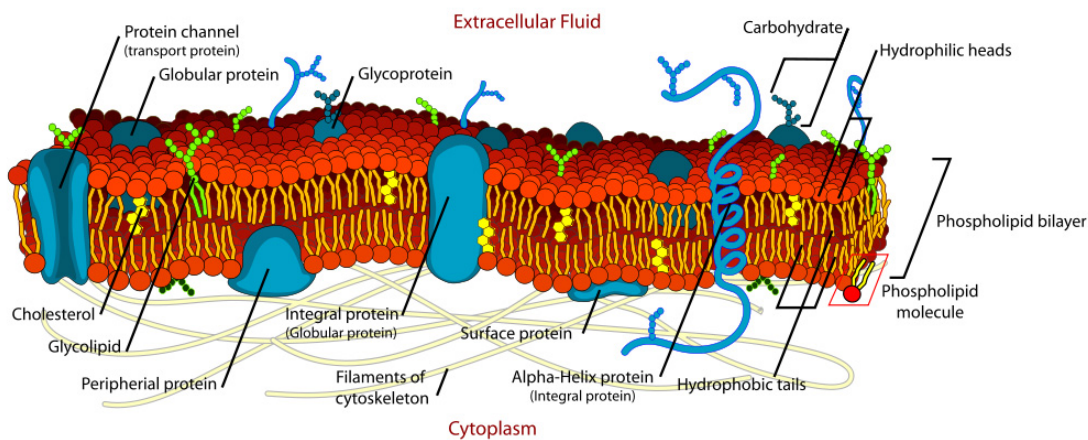
The laws of physics help us understand how common appliances work. For example, the laws of physics can help explain how microwave ovens heat up food, and they also help us understand why it is dangerous to place metal objects in a microwave oven. (credit: MoneyBlogNewz)



These two applications of physics have more in common than meets the eye. Microwave ovens use electromagnetic waves to heat food. Magnetic resonance imaging (MRI) also uses electromagnetic waves to yield an image of the brain, from which the exact location of tumors can be determined.
(credit: Rashmi Chawla, Daniel Smith, and Paul E. Marik)



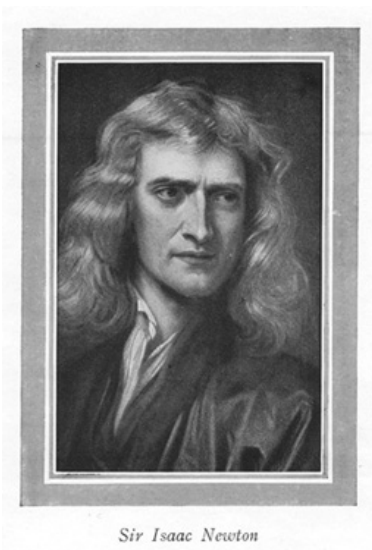
Physics, chemistry,
and biology help
describe the properties
of cell walls in plant
cells, such as the onion
cells seen here. (credit:
Umberto Salvagnin)



An artist's rendition of the the structure of a cell membrane.
Membranes form the boundaries of animal cells and are
complex in structure and function. Many of the most
fundamental properties of life, such as the firing of nerve cells,
are related to membranes. The disciplines of biology,
chemistry, and physics all help us understand the membranes of
animal cells. (credit: Mariana Ruiz)

Models, Theories, and Laws; The Role of Experimentation

The laws of nature are concise descriptions of the universe around us; they are human statements of the underlying laws or rules that all natural processes follow. Such laws are intrinsic to the universe; humans did not create them and so cannot change them. We can only discover and understand them. Their discovery is a very human endeavor, with all the elements of mystery, imagination, struggle, triumph, and disappointment inherent in any creative effort. (See [\[link\]](#) and [\[link\]](#).) The cornerstone of discovering natural laws is observation; science must describe the universe as it is, not as we may imagine it to be.



Isaac Newton
(1642–1727) was
very reluctant to
publish his
revolutionary
work and had to
be convinced to

do so. In his later years, he stepped down from his academic post and became exchequer of the Royal Mint. He took this post seriously, inventing reeding (or creating ridges) on the edge of coins to prevent unscrupulous people from trimming the silver off of them before using them as currency. (credit: Arthur Shuster and Arthur E. Shipley: *Britain's Heritage of Science*. London, 1917.)



Marie Curie
(1867–1934)
sacrificed
monetary assets
to help finance
her early
research and
damaged her
physical well-
being with
radiation
exposure. She is
the only person
to win Nobel
prizes in both
physics and
chemistry. One
of her daughters
also won a
Nobel Prize.
(credit:
Wikimedia
Commons)

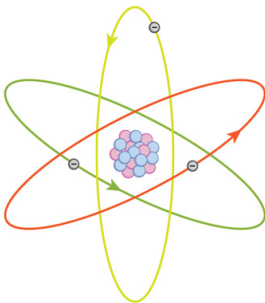
We all are curious to some extent. We look around, make generalizations, and try to understand what we see—for example, we look up and wonder whether one type of cloud signals an oncoming storm. As we become serious about exploring nature, we become more organized and formal in collecting and analyzing data. We attempt greater precision, perform controlled experiments (if we can), and write down ideas about how the data may be organized and unified. We then formulate models, theories, and laws based on the data we have collected and analyzed to generalize and communicate the results of these experiments.

A **model** is a representation of something that is often too difficult (or impossible) to display directly. While a model is justified with experimental proof, it is only accurate under limited situations. An example is the planetary model of the atom in which electrons are pictured as orbiting the nucleus, analogous to the way planets orbit the Sun. (See [\[link\]](#).) We cannot observe electron orbits directly, but the mental image helps explain the observations we can make, such as the emission of light from hot gases (atomic spectra). Physicists use models for a variety of purposes. For example, models can help physicists analyze a scenario and perform a calculation, or they can be used to represent a situation in the form of a computer simulation. A **theory** is an explanation for patterns in nature that is supported by scientific evidence and verified multiple times by various groups of researchers. Some theories include models to help visualize phenomena, whereas others do not. Newton's theory of gravity, for example, does not require a model or mental image, because we can observe the objects directly with our own senses. The kinetic theory of gases, on the other hand, is a model in which a gas is viewed as being composed of atoms and molecules. Atoms and molecules are too small to be observed directly with our senses—thus, we picture them mentally to understand what our instruments tell us about the behavior of gases.

A **law** uses concise language to describe a generalized pattern in nature that is supported by scientific evidence and repeated experiments. Often, a law can be expressed in the form of a single mathematical equation. Laws and theories are similar in that they are both scientific statements that result from a tested hypothesis and are supported by scientific evidence. However, the designation *law* is reserved for a concise and very general statement that

describes phenomena in nature, such as the law that energy is conserved during any process, or Newton's second law of motion, which relates force, mass, and acceleration by the simple equation $\mathbf{F} = m\mathbf{a}$. A theory, in contrast, is a less concise statement of observed phenomena. For example, the Theory of Evolution and the Theory of Relativity cannot be expressed concisely enough to be considered a law. The biggest difference between a law and a theory is that a theory is much more complex and dynamic. A law describes a single action, whereas a theory explains an entire group of related phenomena. And, whereas a law is a postulate that forms the foundation of the scientific method, a theory is the end result of that process.

Less broadly applicable statements are usually called principles (such as Pascal's principle, which is applicable only in fluids), but the distinction between laws and principles often is not carefully made.



What is a
model?
This
planetary
model of
the atom
shows
electrons
orbiting the
nucleus. It
is a
drawing

that we use
to form a
mental
image of
the atom
that we
cannot see
directly
with our
eyes
because it
is too
small.

Note:

Models, Theories, and Laws

Models, theories, and laws are used to help scientists analyze the data they have already collected. However, often after a model, theory, or law has been developed, it points scientists toward new discoveries they would not otherwise have made.

The models, theories, and laws we devise sometimes *imply the existence of objects or phenomena as yet unobserved*. These predictions are remarkable triumphs and tributes to the power of science. It is the underlying order in the universe that enables scientists to make such spectacular predictions. However, if *experiment* does not verify our predictions, then the theory or law is wrong, no matter how elegant or convenient it is. Laws can never be known with absolute certainty because it is impossible to perform every imaginable experiment in order to confirm a law in every possible scenario. Physicists operate under the assumption that all scientific laws and theories are valid until a counterexample is observed. If a good-quality, verifiable

experiment contradicts a well-established law, then the law must be modified or overthrown completely.

The study of science in general and physics in particular is an adventure much like the exploration of uncharted ocean. Discoveries are made; models, theories, and laws are formulated; and the beauty of the physical universe is made more sublime for the insights gained.

Note:

The Scientific Method

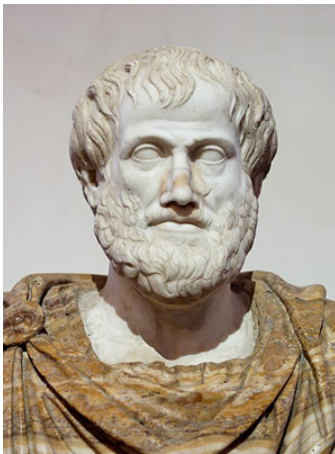
As scientists inquire and gather information about the world, they follow a process called the **scientific method**. This process typically begins with an observation and question that the scientist will research. Next, the scientist typically performs some research about the topic and then devises a hypothesis. Then, the scientist will test the hypothesis by performing an experiment. Finally, the scientist analyzes the results of the experiment and draws a conclusion. Note that the scientific method can be applied to many situations that are not limited to science, and this method can be modified to suit the situation.

Consider an example. Let us say that you try to turn on your car, but it will not start. You undoubtedly wonder: Why will the car not start? You can follow a scientific method to answer this question. First off, you may perform some research to determine a variety of reasons why the car will not start. Next, you will state a hypothesis. For example, you may believe that the car is not starting because it has no engine oil. To test this, you open the hood of the car and examine the oil level. You observe that the oil is at an acceptable level, and you thus conclude that the oil level is not contributing to your car issue. To troubleshoot the issue further, you may devise a new hypothesis to test and then repeat the process again.

The Evolution of Natural Philosophy into Modern Physics

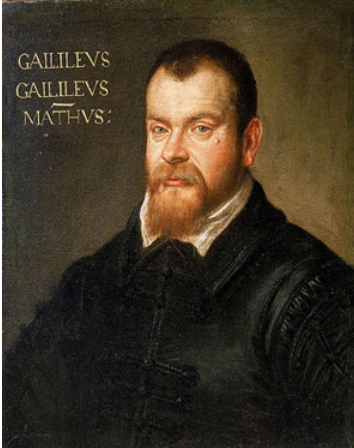
Physics was not always a separate and distinct discipline. It remains connected to other sciences to this day. The word *physics* comes from

Greek, meaning nature. The study of nature came to be called “natural philosophy.” From ancient times through the Renaissance, natural philosophy encompassed many fields, including astronomy, biology, chemistry, physics, mathematics, and medicine. Over the last few centuries, the growth of knowledge has resulted in ever-increasing specialization and branching of natural philosophy into separate fields, with physics retaining the most basic facets. (See [\[link\]](#), [\[link\]](#), and [\[link\]](#).) Physics as it developed from the Renaissance to the end of the 19th century is called **classical physics**. It was transformed into modern physics by revolutionary discoveries made starting at the beginning of the 20th century.



Over the centuries, natural philosophy has evolved into more specialized disciplines, as illustrated by the contributions of some of the greatest minds in history. The

Greek
philosopher
Aristotle (384–
322 B.C.) wrote
on a broad
range of topics
including
physics,
animals, the
soul, politics,
and poetry.
(credit: Jastrow
(2006)/Ludovisi
Collection)



Galileo Galilei
(1564–1642) laid
the foundation of
modern
experimentation
and made
contributions in

mathematics,
physics, and
astronomy.

(credit:
Domenico
Tintoretto)

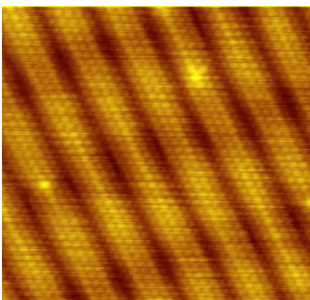


Niels Bohr
(1885–1962)
made
fundamental
contributions to
the development
of quantum
mechanics, one
part of modern
physics. (credit:
United States
Library of
Congress Prints
and Photographs
Division)

Classical physics is not an exact description of the universe, but it is an excellent approximation under the following conditions: Matter must be moving at speeds less than about 1% of the speed of light, the objects dealt with must be large enough to be seen with a microscope, and only weak gravitational fields, such as the field generated by the Earth, can be involved. Because humans live under such circumstances, classical physics seems intuitively reasonable, while many aspects of modern physics seem bizarre. This is why models are so useful in modern physics—they let us conceptualize phenomena we do not ordinarily experience. We can relate to models in human terms and visualize what happens when objects move at high speeds or imagine what objects too small to observe with our senses might be like. For example, we can understand an atom's properties because we can picture it in our minds, although we have never seen an atom with our eyes. New tools, of course, allow us to better picture phenomena we cannot see. In fact, new instrumentation has allowed us in recent years to actually “picture” the atom.

Note:**Limits on the Laws of Classical Physics**

For the laws of classical physics to apply, the following criteria must be met: Matter must be moving at speeds less than about 1% of the speed of light, the objects dealt with must be large enough to be seen with a microscope, and only weak gravitational fields (such as the field generated by the Earth) can be involved.



Using a
scanning
tunneling
microscope
(STM),
scientists can
see the
individual
atoms that
compose this
sheet of gold.
(credit:
Erwinrossen)

Some of the most spectacular advances in science have been made in modern physics. Many of the laws of classical physics have been modified or rejected, and revolutionary changes in technology, society, and our view of the universe have resulted. Like science fiction, modern physics is filled with fascinating objects beyond our normal experiences, but it has the advantage over science fiction of being very real. Why, then, is the majority of this text devoted to topics of classical physics? There are two main reasons: Classical physics gives an extremely accurate description of the universe under a wide range of everyday circumstances, and knowledge of classical physics is necessary to understand modern physics.

Modern physics itself consists of the two revolutionary theories, relativity and quantum mechanics. These theories deal with the very fast and the very small, respectively. **Relativity** must be used whenever an object is traveling at greater than about 1% of the speed of light or experiences a strong gravitational field such as that near the Sun. **Quantum mechanics** must be used for objects smaller than can be seen with a microscope. The combination of these two theories is *relativistic quantum mechanics*, and it describes the behavior of small objects traveling at high speeds or experiencing a strong gravitational field. Relativistic quantum mechanics is the best universally applicable theory we have. Because of its mathematical

complexity, it is used only when necessary, and the other theories are used whenever they will produce sufficiently accurate results. We will find, however, that we can do a great deal of modern physics with the algebra and trigonometry used in this text.

Exercise:

Check Your Understanding

Problem:

A friend tells you he has learned about a new law of nature. What can you know about the information even before your friend describes the law? How would the information be different if your friend told you he had learned about a scientific theory rather than a law?

Solution:

Without knowing the details of the law, you can still infer that the information your friend has learned conforms to the requirements of all laws of nature: it will be a concise description of the universe around us; a statement of the underlying rules that all natural processes follow. If the information had been a theory, you would be able to infer that the information will be a large-scale, broadly applicable generalization.

Note:

PhET Explorations: Equation Grapher

Learn about graphing polynomials. The shape of the curve changes as the constants are adjusted. View the curves for the individual terms (e.g. $y = bx$) to see how they add to generate the polynomial curve.

[Equatio](#)
[n](#)
[Grapher](#)

Summary

- Science seeks to discover and describe the underlying order and simplicity in nature.
- Physics is the most basic of the sciences, concerning itself with energy, matter, space and time, and their interactions.
- Scientific laws and theories express the general truths of nature and the body of knowledge they encompass. These laws of nature are rules that all natural processes appear to follow.

Conceptual Questions

Exercise:

Problem:

Models are particularly useful in relativity and quantum mechanics, where conditions are outside those normally encountered by humans. What is a model?

Exercise:

Problem: How does a model differ from a theory?

Exercise:

Problem:

If two different theories describe experimental observations equally well, can one be said to be more valid than the other (assuming both use accepted rules of logic)?

Exercise:

Problem: What determines the validity of a theory?

Exercise:

Problem:

Certain criteria must be satisfied if a measurement or observation is to be believed. Will the criteria necessarily be as strict for an expected result as for an unexpected result?

Exercise:**Problem:**

Can the validity of a model be limited, or must it be universally valid? How does this compare to the required validity of a theory or a law?

Exercise:**Problem:**

Classical physics is a good approximation to modern physics under certain circumstances. What are they?

Exercise:

Problem: When is it *necessary* to use relativistic quantum mechanics?

Exercise:**Problem:**

Can classical physics be used to accurately describe a satellite moving at a speed of 7500 m/s? Explain why or why not.

Glossary

classical physics

physics that was developed from the Renaissance to the end of the 19th century

physics

the science concerned with describing the interactions of energy, matter, space, and time; it is especially interested in what fundamental

mechanisms underlie every phenomenon

model

representation of something that is often too difficult (or impossible) to display directly

theory

an explanation for patterns in nature that is supported by scientific evidence and verified multiple times by various groups of researchers

law

a description, using concise language or a mathematical formula, a generalized pattern in nature that is supported by scientific evidence and repeated experiments

scientific method

a method that typically begins with an observation and question that the scientist will research; next, the scientist typically performs some research about the topic and then devises a hypothesis; then, the scientist will test the hypothesis by performing an experiment; finally, the scientist analyzes the results of the experiment and draws a conclusion

modern physics

the study of relativity, quantum mechanics, or both

relativity

the study of objects moving at speeds greater than about 1% of the speed of light, or of objects being affected by a strong gravitational field

quantum mechanics

the study of objects smaller than can be seen with a microscope

Physical Quantities and Units



The distance from Earth to the Moon may seem immense, but it is just a tiny fraction of the distances from Earth to other celestial bodies. (credit: NASA)

LEARNING OBJECTIVES

By the end of this section, you will be able to:

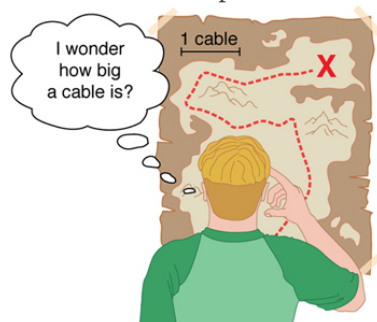
- Perform unit conversions both in the SI and English units.
- Explain the most common prefixes in the SI units and be able to write them in scientific notation.

The range of objects and phenomena studied in physics is immense. From the incredibly short lifetime of a nucleus to the age of Earth, from the tiny sizes of sub-nuclear particles to the vast distance to the edges of the known universe, from the force exerted by a jumping flea to the force between Earth and the Sun, there are enough factors of 10 to challenge the imagination of even the most experienced scientist. Giving numerical values for physical quantities and equations for physical principles allows us to understand nature much more deeply than does qualitative description alone. To comprehend these vast ranges, we must also have accepted units in which to express them. And we shall find that (even in the potentially mundane discussion of meters, kilograms, and seconds) a profound simplicity of nature appears—all physical quantities can be expressed as combinations of only four fundamental physical quantities: length, mass, time, and electric current.

We define a **physical quantity** either by *specifying how it is measured* or by *stating how it is calculated* from other measurements. For example, we define distance and time by specifying methods for measuring them, whereas we define *average speed* by stating that it is calculated as distance traveled divided by time of travel.

Measurements of physical quantities are expressed in terms of **units**, which are standardized values. For example, the length of a race, which is a physical quantity, can be expressed in units of meters (for

sprinters) or kilometers (for distance runners). Without standardized units, it would be extremely difficult for scientists to express and compare measured values in a meaningful way. (See [\[link\]](#).)



Distances given in unknown units are maddeningly useless.

There are two major systems of units used in the world: **SI units** (also known as the metric system) and **English units** (also known as the customary or imperial system). **English units** were historically used in nations once ruled by the British Empire and are still widely used in the United States. Virtually every other country in the world now uses SI units as the standard; the metric system is also the standard system agreed upon by scientists and mathematicians. The acronym “SI” is derived from the French *Système International*.

SI Units: Fundamental and Derived Units

[\[link\]](#) gives the fundamental SI units that are used throughout this textbook. This text uses non-SI units in a few applications where they are in very common use, such as the measurement of blood pressure in millimeters of mercury (mm Hg). Whenever non-SI units are discussed, they will be tied to SI units through conversions.

Length	Mass	Time	Electric Charge
meter (m)	kilogram (kg)	second (s)	coulomb (c)

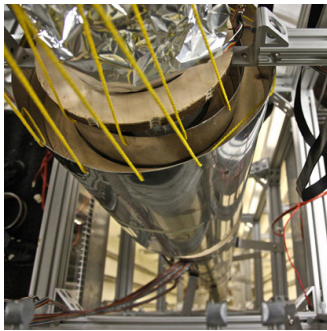
Fundamental SI Units

It is an intriguing fact that some physical quantities are more fundamental than others and that the most fundamental physical quantities can be defined *only* in terms of the procedure used to measure them. The units in which they are measured are thus called **fundamental units**. In this textbook, the fundamental physical quantities are taken to be length, mass, time, and electric charge. (Note that electric current will not be introduced until much later in this text.) All other physical quantities, such as force and electric current, can be expressed as algebraic combinations of length, mass, time, and current (for example, speed is length divided by time); these units are called **derived units**.

Units of Time, Length, and Mass: The Second, Meter, and Kilogram

The Second

The SI unit for time, the **second** (abbreviated s), has a long history. For many years it was defined as $1/86,400$ of a mean solar day. More recently, a new standard was adopted to gain greater accuracy and to define the second in terms of a non-varying, or constant, physical phenomenon (because the solar day is getting longer due to very gradual slowing of the Earth's rotation). Cesium atoms can be made to vibrate in a very steady way, and these vibrations can be readily observed and counted. In 1967 the second was redefined as the time required for 9,192,631,770 of these vibrations. (See [\[link\]](#).) Accuracy in the fundamental units is essential, because all measurements are ultimately expressed in terms of fundamental units and can be no more accurate than are the fundamental units themselves.



An atomic clock such as this one uses the vibrations of cesium atoms to keep time to a precision of better than a microsecond per year. The fundamental unit of time, the second, is based on such clocks. This image is looking down from the top of an atomic fountain nearly 30 feet tall!
(credit: Steve Jurvetson/Flickr)

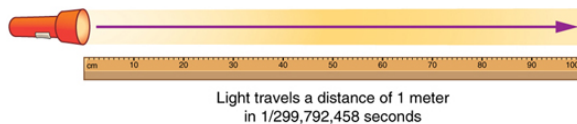
The Meter

The SI unit for length is the **meter** (abbreviated m); its definition has also changed over time to become more accurate and precise. The meter was first defined in 1791 as $1/10,000,000$ of the distance from the equator to the North Pole. This measurement was improved in 1889 by redefining the meter to be the

distance between two engraved lines on a platinum-iridium bar now kept near Paris. By 1960, it had become possible to define the meter even more accurately in terms of the wavelength of light, so it was again redefined as 1,650,763.73 wavelengths of orange light emitted by krypton atoms. In 1983, the meter was given its present definition (partly for greater accuracy) as the distance light travels in a vacuum in $1/299,792,458$ of a second. (See [\[link\]](#).) This change defines the speed of light to be exactly 299,792,458 meters per second. The length of the meter will change if the speed of light is someday measured with greater accuracy.

The Kilogram

The SI unit for mass is the **kilogram** (abbreviated kg); it is defined to be the mass of a platinum-iridium cylinder kept with the old meter standard at the International Bureau of Weights and Measures near Paris. Exact replicas of the standard kilogram are also kept at the United States' National Institute of Standards and Technology, or NIST, located in Gaithersburg, Maryland outside of Washington D.C., and at other locations around the world. The determination of all other masses can be ultimately traced to a comparison with the standard mass.



The meter is defined to be the distance light travels in $1/299,792,458$ of a second in a vacuum. Distance traveled is speed multiplied by time.

Electric current and its accompanying unit, the ampere, will be introduced in [Introduction to Electric Current, Resistance, and Ohm's Law](#) when electricity and magnetism are covered. The initial modules in this textbook are concerned with mechanics, fluids, heat, and waves. In these subjects all pertinent physical quantities can be expressed in terms of the fundamental units of length, mass, and time.

Metric Prefixes

SI units are part of the **metric system**. The metric system is convenient for scientific and engineering calculations because the units are categorized by factors of 10. [\[link\]](#) gives metric prefixes and symbols used to denote various factors of 10.

Metric systems have the advantage that conversions of units involve only powers of 10. There are 100 centimeters in a meter, 1000 meters in a kilometer, and so on. In nonmetric systems, such as the system of U.S. customary units, the relationships are not as simple—there are 12 inches in a foot, 5280 feet in a mile, and so on. Another advantage of the metric system is that the same unit can be used over extremely large ranges of values simply by using an appropriate metric prefix. For example, distances in meters are suitable in construction, while distances in kilometers are appropriate for air travel, and the tiny measure of nanometers are convenient in optical design. With the metric system there is no need to invent new units for particular applications.

The term **order of magnitude** refers to the scale of a value expressed in the metric system. Each power of 10 in the metric system represents a different order of magnitude. For example, 10^1 , 10^2 , 10^3 , and so forth are all different orders of magnitude. All quantities that can be expressed as a product of a specific power of 10 are said to be of the *same* order of magnitude. For example, the number 800 can be written as 8×10^2 , and the number 450 can be written as 4.5×10^2 . Thus, the numbers 800 and 450 are of the same order of magnitude: 10^2 . Order of magnitude can be thought of as a ballpark estimate for the scale of a value. The diameter of an atom is on the order of 10^{-9} m, while the diameter of the Sun is on the order of 10^9 m.

Note:

The Quest for Microscopic Standards for Basic Units

The fundamental units described in this chapter are those that produce the greatest accuracy and precision in measurement. There is a sense among physicists that, because there is an underlying microscopic substructure to matter, it would be most satisfying to base our standards of measurement on microscopic objects and fundamental physical phenomena such as the speed of light. A microscopic standard has been accomplished for the standard of time, which is based on the oscillations of the cesium atom.

The standard for length was once based on the wavelength of light (a small-scale length) emitted by a certain type of atom, but it has been supplanted by the more precise measurement of the speed of light. If it becomes possible to measure the mass of atoms or a particular arrangement of atoms such as a silicon sphere to greater precision than the kilogram standard, it may become possible to base mass measurements on the small scale. There are also possibilities that electrical phenomena on the small scale may someday allow us to base a unit of charge on the charge of electrons and protons, but at present current and charge are related to large-scale currents and forces between wires.

Prefix	Symbol	Value ^{[footnote]} See [link] for a discussion of powers of 10.	Example (some are approximate)			
exa	E	10^{18}	exameter	Em	10^{18} m	distance light travels in a century
peta	P	10^{15}	petasecond	Ps	10^{15} s	30 million years
tera	T	10^{12}	terawatt	TW	10^{12} W	powerful laser output
giga	G	10^9	gigahertz	GHz	10^9 Hz	a microwave frequency

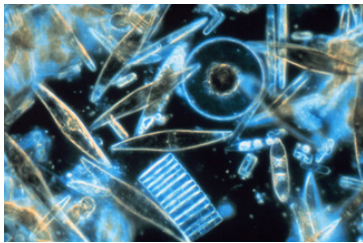
Prefix	Symbol	Value ^{[footnote]} See [link] for a discussion of powers of 10.	Example (some are approximate)			
mega	M	10^6	megacurie	MCi	10^6 Ci	high radioactivity
kilo	k	10^3	kilometer	km	10^3 m	about 6/10 mile
hecto	h	10^2	hectoliter	hL	10^2 L	26 gallons
deka	da	10^1	dekagram	dag	10^1 g	teaspoon of butter
—	—	$10^0 (= 1)$				
deci	d	10^{-1}	deciliter	dL	10^{-1} L	less than half a soda
centi	c	10^{-2}	centimeter	cm	10^{-2} m	fingertip thickness
milli	m	10^{-3}	millimeter	mm	10^{-3} m	flea at its shoulders
micro	μ	10^{-6}	micrometer	μm	10^{-6} m	detail in microscope
nano	n	10^{-9}	nanogram	ng	10^{-9} g	small speck of dust
pico	p	10^{-12}	picofarad	pF	10^{-12} F	small capacitor in radio
femto	f	10^{-15}	femtometer	fm	10^{-15} m	size of a proton

Prefix	Symbol	Value ^{[footnote]} See [link] for a discussion of powers of 10.	Example (some are approximate)			
atto	a	10^{-18}	attosecond	as	10^{-18} s	time light crosses an atom

Metric Prefixes for Powers of 10 and their Symbols

Known Ranges of Length, Mass, and Time

The vastness of the universe and the breadth over which physics applies are illustrated by the wide range of examples of known lengths, masses, and times in [\[link\]](#). Examination of this table will give you some feeling for the range of possible topics and numerical values. (See [\[link\]](#) and [\[link\]](#).)

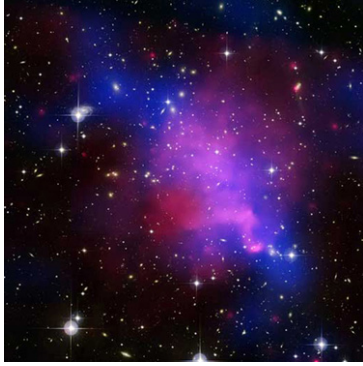


Tiny phytoplankton
swims among crystals of
ice in the Antarctic Sea.

They range from a few
micrometers to as much
as 2 millimeters in length.

(credit: Prof. Gordon T.

Taylor, Stony Brook
University; NOAA Corps
Collections)



Galaxies collide 2.4 billion light years away from Earth. The tremendous range of observable phenomena in nature challenges the imagination. (credit: NASA/CXC/UVic./A. Mahdavi et al. Optical/lensing: CFHT/UVic./H. Hoekstra et al.)

Unit Conversion and Dimensional Analysis

It is often necessary to convert from one type of unit to another. For example, if you are reading a European cookbook, some quantities may be expressed in units of liters and you need to convert them to cups. Or, perhaps you are reading walking directions from one location to another and you are interested in how many miles you will be walking. In this case, you will need to convert units of feet to miles.

Let us consider a simple example of how to convert units. Let us say that we want to convert 80 meters (m) to kilometers (km).

The first thing to do is to list the units that you have and the units that you want to convert to. In this case, we have units in *meters* and we want to convert to *kilometers*.

Next, we need to determine a **conversion factor** relating meters to kilometers. A conversion factor is a ratio expressing how many of one unit are equal to another unit. For example, there are 12 inches in 1 foot, 100 centimeters in 1 meter, 60 seconds in 1 minute, and so on. In this case, we know that there are 1,000 meters in 1 kilometer.

Now we can set up our unit conversion. We will write the units that we have and then multiply them by the conversion factor so that the units cancel out, as shown:

Equation:

$$80 \cancel{\text{m}} \times \frac{1 \text{ km}}{1000 \cancel{\text{m}}} = 0.080 \text{ km.}$$

Note that the unwanted m unit cancels, leaving only the desired km unit. You can use this method to convert between any types of unit.

See [Appendix C](#) for a more complete list of conversion factors.

Lengths in meters		Masses in kilograms (more precise values in parentheses)		Times in seconds (more precise values in parentheses)	
10^{-18}	Present experimental limit to smallest observable detail	10^{-30}	Mass of an electron (9.11×10^{-31} kg)	10^{-23}	Time for light to cross a proton
10^{-15}	Diameter of a proton	10^{-27}	Mass of a hydrogen atom (1.67×10^{-27} kg)	10^{-22}	Mean life of an extremely unstable nucleus
10^{-14}	Diameter of a uranium nucleus	10^{-15}	Mass of a bacterium	10^{-15}	Time for one oscillation of visible light
10^{-10}	Diameter of a hydrogen atom	10^{-5}	Mass of a mosquito	10^{-13}	Time for one vibration of an atom in a solid
10^{-8}	Thickness of membranes in cells of living organisms	10^{-2}	Mass of a hummingbird	10^{-8}	Time for one oscillation of an FM radio wave
10^{-6}	Wavelength of visible light	1	Mass of a liter of water (about a quart)	10^{-3}	Duration of a nerve impulse
10^{-3}	Size of a grain of sand	10^2	Mass of a person	1	Time for one heartbeat
1	Height of a 4-year-old child	10^3	Mass of a car	10^5	One day (8.64×10^4 s)
10^2	Length of a football field	10^8	Mass of a large ship	10^7	One year (y) (3.16×10^7 s)

Lengths in meters		Masses in kilograms (more precise values in parentheses)		Times in seconds (more precise values in parentheses)	
10^4	Greatest ocean depth	10^{12}	Mass of a large iceberg	10^9	About half the life expectancy of a human
10^7	Diameter of the Earth	10^{15}	Mass of the nucleus of a comet	10^{11}	Recorded history
10^{11}	Distance from the Earth to the Sun	10^{23}	Mass of the Moon (7.35×10^{22} kg)	10^{17}	Age of the Earth
10^{16}	Distance traveled by light in 1 year (a light year)	10^{25}	Mass of the Earth (5.97×10^{24} kg)	10^{18}	Age of the universe
10^{21}	Diameter of the Milky Way galaxy	10^{30}	Mass of the Sun (1.99×10^{30} kg)		
10^{22}	Distance from the Earth to the nearest large galaxy (Andromeda)	10^{42}	Mass of the Milky Way galaxy (current upper limit)		
10^{26}	Distance from the Earth to the edges of the known universe	10^{53}	Mass of the known universe (current upper limit)		

Approximate Values of Length, Mass, and Time

Example:

Unit Conversions: A Short Drive Home

Suppose that you drive the 10.0 km from your university to home in 20.0 min. Calculate your average speed (a) in kilometers per hour (km/h) and (b) in meters per second (m/s). (Note: Average speed is distance traveled divided by time of travel.)

Strategy

First we calculate the average speed using the given units. Then we can get the average speed into the desired units by picking the correct conversion factor and multiplying by it. The correct conversion factor is the one that cancels the unwanted unit and leaves the desired unit in its place.

Solution for (a)

(1) Calculate average speed. Average speed is distance traveled divided by time of travel. (Take this definition as a given for now—average speed and other motion concepts will be covered in a later module.) In equation form,

Equation:

$$\text{average speed} = \frac{\text{distance}}{\text{time}}.$$

(2) Substitute the given values for distance and time.

Equation:

$$\text{average speed} = \frac{10.0 \text{ km}}{20.0 \text{ min}} = 0.500 \frac{\text{km}}{\text{min}}.$$

(3) Convert km/min to km/h: multiply by the conversion factor that will cancel minutes and leave hours. That conversion factor is 60 min/hr. Thus,

Equation:

$$\text{average speed} = 0.500 \frac{\text{km}}{\text{min}} \times \frac{60 \text{ min}}{1 \text{ h}} = 30.0 \frac{\text{km}}{\text{h}}.$$

Discussion for (a)

To check your answer, consider the following:

(1) Be sure that you have properly cancelled the units in the unit conversion. If you have written the unit conversion factor upside down, the units will not cancel properly in the equation. If you accidentally get the ratio upside down, then the units will not cancel; rather, they will give you the wrong units as follows:

Equation:

$$\frac{\text{km}}{\text{min}} \times \frac{1 \text{ hr}}{60 \text{ min}} = \frac{1}{60} \frac{\text{km} \cdot \text{hr}}{\text{min}^2},$$

which are obviously not the desired units of km/h.

(2) Check that the units of the final answer are the desired units. The problem asked us to solve for average speed in units of km/h and we have indeed obtained these units.

(3) Check the significant figures. Because each of the values given in the problem has three significant figures, the answer should also have three significant figures. The answer 30.0 km/hr does indeed have three significant figures, so this is appropriate. Note that the significant figures in the conversion factor are not relevant because an hour is *defined* to be 60 minutes, so the precision of the conversion factor is perfect.

(4) Next, check whether the answer is reasonable. Let us consider some information from the problem—if you travel 10 km in a third of an hour (20 min), you would travel three times that far in an hour. The answer does seem reasonable.

Solution for (b)

There are several ways to convert the average speed into meters per second.

(1) Start with the answer to (a) and convert km/h to m/s. Two conversion factors are needed—one to convert hours to seconds, and another to convert kilometers to meters.

(2) Multiplying by these yields

Equation:

$$\text{Average speed} = 30.0 \frac{\text{km}}{\text{h}} \times \frac{1 \text{ h}}{3,600 \text{ s}} \times \frac{1,000 \text{ m}}{1 \text{ km}},$$

Equation:

$$\text{Average speed} = 8.33 \frac{\text{m}}{\text{s}}.$$

Discussion for (b)

If we had started with 0.500 km/min, we would have needed different conversion factors, but the answer would have been the same: 8.33 m/s.

You may have noted that the answers in the worked example just covered were given to three digits.

Why? When do you need to be concerned about the number of digits in something you calculate? Why not write down all the digits your calculator produces? The module [Accuracy, Precision, and Significant Figures](#) will help you answer these questions.

Note:

Nonstandard Units

While there are numerous types of units that we are all familiar with, there are others that are much more obscure. For example, a **firkin** is a unit of volume that was once used to measure beer. One firkin equals about 34 liters. To learn more about nonstandard units, use a dictionary or encyclopedia to research different “weights and measures.” Take note of any unusual units, such as a barleycorn, that are not listed in the text. Think about how the unit is defined and state its relationship to SI units.

Exercise:

Check Your Understanding

Problem:

Some hummingbirds beat their wings more than 50 times per second. A scientist is measuring the time it takes for a hummingbird to beat its wings once. Which fundamental unit should the scientist use to describe the measurement? Which factor of 10 is the scientist likely to use to describe the motion precisely? Identify the metric prefix that corresponds to this factor of 10.

Solution:

The scientist will measure the time between each movement using the fundamental unit of seconds. Because the wings beat so fast, the scientist will probably need to measure in milliseconds, or 10^{-3} seconds. (50 beats per second corresponds to 20 milliseconds per beat.)

Exercise:

Check Your Understanding

Problem:

One cubic centimeter is equal to one milliliter. What does this tell you about the different units in the SI metric system?

Solution:

The fundamental unit of length (meter) is probably used to create the derived unit of volume (liter). The measure of a milliliter is dependent on the measure of a centimeter.

Summary

- Physical quantities are a characteristic or property of an object that can be measured or calculated from other measurements.

- Units are standards for expressing and comparing the measurement of physical quantities. All units can be expressed as combinations of four fundamental units.
- The four fundamental units we will use in this text are the meter (for length), the kilogram (for mass), the second (for time), and the ampere (for electric current). These units are part of the metric system, which uses powers of 10 to relate quantities over the vast ranges encountered in nature.
- The four fundamental units are abbreviated as follows: meter, m; kilogram, kg; second, s; and ampere, A. The metric system also uses a standard set of prefixes to denote each order of magnitude greater than or lesser than the fundamental unit itself.
- Unit conversions involve changing a value expressed in one type of unit to another type of unit. This is done by using conversion factors, which are ratios relating equal quantities of different units.

Conceptual Questions

Exercise:

Problem: Identify some advantages of metric units.

Problems & Exercises

Exercise:

Problem:

The speed limit on some interstate highways is roughly 100 km/h. (a) What is this in meters per second? (b) How many miles per hour is this?

Solution:

- a. 27.8 m/s
- b. 62.1 mph

Exercise:

Problem:

A car is traveling at a speed of 33 m/s. (a) What is its speed in kilometers per hour? (b) Is it exceeding the 90 km/h speed limit?

Exercise:

Problem:

Show that $1.0 \text{ m/s} = 3.6 \text{ km/h}$. Hint: Show the explicit steps involved in converting $1.0 \text{ m/s} = 3.6 \text{ km/h}$.

Solution:

$$\begin{aligned} \frac{1.0 \text{ m}}{\text{s}} &= \frac{1.0 \text{ m}}{\text{s}} \times \frac{3600 \text{ s}}{1 \text{ hr}} \times \frac{1 \text{ km}}{1000 \text{ m}} \\ &= 3.6 \text{ km/h.} \end{aligned}$$

Exercise:

Problem:

American football is played on a 100-yd-long field, excluding the end zones. How long is the field in meters? (Assume that 1 meter equals 3.281 feet.)

Exercise:**Problem:**

Soccer fields vary in size. A large soccer field is 115 m long and 85 m wide. What are its dimensions in feet and inches? (Assume that 1 meter equals 3.281 feet.)

Solution:

length: 377 ft; 4.53×10^3 in. width: 280 ft; 3.3×10^3 in.

Exercise:**Problem:**

What is the height in meters of a person who is 6 ft 1.0 in. tall? (Assume that 1 meter equals 39.37 in.)

Exercise:**Problem:**

Mount Everest, at 29,028 feet, is the tallest mountain on the Earth. What is its height in kilometers? (Assume that 1 kilometer equals 3,281 feet.)

Solution:

8.847 km

Exercise:

Problem: The speed of sound is measured to be 342 m/s on a certain day. What is this in km/h?

Exercise:**Problem:**

Tectonic plates are large segments of the Earth's crust that move slowly. Suppose that one such plate has an average speed of 4.0 cm/year. (a) What distance does it move in 1 s at this speed? (b) What is its speed in kilometers per million years?

Solution:

(a) 1.3×10^{-9} m

(b) 40 km/My

Exercise:

Problem:

(a) Refer to [\[link\]](#) to determine the average distance between the Earth and the Sun. Then calculate the average speed of the Earth in its orbit in kilometers per second. (b) What is this in meters per second?

Glossary

physical quantity

a characteristic or property of an object that can be measured or calculated from other measurements

units

a standard used for expressing and comparing measurements

SI units

the international system of units that scientists in most countries have agreed to use; includes units such as meters, liters, and grams

English units

system of measurement used in the United States; includes units of measurement such as feet, gallons, and pounds

fundamental units

units that can only be expressed relative to the procedure used to measure them

derived units

units that can be calculated using algebraic combinations of the fundamental units

second

the SI unit for time, abbreviated (s)

meter

the SI unit for length, abbreviated (m)

kilogram

the SI unit for mass, abbreviated (kg)

metric system

a system in which values can be calculated in factors of 10

order of magnitude

refers to the size of a quantity as it relates to a power of 10

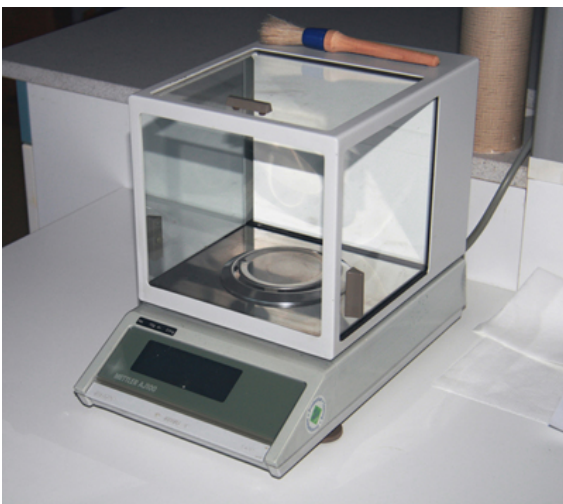
conversion factor

a ratio expressing how many of one unit are equal to another unit

Accuracy, Precision, and Significant Figures



A double-pan mechanical balance is used to compare different masses. Usually an object with unknown mass is placed in one pan and objects of known mass are placed in the other pan. When the bar that connects the two pans is horizontal, then the masses in both pans are equal. The “known masses” are typically metal cylinders of standard mass such as 1 gram, 10 grams, and 100 grams.
(credit: Serge Melki)



Many mechanical balances, such as double-pan balances, have been replaced by digital scales, which can typically measure the mass of an object more precisely. Whereas a mechanical balance may only read the mass of an object to the nearest tenth of a gram, many digital scales can measure the mass of an object up to the nearest thousandth of a gram. (credit: Karel Jakubec)

LEARNING OBJECTIVES

By the end of this section, you will be able to:

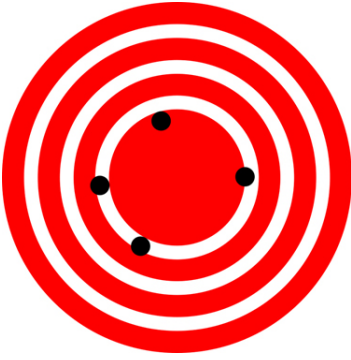
- Determine the appropriate number of significant figures in both addition and subtraction, as well as multiplication and division calculations.
- Calculate the percent uncertainty of a measurement.

Accuracy and Precision of a Measurement

Science is based on observation and experiment—that is, on measurements. **Accuracy** is how close a measurement is to the correct value for that measurement. For example, let us say that you are measuring the length of standard computer paper. The packaging in which you purchased the paper states that it is 11.0 inches long. You measure the length of the paper three times and obtain the following measurements: 11.1 in., 11.2 in., and 10.9 in. These measurements are quite accurate because they are very close to the correct value of 11.0 inches. In contrast, if you had obtained a measurement of 12 inches, your measurement would not be very accurate.

The **precision** of a measurement system is refers to how close the agreement is between repeated measurements (which are repeated under the same conditions). Consider the example of the paper measurements. The precision of the measurements refers to the spread of the measured values. One way to analyze the precision of the measurements would be to determine the range, or difference, between the lowest and the highest measured values. In that case, the lowest value was 10.9 in. and the highest value was 11.2 in. Thus, the measured values deviated from each other by at most 0.3 in. These measurements were relatively precise because they did not vary too much in value. However, if the measured values had been 10.9, 11.1, and 11.9, then the measurements would not be very precise because there would be significant variation from one measurement to another.

The measurements in the paper example are both accurate and precise, but in some cases, measurements are accurate but not precise, or they are precise but not accurate. Let us consider an example of a GPS system that is attempting to locate the position of a restaurant in a city. Think of the restaurant location as existing at the center of a bull's-eye target, and think of each GPS attempt to locate the restaurant as a black dot. In [\[link\]](#), you can see that the GPS measurements are spread out far apart from each other, but they are all relatively close to the actual location of the restaurant at the center of the target. This indicates a low precision, high accuracy measuring system. However, in [\[link\]](#), the GPS measurements are concentrated quite closely to one another, but they are far away from the target location. This indicates a high precision, low accuracy measuring system.



A GPS system attempts to locate a restaurant at the center of the bull's-eye. The black dots represent each attempt to pinpoint the location of the restaurant. The dots are spread out quite far apart from one another, indicating low precision, but they are each rather close to the actual location of the restaurant, indicating high accuracy.
(credit: Dark Evil)



In this figure,
the dots are
concentrated
rather closely to
one another,
indicating high
precision, but
they are rather
far away from
the actual
location of the
restaurant,
indicating low
accuracy.
(credit: Dark
Evil)

Accuracy, Precision, and Uncertainty

The degree of accuracy and precision of a measuring system are related to the **uncertainty** in the measurements. Uncertainty is a quantitative measure of how much your measured values deviate from a standard or expected value. If your measurements are not very accurate or precise, then the

uncertainty of your values will be very high. In more general terms, uncertainty can be thought of as a disclaimer for your measured values. For example, if someone asked you to provide the mileage on your car, you might say that it is 45,000 miles, plus or minus 500 miles. The plus or minus amount is the uncertainty in your value. That is, you are indicating that the actual mileage of your car might be as low as 44,500 miles or as high as 45,500 miles, or anywhere in between. All measurements contain some amount of uncertainty. In our example of measuring the length of the paper, we might say that the length of the paper is 11 in., plus or minus 0.2 in. The uncertainty in a measurement, A , is often denoted as δA (“delta A ”), so the measurement result would be recorded as $A \pm \delta A$. In our paper example, the length of the paper could be expressed as $11 \text{ in.} \pm 0.2$.

The factors contributing to uncertainty in a measurement include:

1. Limitations of the measuring device,
2. The skill of the person making the measurement,
3. Irregularities in the object being measured,
4. Any other factors that affect the outcome (highly dependent on the situation).

In our example, such factors contributing to the uncertainty could be the following: the smallest division on the ruler is 0.1 in., the person using the ruler has bad eyesight, or one side of the paper is slightly longer than the other. At any rate, the uncertainty in a measurement must be based on a careful consideration of all the factors that might contribute and their possible effects.

Note:

Making Connections: Real-World Connections—Fevers or Chills?

Uncertainty is a critical piece of information, both in physics and in many other real-world applications. Imagine you are caring for a sick child. You suspect the child has a fever, so you check his or her temperature with a thermometer. What if the uncertainty of the thermometer were 3.0°C ? If the child's temperature reading was 37.0°C (which is normal body temperature), the “true” temperature could be anywhere from a

hypothermic 34.0°C to a dangerously high 40.0°C. A thermometer with an uncertainty of 3.0°C would be useless.

Percent Uncertainty

One method of expressing uncertainty is as a percent of the measured value. If a measurement A is expressed with uncertainty, δA , the **percent uncertainty** (%unc) is defined to be

Equation:

$$\% \text{ unc} = \frac{\delta A}{A} \times 100\%.$$

Example:

Calculating Percent Uncertainty: A Bag of Apples

A grocery store sells 5 lb bags of apples. You purchase four bags over the course of a month and weigh the apples each time. You obtain the following measurements:

Week 1 weight: 4.8 lb

Week 2 weight: 5.3 lb

Week 3 weight: 4.9 lb

Week 4 weight: 5.4 lb

You determine that the weight of the 5 lb bag has an uncertainty of ± 0.4 lb. What is the percent uncertainty of the bag's weight?

Strategy

First, observe that the expected value of the bag's weight, A , is 5 lb. The uncertainty in this value, δA , is 0.4 lb. We can use the following equation to determine the percent uncertainty of the weight:

Equation:

$$\% \text{ unc} = \frac{\delta A}{A} \times 100\%.$$

Solution

Plug the known values into the equation:

Equation:

$$\% \text{ unc} = \frac{0.4 \text{ lb}}{5 \text{ lb}} \times 100\% = 8\%.$$

Discussion

We can conclude that the weight of the apple bag is $5 \text{ lb} \pm 8\%$. Consider how this percent uncertainty would change if the bag of apples were half as heavy, but the uncertainty in the weight remained the same. Hint for future calculations: when calculating percent uncertainty, always remember that you must multiply the fraction by 100%. If you do not do this, you will have a decimal quantity, not a percent value.

Uncertainties in Calculations

There is an uncertainty in anything calculated from measured quantities. For example, the area of a floor calculated from measurements of its length and width has an uncertainty because the length and width have uncertainties. How big is the uncertainty in something you calculate by multiplication or division? If the measurements going into the calculation have small uncertainties (a few percent or less), then the **method of adding percents** can be used for multiplication or division. This method says that *the percent uncertainty in a quantity calculated by multiplication or division is the sum of the percent uncertainties in the items used to make the calculation*. For example, if a floor has a length of 4.00 m and a width of 3.00 m, with uncertainties of 2% and 1%, respectively, then the area of the floor is 12.0 m^2 and has an uncertainty of 3%. (Expressed as an area this is 0.36 m^2 , which we round to 0.4 m^2 since the area of the floor is given to a tenth of a square meter.)

Exercise:

Check Your Understanding

Problem:

A high school track coach has just purchased a new stopwatch. The stopwatch manual states that the stopwatch has an uncertainty of ± 0.05 s. Runners on the track coach's team regularly clock 100 m sprints of 11.49 s to 15.01 s. At the school's last track meet, the first-place sprinter came in at 12.04 s and the second-place sprinter came in at 12.07 s. Will the coach's new stopwatch be helpful in timing the sprint team? Why or why not?

Solution:

No, the uncertainty in the stopwatch is too great to effectively differentiate between the sprint times.

Precision of Measuring Tools and Significant Figures

An important factor in the accuracy and precision of measurements involves the precision of the measuring tool. In general, a precise measuring tool is one that can measure values in very small increments. For example, a standard ruler can measure length to the nearest millimeter, while a caliper can measure length to the nearest 0.01 millimeter. The caliper is a more precise measuring tool because it can measure extremely small differences in length. The more precise the measuring tool, the more precise and accurate the measurements can be.

When we express measured values, we can only list as many digits as we initially measured with our measuring tool. For example, if you use a standard ruler to measure the length of a stick, you may measure it to be 36.7 cm. You could not express this value as 36.71 cm because your measuring tool was not precise enough to measure a hundredth of a centimeter. It should be noted that the last digit in a measured value has been estimated in some way by the person performing the measurement. For example, the person measuring the length of a stick with a ruler notices that the stick length seems to be somewhere in between 36.6 cm and 36.7 cm, and he or she must estimate the value of the last digit. Using the

method of **significant figures**, the rule is that *the last digit written down in a measurement is the first digit with some uncertainty*. In order to determine the number of significant digits in a value, start with the first measured value at the left and count the number of digits through the last digit written on the right. For example, the measured value 36.7 cm has three digits, or significant figures. Significant figures indicate the precision of a measuring tool that was used to measure a value.

Zeros

Special consideration is given to zeros when counting significant figures. The zeros in 0.053 are not significant, because they are only placekeepers that locate the decimal point. There are two significant figures in 0.053. The zeros in 10.053 are not placekeepers but are significant—this number has five significant figures. The zeros in 1300 may or may not be significant depending on the style of writing numbers. They could mean the number is known to the last digit, or they could be placekeepers. So 1300 could have two, three, or four significant figures. (To avoid this ambiguity, write 1300 in scientific notation.) *Zeros are significant except when they serve only as placekeepers.*

Exercise:

Check Your Understanding

Problem:

Determine the number of significant figures in the following measurements:

- a. 0.0009
- b. 15,450.0
- c. 6×10^3
- d. 87.990
- e. 30.42

Solution:

- (a) 1; the zeros in this number are placekeepers that indicate the decimal point
- (b) 6; here, the zeros indicate that a measurement was made to the 0.1 decimal point, so the zeros are significant
- (c) 1; the value 10^3 signifies the decimal place, not the number of measured values
- (d) 5; the final zero indicates that a measurement was made to the 0.001 decimal point, so it is significant
- (e) 4; any zeros located in between significant figures in a number are also significant

Significant Figures in Calculations

When combining measurements with different degrees of accuracy and precision, *the number of significant digits in the final answer can be no greater than the number of significant digits in the least precise measured value*. There are two different rules, one for multiplication and division and the other for addition and subtraction, as discussed below.

1. For multiplication and division: *The result should have the same number of significant figures as the quantity having the least significant figures entering into the calculation.* For example, the area of a circle can be calculated from its radius using $A = \pi r^2$. Let us see how many significant figures the area has if the radius has only two—say, $r = 1.2$ m. Then,

Equation:

$$A = \pi r^2 = (3.1415927...) \times (1.2 \text{ m})^2 = 4.5238934 \text{ m}^2$$

is what you would get using a calculator that has an eight-digit output. But because the radius has only two significant figures, it limits the calculated

quantity to two significant figures or

Equation:

$$A=4.5 \text{ m}^2,$$

even though π is good to at least eight digits.

2. For addition and subtraction: *The answer can contain no more decimal places than the least precise measurement.* Suppose that you buy 7.56 kg of potatoes in a grocery store as measured with a scale with precision 0.01 kg. Then you drop off 6.052 kg of potatoes at your laboratory as measured by a scale with precision 0.001 kg. Finally, you go home and add 13.7 kg of potatoes as measured by a bathroom scale with precision 0.1 kg. How many kilograms of potatoes do you now have, and how many significant figures are appropriate in the answer? The mass is found by simple addition and subtraction:

Equation:

$$\begin{array}{r} 7.56 \text{ kg} \\ - 6.052 \text{ kg} \\ \hline \end{array} \quad \begin{array}{l} +13.7 \text{ kg} \\ 15.208 \text{ kg} \end{array} = 15.2 \text{ kg}.$$

Next, we identify the least precise measurement: 13.7 kg. This measurement is expressed to the 0.1 decimal place, so our final answer must also be expressed to the 0.1 decimal place. Thus, the answer is rounded to the tenths place, giving us 15.2 kg.

Significant Figures in this Text

In this text, most numbers are assumed to have three significant figures. Furthermore, consistent numbers of significant figures are used in all worked examples. You will note that an answer given to three digits is based on input good to at least three digits, for example. If the input has fewer significant figures, the answer will also have fewer significant

figures. Care is also taken that the number of significant figures is reasonable for the situation posed. In some topics, particularly in optics, more accurate numbers are needed and more than three significant figures will be used. Finally, if a number is *exact*, such as the two in the formula for the circumference of a circle, $c = 2\pi r$, it does not affect the number of significant figures in a calculation.

Exercise:

Check Your Understanding

Problem:

Perform the following calculations and express your answer using the correct number of significant digits.

- (a) A woman has two bags weighing 13.5 pounds and one bag with a weight of 10.2 pounds. What is the total weight of the bags?
- (b) The force F on an object is equal to its mass m multiplied by its acceleration a . If a wagon with mass 55 kg accelerates at a rate of 0.0255 m/s^2 , what is the force on the wagon? (The unit of force is called the newton, and it is expressed with the symbol N.)

Solution:

- (a) 37.2 pounds; Because the number of bags is an exact value, it is not considered in the significant figures.
- (b) 1.4 N; Because the value 55 kg has only two significant figures, the final value must also contain two significant figures.

Note:

PhET Explorations: Estimation

Explore size estimation in one, two, and three dimensions! Multiple levels of difficulty allow for progressive skill improvement.

Summary

- Accuracy of a measured value refers to how close a measurement is to the correct value. The uncertainty in a measurement is an estimate of the amount by which the measurement result may differ from this value.
- Precision of measured values refers to how close the agreement is between repeated measurements.
- The precision of a *measuring tool* is related to the size of its measurement increments. The smaller the measurement increment, the more precise the tool.
- Significant figures express the precision of a measuring tool.
- When multiplying or dividing measured values, the final answer can contain only as many significant figures as the least precise value.
- When adding or subtracting measured values, the final answer cannot contain more decimal places than the least precise value.

Conceptual Questions

Exercise:

Problem:

What is the relationship between the accuracy and uncertainty of a measurement?

Exercise:

Problem:

Prescriptions for vision correction are given in units called *diopters* (D). Determine the meaning of that unit. Obtain information (perhaps by calling an optometrist or performing an internet search) on the minimum uncertainty with which corrections in diopters are determined and the accuracy with which corrective lenses can be produced. Discuss the sources of uncertainties in both the prescription and accuracy in the manufacture of lenses.

Problems & Exercises

Express your answers to problems in this section to the correct number of significant figures and proper units.

Exercise:**Problem:**

Suppose that your bathroom scale reads your mass as 65 kg with a 3% uncertainty. What is the uncertainty in your mass (in kilograms)?

Solution:

2 kg

Exercise:**Problem:**

A good-quality measuring tape can be off by 0.50 cm over a distance of 20 m. What is its percent uncertainty?

Exercise:**Problem:**

(a) A car speedometer has a 5.0% uncertainty. What is the range of possible speeds when it reads 90 km/h? (b) Convert this range to miles per hour. (1 km = 0.6214 mi)

Solution:

- a. 85.5 to 94.5 km/h
- b. 53.1 to 58.7 mi/h

Exercise:**Problem:**

An infant's pulse rate is measured to be 130 ± 5 beats/min. What is the percent uncertainty in this measurement?

Exercise:**Problem:**

(a) Suppose that a person has an average heart rate of 72.0 beats/min. How many beats does he or she have in 2.0 y? (b) In 2.00 y? (c) In 2.000 y?

Solution:

- (a) 7.6×10^7 beats
- (b) 7.57×10^7 beats
- (c) 7.57×10^7 beats

Exercise:**Problem:**

A can contains 375 mL of soda. How much is left after 308 mL is removed?

Exercise:

Problem:

State how many significant figures are proper in the results of the following calculations: (a) $(106.7)(98.2)/(46.210)(1.01)$ (b) $(18.7)^2$ (c) $(1.60 \times 10^{-19})(3712)$.

Solution:

- a. 3
- b. 3
- c. 3

Exercise:**Problem:**

(a) How many significant figures are in the numbers 99 and 100? (b) If the uncertainty in each number is 1, what is the percent uncertainty in each? (c) Which is a more meaningful way to express the accuracy of these two numbers, significant figures or percent uncertainties?

Exercise:**Problem:**

(a) If your speedometer has an uncertainty of 2.0 km/h at a speed of 90 km/h, what is the percent uncertainty? (b) If it has the same percent uncertainty when it reads 60 km/h, what is the range of speeds you could be going?

Solution:

- a) 2.2%
- (b) 59 to 61 km/h

Exercise:

Problem:

(a) A person's blood pressure is measured to be 120 ± 2 mm Hg. What is its percent uncertainty? (b) Assuming the same percent uncertainty, what is the uncertainty in a blood pressure measurement of 80 mm Hg?

Exercise:**Problem:**

A person measures his or her heart rate by counting the number of beats in 30 s. If 40 ± 1 beats are counted in 30.0 ± 0.5 s, what is the heart rate and its uncertainty in beats per minute?

Solution:

80 ± 3 beats/min

Exercise:

Problem: What is the area of a circle 3.102 cm in diameter?

Exercise:**Problem:**

If a marathon runner averages 9.5 mi/h, how long does it take him or her to run a 26.22 mi marathon?

Solution:

2.8 h

Exercise:

Problem:

A marathon runner completes a 42.188 km course in 2 h, 30 min, and 12 s. There is an uncertainty of 25 m in the distance traveled and an uncertainty of 1 s in the elapsed time. (a) Calculate the percent uncertainty in the distance. (b) Calculate the uncertainty in the elapsed time. (c) What is the average speed in meters per second? (d) What is the uncertainty in the average speed?

Exercise:**Problem:**

The sides of a small rectangular box are measured to be 1.80 ± 0.01 cm, 2.05 ± 0.02 cm, and 3.1 ± 0.1 cm long. Calculate its volume and uncertainty in cubic centimeters.

Solution:

$$11 \pm 1 \text{ cm}^3$$

Exercise:**Problem:**

When non-metric units were used in the United Kingdom, a unit of mass called the *pound-mass* (lbm) was employed, where $1 \text{ lbm} = 0.4539 \text{ kg}$. (a) If there is an uncertainty of 0.0001 kg in the pound-mass unit, what is its percent uncertainty? (b) Based on that percent uncertainty, what mass in pound-mass has an uncertainty of 1 kg when converted to kilograms?

Exercise:**Problem:**

The length and width of a rectangular room are measured to be 3.955 ± 0.005 m and 3.050 ± 0.005 m. Calculate the area of the room and its uncertainty in square meters.

Solution:

$$12.06 \pm 0.04 \text{ m}^2$$

Exercise:**Problem:**

A car engine moves a piston with a circular cross section of 7.500 ± 0.002 cm diameter a distance of 3.250 ± 0.001 cm to compress the gas in the cylinder. (a) By what amount is the gas decreased in volume in cubic centimeters? (b) Find the uncertainty in this volume.

Glossary

accuracy

the degree to which a measured value agrees with correct value for that measurement

method of adding percents

the percent uncertainty in a quantity calculated by multiplication or division is the sum of the percent uncertainties in the items used to make the calculation

percent uncertainty

the ratio of the uncertainty of a measurement to the measured value, expressed as a percentage

precision

the degree to which repeated measurements agree with each other

significant figures

express the precision of a measuring tool used to measure a value

uncertainty

a quantitative measure of how much your measured values deviate from a standard or expected value

Approximation

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Make reasonable approximations based on given data.

On many occasions, physicists, other scientists, and engineers need to make **approximations** or “guesstimates” for a particular quantity. What is the distance to a certain destination? What is the approximate density of a given item? About how large a current will there be in a circuit? Many approximate numbers are based on formulae in which the input quantities are known only to a limited accuracy. As you develop problem-solving skills (that can be applied to a variety of fields through a study of physics), you will also develop skills at approximating. You will develop these skills through thinking more quantitatively, and by being willing to take risks. As with any endeavor, experience helps, as well as familiarity with units. These approximations allow us to rule out certain scenarios or unrealistic numbers. Approximations also allow us to challenge others and guide us in our approaches to our scientific world. Let us do two examples to illustrate this concept.

Example:

Approximate the Height of a Building

Can you approximate the height of one of the buildings on your campus, or in your neighborhood? Let us make an approximation based upon the

height of a person. In this example, we will calculate the height of a 39-story building.

Strategy

Think about the average height of an adult male. We can approximate the height of the building by scaling up from the height of a person.

Solution

Based on information in the example, we know there are 39 stories in the building. If we use the fact that the height of one story is approximately equal to about the length of two adult humans (each human is about 2 m tall), then we can estimate the total height of the building to be

Equation:

$$\frac{2 \text{ m}}{1 \text{ person}} \times \frac{2 \text{ person}}{1 \text{ story}} \times 39 \text{ stories} = 156 \text{ m.}$$

Discussion

You can use known quantities to determine an approximate measurement of unknown quantities. If your hand measures 10 cm across, how many hand lengths equal the width of your desk? What other measurements can you approximate besides length?

Example:

Approximating Vast Numbers: a Trillion Dollars



A bank stack contains one-hundred

\$100 bills, and is worth \$10,000. How many bank stacks make up a trillion dollars? (credit: Andrew Magill)

The U.S. federal deficit in the 2008 fiscal year was a little greater than \$10 trillion. Most of us do not have any concept of how much even one trillion actually is. Suppose that you were given a trillion dollars in \$100 bills. If you made 100-bill stacks and used them to evenly cover a football field (between the end zones), make an approximation of how high the money pile would become. (We will use feet/inches rather than meters here because football fields are measured in yards.) One of your friends says 3 in., while another says 10 ft. What do you think?

Strategy

When you imagine the situation, you probably envision thousands of small stacks of 100 wrapped \$100 bills, such as you might see in movies or at a bank. Since this is an easy-to-approximate quantity, let us start there. We can find the volume of a stack of 100 bills, find out how many stacks make up one trillion dollars, and then set this volume equal to the area of the football field multiplied by the unknown height.

Solution

(1) Calculate the volume of a stack of 100 bills. The dimensions of a single bill are approximately 3 in. by 6 in. A stack of 100 of these is about 0.5 in. thick. So the total volume of a stack of 100 bills is:

Equation:

$$\begin{aligned}\text{volume of stack} &= \text{length} \times \text{width} \times \text{height}, \\ \text{volume of stack} &= 6 \text{ in.} \times 3 \text{ in.} \times 0.5 \text{ in.}, \\ \text{volume of stack} &= 9 \text{ in.}^3.\end{aligned}$$

(2) Calculate the number of stacks. Note that a trillion dollars is equal to $\$1 \times 10^{12}$, and a stack of one-hundred \$100 bills is equal to \$10,000, or $\$1 \times 10^4$. The number of stacks you will have is:

Equation:

$$\$1 \times 10^{12} (\text{a trillion dollars}) / \$1 \times 10^4 \text{ per stack} = 1 \times 10^8 \text{ stacks.}$$

(3) Calculate the area of a football field in square inches. The area of a football field is $100 \text{ yd} \times 50 \text{ yd}$, which gives $5,000 \text{ yd}^2$. Because we are working in inches, we need to convert square yards to square inches:

Equation:

$$\text{Area} = 5,000 \text{ yd}^2 \times \frac{3 \text{ ft}}{1 \text{ yd}} \times \frac{3 \text{ ft}}{1 \text{ yd}} \times \frac{12 \text{ in.}}{1 \text{ ft}} \times \frac{12 \text{ in.}}{1 \text{ ft}} = 6,480,000 \text{ in.}^2,$$
$$\text{Area} \approx 6 \times 10^6 \text{ in.}^2.$$

This conversion gives us $6 \times 10^6 \text{ in.}^2$ for the area of the field. (Note that we are using only one significant figure in these calculations.)

(4) Calculate the total volume of the bills. The volume of all the \$100-bill stacks is $9 \text{ in.}^3/\text{stack} \times 10^8 \text{ stacks} = 9 \times 10^8 \text{ in.}^3$.

(5) Calculate the height. To determine the height of the bills, use the equation:

Equation:

$$\text{volume of bills} = \text{area of field} \times \text{height of money:}$$

$$\text{Height of money} = \frac{\text{volume of bills}}{\text{area of field}},$$

$$\text{Height of money} = \frac{9 \times 10^8 \text{ in.}^3}{6 \times 10^6 \text{ in.}^2} = 1.33 \times 10^2 \text{ in.},$$

$$\text{Height of money} \approx 1 \times 10^2 \text{ in.} = 100 \text{ in.}$$

The height of the money will be about 100 in. high. Converting this value to feet gives

Equation:

$$100 \text{ in.} \times \frac{1 \text{ ft}}{12 \text{ in.}} = 8.33 \text{ ft} \approx 8 \text{ ft.}$$

Discussion

The final approximate value is much higher than the early estimate of 3 in., but the other early estimate of 10 ft (120 in.) was roughly correct. How did the approximation measure up to your first guess? What can this exercise tell you in terms of rough “guesstimates” versus carefully calculated approximations?

Exercise:
Check Your Understanding

Problem:

Using mental math and your understanding of fundamental units, approximate the area of a regulation basketball court. Describe the process you used to arrive at your final approximation.

Solution:

An average male is about two meters tall. It would take approximately 15 men laid out end to end to cover the length, and about 7 to cover the width. That gives an approximate area of 420 m^2 .

Summary

Scientists often approximate the values of quantities to perform calculations and analyze systems.

Problems & Exercises

Exercise:

Problem: How many heartbeats are there in a lifetime?

Solution:

Sample answer: 2×10^9 heartbeats

Exercise:

Problem:

A generation is about one-third of a lifetime. Approximately how many generations have passed since the year 0 AD?

Exercise:

Problem:

How many times longer than the mean life of an extremely unstable atomic nucleus is the lifetime of a human? (Hint: The lifetime of an unstable atomic nucleus is on the order of 10^{-22} s.)

Solution:

Sample answer: 2×10^{31} if an average human lifetime is taken to be about 70 years.

Exercise:**Problem:**

Calculate the approximate number of atoms in a bacterium. Assume that the average mass of an atom in the bacterium is ten times the mass of a hydrogen atom. (Hint: The mass of a hydrogen atom is on the order of 10^{-27} kg and the mass of a bacterium is on the order of 10^{-15} kg.)



This color-enhanced photo shows *Salmonella typhimurium* (red) attacking human cells.

These bacteria are commonly known for causing foodborne illness. Can you estimate the number of atoms in each bacterium? (credit: Rocky Mountain Laboratories, NIAID, NIH)

Exercise:

Problem:

Approximately how many atoms thick is a cell membrane, assuming all atoms there average about twice the size of a hydrogen atom?

Solution:

Sample answer: 50 atoms

Exercise:

Problem:

(a) What fraction of Earth's diameter is the greatest ocean depth? (b) The greatest mountain height?

Exercise:

Problem:

(a) Calculate the number of cells in a hummingbird assuming the mass of an average cell is ten times the mass of a bacterium. (b) Making the same assumption, how many cells are there in a human?

Solution:

Sample answers:

(a) 10^{12} cells/hummingbird

(b) 10^{16} cells/human

Exercise:

Problem:

Assuming one nerve impulse must end before another can begin, what is the maximum firing rate of a nerve in impulses per second?

Glossary

approximation

an estimated value based on prior experience and reasoning

Connection for AP® Courses
class="introduction"

The motion
of an
American
kestrel
through the
air can be
described by
the bird's
displacement
, speed,
velocity, and
acceleration.
When it flies
in a straight
line without
any change
in direction,
its motion is
said to be
one
dimensional.
(credit: Vince
Maidens,
Wikimedia
Commons)



Objects are in motion everywhere we look. Everything from a tennis game to a space-probe flyby of the planet Neptune involves motion. When you are resting, your heart moves blood through your veins. Even in inanimate objects, there is a continuous motion in the vibrations of atoms and molecules. Questions about motion are interesting in and of themselves: *How long will it take for a space probe to get to Mars? Where will a football land if it is thrown at a certain angle?*

Understanding motion will not only provide answers to these questions, but will be key to understanding more advanced concepts in physics. For example, the discussion of force in Chapter 4 will not fully make sense until you understand acceleration. This relationship between force and acceleration is also critical to understanding Big Idea 3.

Additionally, this unit will explore the topic of reference frames, a critical component to quantifying how things move. If you have ever waved to a departing friend at a train station, you are likely familiar with this idea. While you see your friend move away from you at a considerable rate, those sitting with her will likely see her as not moving. The effect that the chosen reference frame has on your observations is substantial, and an understanding of this is needed to grasp both Enduring Understanding 3.A and Essential Knowledge 3.A.1.

Our formal study of physics begins with **kinematics**, which is defined as the *study of motion without considering its causes*. In one- and two-dimensional kinematics we will study only the *motion* of a football, for example, without worrying about what forces cause or change its motion. In this chapter, we examine the simplest type of motion—namely, motion along a straight line, or one-dimensional motion. Later, in two-dimensional kinematics, we apply concepts developed here to study motion along curved paths (two- and three-dimensional motion), for example, that of a car rounding a curve.

The content in this chapter supports:

Big Idea 3 The interactions of an object with other objects can be described by forces.

Enduring Understanding 3.A All forces share certain common characteristics when considered by observers in inertial reference frames.

Essential Knowledge 3.A.1 An observer in a particular reference frame can describe the motion of an object using such quantities as position, displacement, distance, velocity, speed, and acceleration.

Displacement



These cyclists in Vietnam can be described by their position relative to buildings and a canal. Their motion can be described by their change in position, or displacement, in the frame of reference. (credit: Suzan Black, Fotopedia)

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define position, displacement, distance, and distance traveled in a particular frame of reference.
- Explain the relationship between position and displacement.
- Distinguish between displacement and distance traveled.
- Calculate displacement and distance given initial position, final position, and the path between the two.

The information presented in this section supports the following AP® learning objectives and science practices:

- **3.A.1.1** The student is able to express the motion of an object using narrative, mathematical, and graphical representations. (S.P. 1.5, 2.1, 2.2)
- **3.A.1.3** The student is able to analyze experimental data describing the motion of an object and is able to express the results of the analysis using narrative, mathematical, and graphical representations. (S.P. 5.1)

Position

In order to describe the motion of an object, you must first be able to describe its **position**—where it is at any particular time. More precisely, you need to specify its position relative to a convenient reference frame. Earth is often used as a reference frame, and we often describe the position of an object as it relates to stationary objects in that reference frame. For example, a rocket launch would be described in terms of the position of the rocket with respect to the Earth as a whole, while a professor's position could be described in terms of where she is in relation to the nearby white board. (See [\[link\]](#).) In other cases, we use reference frames that are not stationary but are in motion relative to the Earth. To describe the position of a person in an airplane, for example, we use the airplane, not the Earth, as the reference frame. (See [\[link\]](#).)

Displacement

If an object moves relative to a reference frame (for example, if a professor moves to the right relative to a white board or a passenger moves toward the rear of an airplane), then the object's position changes. This change in position is known as **displacement**. The word “displacement” implies that an object has moved, or has been displaced.

Note:**Displacement**

Displacement is the *change in position* of an object:

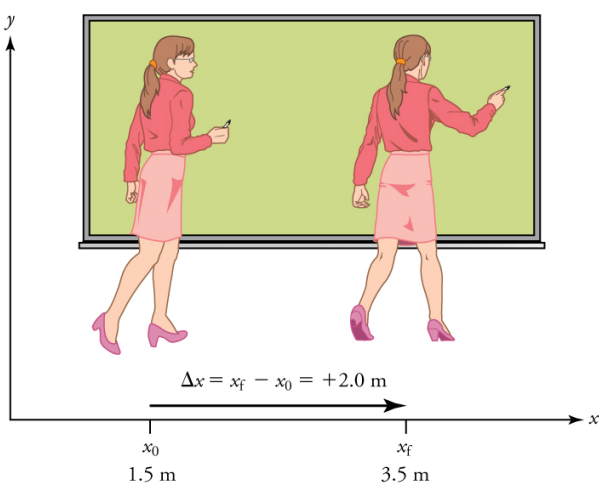
Equation:

$$\Delta x = x_f - x_0,$$

where Δx is displacement, x_f is the final position, and x_0 is the initial position.

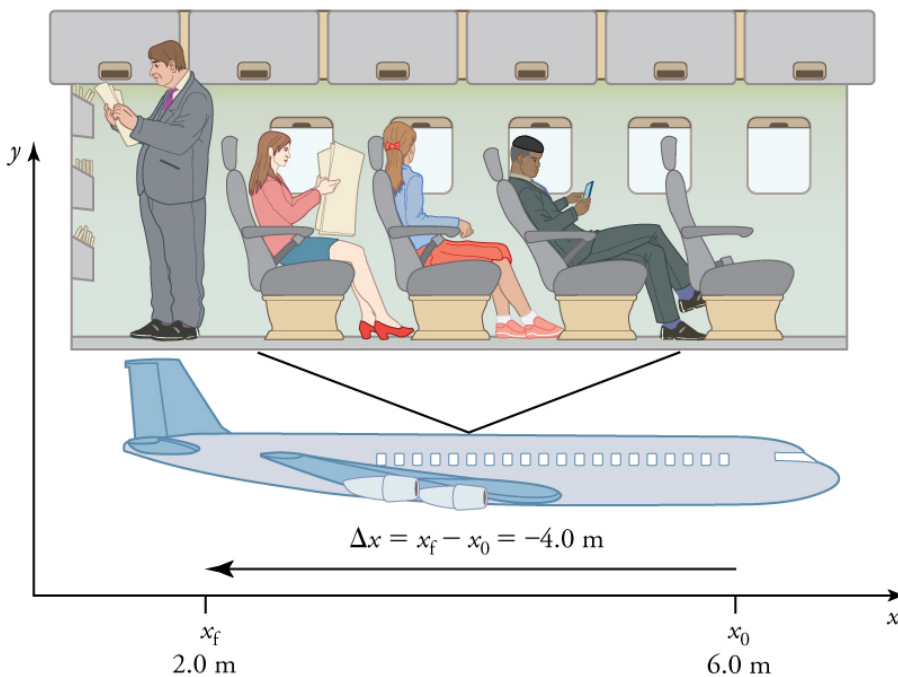
In this text the upper case Greek letter Δ (delta) always means “change in” whatever quantity follows it; thus, Δx means *change in position*. Always solve for displacement by subtracting initial position x_0 from final position x_f .

Note that the SI unit for displacement is the meter (m) (see [Physical Quantities and Units](#)), but sometimes kilometers, miles, feet, and other units of length are used. Keep in mind that when units other than the meter are used in a problem, you may need to convert them into meters to complete the calculation.



A professor paces left and right while lecturing. Her position

relative to the blackboard is given by x . The $+2.0$ m displacement of the professor relative to the blackboard is represented by an arrow pointing to the right.



A passenger moves from his seat to the back of the plane. His location relative to the airplane is given by x . The -4 m displacement of the passenger relative to the plane is represented by an arrow toward the rear of the plane. Notice that the arrow representing his displacement is twice as long as the arrow representing the displacement of the professor (he moves twice as far) in [\[link\]](#).

Note that displacement has a direction as well as a magnitude. The professor's displacement is 2.0 m to the right, and the airline passenger's displacement is 4.0 m toward the rear. In one-dimensional motion, direction can be specified with a plus or minus sign. When you begin a problem, you should select which direction is positive (usually that will be to the right or up, but you are free to select positive as being any direction). The professor's initial position is $x_0 = 1.5$ m and her final position is $x_f = 3.5$ m. Thus her displacement is

Equation:

$$\Delta x = x_f - x_0 = 3.5 \text{ m} - 1.5 \text{ m} = +2.0 \text{ m}.$$

In this coordinate system, motion to the right is positive, whereas motion to the left is negative. Similarly, the airplane passenger's initial position is $x_0 = 6.0$ m and his final position is $x_f = 2.0$ m, so his displacement is

Equation:

$$\Delta x = x_f - x_0 = 2.0 \text{ m} - 6.0 \text{ m} = -4.0 \text{ m}.$$

His displacement is negative because his motion is toward the rear of the plane, or in the negative x direction in our coordinate system.

Distance

Although displacement is described in terms of direction, distance is not. **Distance** is defined to be *the magnitude or size of displacement between two positions*. Note that the distance between two positions is not the same as the distance traveled between them. **Distance traveled** is *the total length of the path traveled between two positions*. Distance has no direction and, thus, no sign. For example, the distance the professor walks is 2.0 m. The distance the airplane passenger walks is 4.0 m.

Note:

Misconception Alert: Distance Traveled vs. Magnitude of Displacement

It is important to note that the *distance traveled*, however, can be greater than the magnitude of the displacement (by magnitude, we mean just the size of the displacement without regard to its direction; that is, just a number with a unit). For example, the professor could pace back and forth many times, perhaps walking a distance of 150 m during a lecture, yet still end up only 2.0 m to the right of her starting point. In this case her displacement would be +2.0 m, the magnitude of her displacement would be 2.0 m, but the distance she traveled would be 150 m. In kinematics we nearly always deal with displacement and magnitude of displacement, and almost never with distance traveled. One way to think about this is to assume you marked the start of the motion and the end of the motion. The displacement is simply the difference in the position of the two marks and is independent of the path taken in traveling between the two marks. The distance traveled, however, is the total length of the path taken between the two marks.

Exercise:

Check Your Understanding

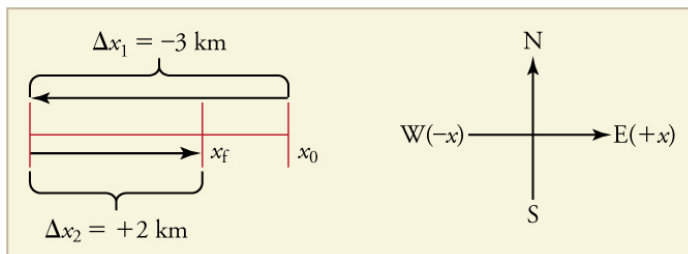
Problem:

A cyclist rides 3 km west and then turns around and rides 2 km east.

(a) What is her displacement? (b) What distance does she ride? (c)

What is the magnitude of her displacement?

Solution:



(a) The rider's displacement is $\Delta x = x_f - x_0 = -1 \text{ km}$. (The displacement is negative because we take east to be positive and west to be negative.)

(b) The distance traveled is $3 \text{ km} + 2 \text{ km} = 5 \text{ km}$.

(c) The magnitude of the displacement is 1 km .

Section Summary

- Kinematics is the study of motion without considering its causes. In this chapter, it is limited to motion along a straight line, called one-dimensional motion.
- Displacement is the change in position of an object.
- In symbols, displacement Δx is defined to be

Equation:

$$\Delta x = x_f - x_0,$$

where x_0 is the initial position and x_f is the final position. In this text, the Greek letter Δ (delta) always means “change in” whatever quantity follows it. The SI unit for displacement is the meter (m). Displacement has a direction as well as a magnitude.

- When you start a problem, assign which direction will be positive.
- Distance is the magnitude of displacement between two positions.
- Distance traveled is the total length of the path traveled between two positions.

Conceptual Questions

Exercise:

Problem:

Give an example in which there are clear distinctions among distance traveled, displacement, and magnitude of displacement. Specifically identify each quantity in your example.

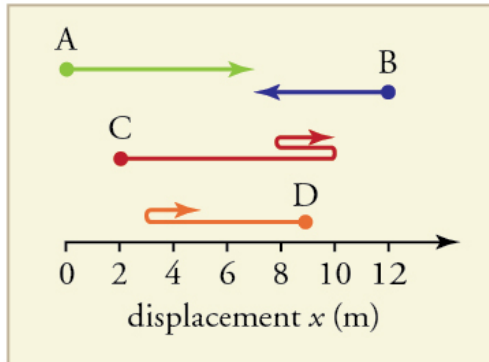
Exercise:

Problem:

Under what circumstances does distance traveled equal magnitude of displacement? What is the only case in which magnitude of displacement and displacement are exactly the the same?

Exercise:**Problem:**

Bacteria move back and forth by using their flagella (structures that look like little tails). Speeds of up to $50 \mu\text{m/s}$ ($50 \times 10^{-6} \text{ m/s}$) have been observed. The total distance traveled by a bacterium is large for its size, while its displacement is small. Why is this?

Problems & Exercises**Exercise:****Problem:**

Find the following for path A in [\[link\]](#): (a) The distance traveled. (b) The magnitude of the displacement from start to finish. (c) The displacement from start to finish.

Solution:

(a) 7 m

(b) 7 m

(c) +7 m

Exercise:

Problem:

Find the following for path B in [\[link\]](#): (a) The distance traveled. (b) The magnitude of the displacement from start to finish. (c) The displacement from start to finish.

Exercise:

Problem:

Find the following for path C in [\[link\]](#): (a) The distance traveled. (b) The magnitude of the displacement from start to finish. (c) The displacement from start to finish.

Solution:

(a) 13 m

(b) 9 m

(c) +9 m

Exercise:

Problem:

Find the following for path D in [\[link\]](#): (a) The distance traveled. (b) The magnitude of the displacement from start to finish. (c) The displacement from start to finish.

Test Prep for AP Courses

Exercise:

Problem:

Which of the following statements comparing position, distance traveled, and displacement is correct?

- a. An object may record a distance traveled of zero while recording a non-zero displacement.
 - b. An object may record a non-zero distance traveled while recording a displacement of zero.
 - c. An object may record a non-zero distance traveled while maintaining a position of zero.
 - d. An object may record a non-zero displacement while maintaining a position of zero.
-

Solution:

(a)

Glossary

kinematics

the study of motion without considering its causes

position

the location of an object at a particular time

displacement

the change in position of an object

distance

the magnitude of displacement between two positions

distance traveled

the total length of the path traveled between two positions

Vectors, Scalars, and Coordinate Systems



The motion of this Eclipse Concept jet can be described in terms of the distance it has traveled (a scalar quantity) or its displacement in a specific direction (a vector quantity). In order to specify the direction of motion, its displacement must be described based on a coordinate system. In this case, it may be convenient to choose motion toward the left as positive motion (it is the forward direction for the plane), although in many cases, the x -coordinate runs from left to right, with motion to the right as positive and motion to the left as negative. (credit: Armchair Aviator, Flickr)

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define and distinguish between scalar and vector quantities.
- Assign a coordinate system for a scenario involving one-dimensional motion.

The information presented in this section supports the following AP® learning objectives and science practices:

- **3.A.1.2** The student is able to design an experimental investigation of the motion of an object.

What is the difference between distance and displacement? Whereas displacement is defined by both direction and magnitude, distance is defined only by magnitude. Displacement is an example of a vector quantity. Distance is an example of a scalar quantity. A **vector** is any quantity with both *magnitude and direction*. Other examples of vectors include a velocity of 90 km/h east and a force of 500 newtons straight down.

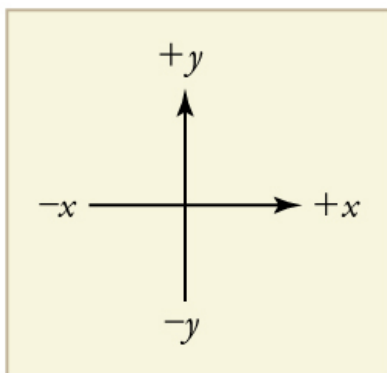
The direction of a vector in one-dimensional motion is given simply by a plus (+) or minus (−) sign. Vectors are represented graphically by arrows. An arrow used to represent a vector has a length proportional to the vector's magnitude (e.g., the larger the magnitude, the longer the length of the vector) and points in the same direction as the vector.

Some physical quantities, like distance, either have no direction or none is specified. A **scalar** is any quantity that has a magnitude, but no direction. For example, a 20°C temperature, the 250 kilocalories (250 Calories) of energy in a candy bar, a 90 km/h speed limit, a person's 1.8 m height, and a distance of 2.0 m are all scalars—quantities with no specified direction.

Note, however, that a scalar can be negative, such as a -20°C temperature. In this case, the minus sign indicates a point on a scale rather than a direction. Scalars are never represented by arrows.

Coordinate Systems for One-Dimensional Motion

In order to describe the direction of a vector quantity, you must designate a coordinate system within the reference frame. For one-dimensional motion, this is a simple coordinate system consisting of a one-dimensional coordinate line. In general, when describing horizontal motion, motion to the right is usually considered positive, and motion to the left is considered negative. With vertical motion, motion up is usually positive and motion down is negative. In some cases, however, as with the jet in [\[link\]](#), it can be more convenient to switch the positive and negative directions. For example, if you are analyzing the motion of falling objects, it can be useful to define downwards as the positive direction. If people in a race are running to the left, it is useful to define left as the positive direction. It does not matter as long as the system is clear and consistent. Once you assign a positive direction and start solving a problem, you cannot change it.



It is usually
convenient to
consider motion
upward or to the
right as positive
(+) and motion

downward or to the
left as negative (−)

.

Exercise:

Check Your Understanding

Problem:

A person's speed can stay the same as he or she rounds a corner and changes direction. Given this information, is speed a scalar or a vector quantity? Explain.

Solution:

Speed is a scalar quantity. It does not change at all with direction changes; therefore, it has magnitude only. If it were a vector quantity, it would change as direction changes (even if its magnitude remained constant).

Section Summary

- A vector is any quantity that has magnitude and direction.
- A scalar is any quantity that has magnitude but no direction.
- Displacement and velocity are vectors, whereas distance and speed are scalars.
- In one-dimensional motion, direction is specified by a plus or minus sign to signify left or right, up or down, and the like.

Conceptual Questions

Exercise:

Problem:

A student writes, “A bird that is diving for prey has a speed of -10 m/s .” What is wrong with the student's statement? What has the student actually described? Explain.

Exercise:

Problem: What is the speed of the bird in [\[link\]](#)?

Exercise:**Problem:**

Acceleration is the change in velocity over time. Given this information, is acceleration a vector or a scalar quantity? Explain.

Exercise:**Problem:**

A weather forecast states that the temperature is predicted to be -5°C the following day. Is this temperature a vector or a scalar quantity? Explain.

Note:**Switching Reference Frames**

A fundamental tenet of physics is that information about an event can be gathered from a variety of reference frames. For example, imagine that you are a passenger walking toward the front of a bus. As you walk, your motion is observed by a fellow bus passenger and by an observer standing on the sidewalk.

Both the bus passenger and sidewalk observer will be able to collect information about you. They can determine how far you moved and how much time it took you to do so. However, while you moved at a consistent pace, both observers will get different results. To the passenger sitting on the bus, you moved forward at what one would consider a normal pace,

something similar to how quickly you would walk outside on a sunny day. To the sidewalk observer though, you will have moved much quicker. Because the bus is also moving forward, the distance you move forward against the sidewalk each second increases, and the sidewalk observer must conclude that you are moving at a greater pace.

To show that you understand this concept, you will need to create an event and think of a way to view this event from two different frames of reference. In order to ensure that the event is being observed simultaneously from both frames, you will need an assistant to help out.

An example of a possible event is to have a friend ride on a skateboard while tossing a ball. How will your friend observe the ball toss, and how will those observations be different from your own?

Your task is to describe your event and the observations of your event from both frames of reference. Answer the following questions below to demonstrate your understanding. For assistance, you can review the information given in the 'Position' paragraph at the start of Section 2.1.

1. What is your event? What object are both you and your assistant observing?
2. What do *you* see as the event takes place?
3. What does *your assistant* see as the event takes place?
4. How do your reference frames cause you and your assistant to have two different sets of observations?

Test Prep for AP Courses

Exercise:

Problem:

A student is trying to determine the acceleration of a feather as she drops it to the ground. If the student is looking to achieve a positive velocity and positive acceleration, what is the *most sensible* way to set up her coordinate system?

- a. Her hand should be a coordinate of zero and the upward direction should be considered positive.
- b. Her hand should be a coordinate of zero and the downward direction should be considered positive.
- c. The floor should be a coordinate of zero and the upward direction should be considered positive.
- d. The floor should be a coordinate of zero and the downward direction should be considered positive.

Glossary

scalar

a quantity that is described by magnitude, but not direction

vector

a quantity that is described by both magnitude and direction

Time, Velocity, and Speed



The motion of these racing snails can be described by their speeds and their velocities.
(credit: tobitasflickr, Flickr)

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain the relationships between instantaneous velocity, average velocity, instantaneous speed, average speed, displacement, and time.
- Calculate velocity and speed given initial position, initial time, final position, and final time.
- Derive a graph of velocity vs. time given a graph of position vs. time.
- Interpret a graph of velocity vs. time.

The information presented in this section supports the following AP® learning objectives and science practices:

- **3.A.1.1** The student is able to express the motion of an object using narrative, mathematical, and graphical representations. (S.P. 1.5, 2.1, 2.2)
- **3.A.1.3** The student is able to analyze experimental data describing the motion of an object and is able to express the results of the analysis using narrative, mathematical, and graphical representations. (S.P. 5.1)

There is more to motion than distance and displacement. Questions such as, “How long does a foot race take?” and “What was the runner's speed?” cannot be answered without an understanding of other concepts. In this section we add definitions of time, velocity, and speed to expand our description of motion.

Time

As discussed in [Physical Quantities and Units](#), the most fundamental physical quantities are defined by how they are measured. This is the case with time. Every measurement of time involves measuring a change in some physical quantity. It may be a number on a digital clock, a heartbeat, or the position of the Sun in the sky. In physics, the definition of time is simple—**time** is *change*, or the interval over which change occurs. It is impossible to know that time has passed unless something changes.

The amount of time or change is calibrated by comparison with a standard. The SI unit for time is the second, abbreviated s. We might, for example, observe that a certain pendulum makes one full swing every 0.75 s. We could then use the pendulum to measure time by counting its swings or, of course, by connecting the pendulum to a clock mechanism that registers time on a dial. This allows us to not only measure the amount of time, but also to determine a sequence of events.

How does time relate to motion? We are usually interested in elapsed time for a particular motion, such as how long it takes an airplane passenger to get from his seat to the back of the plane. To find elapsed time, we note the time at the beginning and end of the motion and subtract the two. For example, a lecture may start at 11:00 A.M. and end at 11:50 A.M., so that the elapsed time would be 50 min. **Elapsed time** Δt is the difference between the ending time and beginning time,

Equation:

$$\Delta t = t_f - t_0,$$

where Δt is the change in time or elapsed time, t_f is the time at the end of the motion, and t_0 is the time at the beginning of the motion. (As usual, the delta symbol, Δ , means the change in the quantity that follows it.)

Life is simpler if the beginning time t_0 is taken to be zero, as when we use a stopwatch. If we were using a stopwatch, it would simply read zero at the start of the lecture and 50 min at the end. If $t_0 = 0$, then $\Delta t = t_f \equiv t$.

In this text, for simplicity's sake,

- motion starts at time equal to zero ($t_0 = 0$)
- the symbol t is used for elapsed time unless otherwise specified ($\Delta t = t_f \equiv t$)

Velocity

Your notion of velocity is probably the same as its scientific definition. You know that if you have a large displacement in a small amount of time you have a large velocity, and that velocity has units of distance divided by time, such as miles per hour or kilometers per hour.

Note:

Average Velocity

Average velocity is *displacement (change in position) divided by the time of travel*,

Equation:

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_0}{t_f - t_0},$$

where \bar{v} is the *average* (indicated by the bar over the v) velocity, Δx is the change in position (or displacement), and x_f and x_0 are the final and beginning positions at times t_f and t_0 , respectively. If the starting time t_0 is taken to be zero, then the average velocity is simply

Equation:

$$\bar{v} = \frac{\Delta x}{t}.$$

Notice that this definition indicates that *velocity is a vector because displacement is a vector*. It has both magnitude and direction. The SI unit for velocity is meters per second or m/s, but many other units, such as km/h, mi/h (also written as mph), and cm/s, are in common use. Suppose, for example, an airplane passenger took 5 seconds to move -4 m (the minus sign indicates that displacement is toward the back of the plane). His average velocity would be

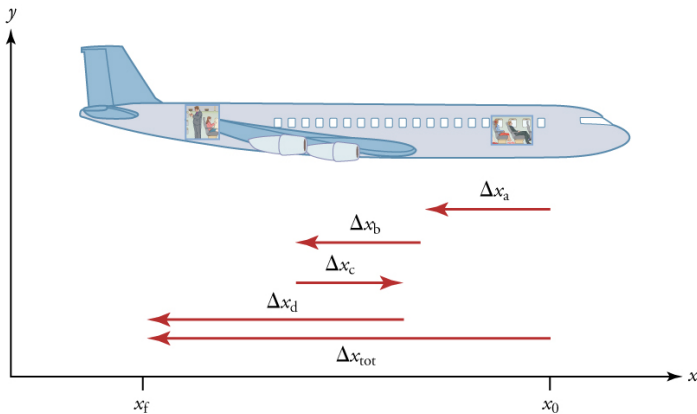
Equation:

$$\bar{v} = \frac{\Delta x}{t} = \frac{-4 \text{ m}}{5 \text{ s}} = -0.8 \text{ m/s}.$$

The minus sign indicates the average velocity is also toward the rear of the plane.

The average velocity of an object does not tell us anything about what happens to it between the starting point and ending point, however. For example, we cannot tell from average velocity whether the airplane

passenger stops momentarily or backs up before he goes to the back of the plane. To get more details, we must consider smaller segments of the trip over smaller time intervals.



A more detailed record of an airplane passenger heading toward the back of the plane, showing smaller segments of his trip.

The smaller the time intervals considered in a motion, the more detailed the information. When we carry this process to its logical conclusion, we are left with an infinitesimally small interval. Over such an interval, the average velocity becomes the *instantaneous velocity* or the *velocity at a specific instant*. A car's speedometer, for example, shows the magnitude (but not the direction) of the instantaneous velocity of the car. (Police give tickets based on instantaneous velocity, but when calculating how long it will take to get from one place to another on a road trip, you need to use average velocity.) **Instantaneous velocity** v is the average velocity at a specific instant in time (or over an infinitesimally small time interval).

Mathematically, finding instantaneous velocity, v , at a precise instant t can involve taking a limit, a calculus operation beyond the scope of this text.

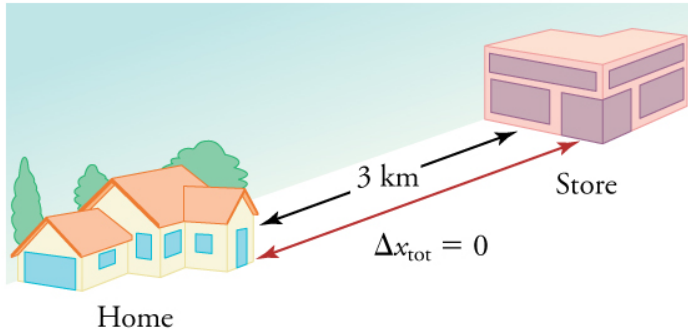
However, under many circumstances, we can find precise values for instantaneous velocity without calculus.

Speed

In everyday language, most people use the terms “speed” and “velocity” interchangeably. In physics, however, they do not have the same meaning and they are distinct concepts. One major difference is that speed has no direction. Thus *speed is a scalar*. Just as we need to distinguish between instantaneous velocity and average velocity, we also need to distinguish between instantaneous speed and average speed.

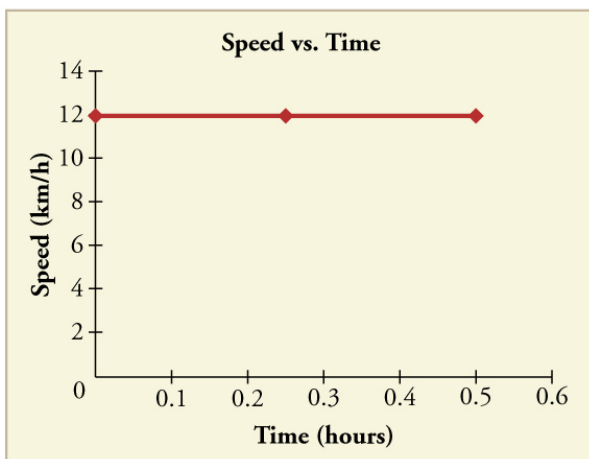
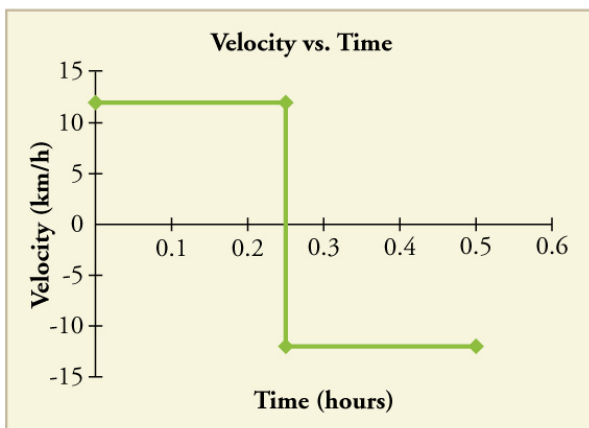
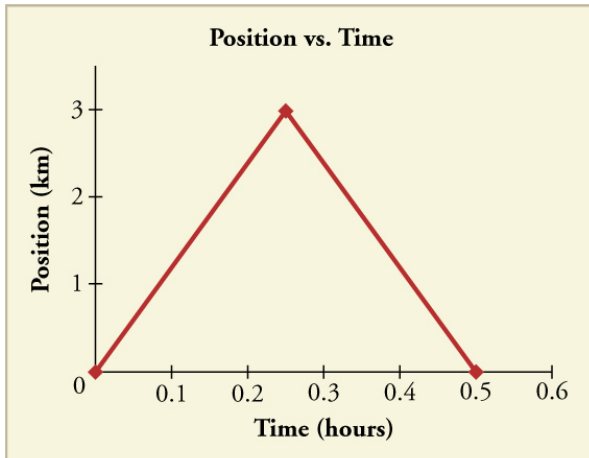
Instantaneous speed is the magnitude of instantaneous velocity. For example, suppose the airplane passenger at one instant had an instantaneous velocity of -3.0 m/s (the minus meaning toward the rear of the plane). At that same time his instantaneous speed was 3.0 m/s. Or suppose that at one time during a shopping trip your instantaneous velocity is 40 km/h due north. Your instantaneous speed at that instant would be 40 km/h—the same magnitude but without a direction. Average speed, however, is very different from average velocity. **Average speed** is the distance traveled divided by elapsed time.

We have noted that distance traveled can be greater than displacement. So average speed can be greater than average velocity, which is displacement divided by time. For example, if you drive to a store and return home in half an hour, and your car's odometer shows the total distance traveled was 6 km, then your average speed was 12 km/h. Your average velocity, however, was zero, because your displacement for the round trip is zero. (Displacement is change in position and, thus, is zero for a round trip.) Thus average speed is *not* simply the magnitude of average velocity.



During a 30-minute round trip to the store, the total distance traveled is 6 km. The average speed is 12 km/h. The displacement for the round trip is zero, since there was no net change in position. Thus the average velocity is zero.

Another way of visualizing the motion of an object is to use a graph. A plot of position or of velocity as a function of time can be very useful. For example, for this trip to the store, the position, velocity, and speed-vs.-time graphs are displayed in [\[link\]](#). (Note that these graphs depict a very simplified **model** of the trip. We are assuming that speed is constant during the trip, which is unrealistic given that we'll probably stop at the store. But for simplicity's sake, we will model it with no stops or changes in speed. We are also assuming that the route between the store and the house is a perfectly straight line.)



Position vs. time, velocity vs. time, and speed vs. time on a trip. Note that the velocity for the return trip is negative.

Note:**Making Connections: Take-Home Investigation—Getting a Sense of Speed**

If you have spent much time driving, you probably have a good sense of speeds between about 10 and 70 miles per hour. But what are these in meters per second? What do we mean when we say that something is moving at 10 m/s? To get a better sense of what these values really mean, do some observations and calculations on your own:

- calculate typical car speeds in meters per second
- estimate jogging and walking speed by timing yourself; convert the measurements into both m/s and mi/h
- determine the speed of an ant, snail, or falling leaf

Exercise:**Check Your Understanding****Problem:**

A commuter train travels from Baltimore to Washington, DC, and back in 1 hour and 45 minutes. The distance between the two stations is approximately 40 miles. What is (a) the average velocity of the train, and (b) the average speed of the train in m/s?

Solution:

(a) The average velocity of the train is zero because $x_f = x_0$; the train ends up at the same place it starts.

(b) The average speed of the train is calculated below. Note that the train travels 40 miles one way and 40 miles back, for a total distance of 80 miles.

Equation:

$$\frac{\text{distance}}{\text{time}} = \frac{80 \text{ miles}}{105 \text{ minutes}}$$

Equation:

$$\frac{80 \text{ miles}}{105 \text{ minutes}} \times \frac{5280 \text{ feet}}{1 \text{ mile}} \times \frac{1 \text{ meter}}{3.28 \text{ feet}} \times \frac{1 \text{ minute}}{60 \text{ seconds}} = 20 \text{ m/s}$$

Section Summary

- Time is measured in terms of change, and its SI unit is the second (s). Elapsed time for an event is

Equation:

$$\Delta t = t_f - t_0,$$

where t_f is the final time and t_0 is the initial time. The initial time is often taken to be zero, as if measured with a stopwatch; the elapsed time is then just t .

- Average velocity \bar{v} is defined as displacement divided by the travel time. In symbols, average velocity is

Equation:

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_0}{t_f - t_0}.$$

- The SI unit for velocity is m/s.
- Velocity is a vector and thus has a direction.
- Instantaneous velocity v is the velocity at a specific instant or the average velocity for an infinitesimal interval.
- Instantaneous speed is the magnitude of the instantaneous velocity.
- Instantaneous speed is a scalar quantity, as it has no direction specified.
- Average speed is the total distance traveled divided by the elapsed time. (Average speed is *not* the magnitude of the average velocity.) Speed is a scalar quantity; it has no direction associated with it.

Conceptual Questions

Exercise:**Problem:**

Give an example (but not one from the text) of a device used to measure time and identify what change in that device indicates a change in time.

Exercise:**Problem:**

There is a distinction between average speed and the magnitude of average velocity. Give an example that illustrates the difference between these two quantities.

Exercise:**Problem:**

Does a car's odometer measure position or displacement? Does its speedometer measure speed or velocity?

Exercise:**Problem:**

If you divide the total distance traveled on a car trip (as determined by the odometer) by the time for the trip, are you calculating the average speed or the magnitude of the average velocity? Under what circumstances are these two quantities the same?

Exercise:**Problem:**

How are instantaneous velocity and instantaneous speed related to one another? How do they differ?

Problems & Exercises**Exercise:**

Problem:

(a) Calculate Earth's average speed relative to the Sun. (b) What is its average velocity over a period of one year?

Solution:

(a) $3.0 \times 10^4 \text{ m/s}$

(b) 0 m/s

Exercise:**Problem:**

A helicopter blade spins at exactly 100 revolutions per minute. Its tip is 5.00 m from the center of rotation. (a) Calculate the average speed of the blade tip in the helicopter's frame of reference. (b) What is its average velocity over one revolution?

Exercise:**Problem:**

The North American and European continents are moving apart at a rate of about 3 cm/y. At this rate how long will it take them to drift 500 km farther apart than they are at present?

Solution:

2×10^7 years

Exercise:

Problem:

Land west of the San Andreas fault in southern California is moving at an average velocity of about 6 cm/y northwest relative to land east of the fault. Los Angeles is west of the fault and may thus someday be at the same latitude as San Francisco, which is east of the fault. How far in the future will this occur if the displacement to be made is 590 km northwest, assuming the motion remains constant?

Exercise:**Problem:**

On May 26, 1934, a streamlined, stainless steel diesel train called the Zephyr set the world's nonstop long-distance speed record for trains. Its run from Denver to Chicago took 13 hours, 4 minutes, 58 seconds, and was witnessed by more than a million people along the route. The total distance traveled was 1633.8 km. What was its average speed in km/h and m/s?

Solution:

$$34.689 \text{ m/s} = 124.88 \text{ km/h}$$

Exercise:**Problem:**

Tidal friction is slowing the rotation of the Earth. As a result, the orbit of the Moon is increasing in radius at a rate of approximately 4 cm/year. Assuming this to be a constant rate, how many years will pass before the radius of the Moon's orbit increases by $3.84 \times 10^6 \text{ m}$ (1%)?

Exercise:

Problem:

A student drove to the university from her home and noted that the odometer reading of her car increased by 12.0 km. The trip took 18.0 min. (a) What was her average speed? (b) If the straight-line distance from her home to the university is 10.3 km in a direction 25.0° south of east, what was her average velocity? (c) If she returned home by the same path 7 h 30 min after she left, what were her average speed and velocity for the entire trip?

Solution:

(a) 40.0 km/h

(b) 34.3 km/h, 25° S of E.

(c) average speed = 3.20 km/h, $\bar{v} = 0$.

Exercise:**Problem:**

The speed of propagation of the action potential (an electrical signal) in a nerve cell depends (inversely) on the diameter of the axon (nerve fiber). If the nerve cell connecting the spinal cord to your feet is 1.1 m long, and the nerve impulse speed is 18 m/s, how long does it take for the nerve signal to travel this distance?

Exercise:

Problem:

Conversations with astronauts on the lunar surface were characterized by a kind of echo in which the earthbound person's voice was so loud in the astronaut's space helmet that it was picked up by the astronaut's microphone and transmitted back to Earth. It is reasonable to assume that the echo time equals the time necessary for the radio wave to travel from the Earth to the Moon and back (that is, neglecting any time delays in the electronic equipment). Calculate the distance from Earth to the Moon given that the echo time was 2.56 s and that radio waves travel at the speed of light (3.00×10^8 m/s).

Solution:

384,000 km

Exercise:**Problem:**

A football quarterback runs 15.0 m straight down the playing field in 2.50 s. He is then hit and pushed 3.00 m straight backward in 1.75 s. He breaks the tackle and runs straight forward another 21.0 m in 5.20 s. Calculate his average velocity (a) for each of the three intervals and (b) for the entire motion.

Exercise:**Problem:**

The planetary model of the atom pictures electrons orbiting the atomic nucleus much as planets orbit the Sun. In this model you can view hydrogen, the simplest atom, as having a single electron in a circular orbit 1.06×10^{-10} m in diameter. (a) If the average speed of the electron in this orbit is known to be 2.20×10^6 m/s, calculate the number of revolutions per second it makes about the nucleus. (b) What is the electron's average velocity?

Solution:

(a) $6.61 \times 10^{15} \text{ rev/s}$

(b) 0 m/s

Test Prep for AP Courses

Exercise:

Problem:

A group of students has two carts, *A* and *B*, with wheels that turn with negligible friction. The two carts travel along a straight horizontal track and eventually collide. Before the collision, cart *A* travels to the right and cart *B* is initially at rest. After the collision, the carts stick together.

- Describe an experimental procedure to determine the velocities of the carts before and after the collision, including all the additional equipment you would need. You may include a labeled diagram of your setup to help in your description. Indicate what measurements you would take and how you would take them. Include enough detail so that another student could carry out your procedure.
- There will be sources of error in the measurements taken in the experiment both before and after the collision. Which velocity will be more greatly affected by this error: the velocity prior to the collision or the velocity after the collision? Or will both sets of data be affected equally? Justify your answer.

Solution:

- Use tape to mark off two distances on the track — one for cart *A* before the collision and one for the combined carts after the collision. Push cart *A* to give it an initial speed. Use a stopwatch to measure the time it takes for the cart(s) to cross the marked distances. The speeds are the distances divided by the times.

- b. If the measurement errors are of the same magnitude, they will have a greater effect after the collision. The speed of the combined carts will be less than the initial speed of cart A. As a result, these errors will be a greater percentage of the actual velocity value after the collision occurs. (Note: Other arguments could properly be made for 'more error before the collision' and error that 'equally affects both sets of measurement'.)

Glossary

average speed

distance traveled divided by time during which motion occurs

average velocity

displacement divided by time over which displacement occurs

instantaneous velocity

velocity at a specific instant, or the average velocity over an infinitesimal time interval

instantaneous speed

magnitude of the instantaneous velocity

time

change, or the interval over which change occurs

model

simplified description that contains only those elements necessary to describe the physics of a physical situation

elapsed time

the difference between the ending time and beginning time

Acceleration



A plane decelerates, or slows down, as it comes in for landing in St. Maarten. Its acceleration is opposite in direction to its velocity. (credit: Steve Conry, Flickr)

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define and distinguish between instantaneous acceleration and average acceleration.
- Calculate acceleration given initial time, initial velocity, final time, and final velocity.

The information presented in this section supports the following AP® learning objectives and science practices:

- **3.A.1.1** The student is able to express the motion of an object using narrative, mathematical, and graphical representations. (S.P. 1.5, 2.1, 2.2)
- **3.A.1.3** The student is able to analyze experimental data describing the motion of an object and is able to express the results of the analysis using narrative, mathematical, and graphical representations. (S.P. 5.1)

In everyday conversation, to accelerate means to speed up. The accelerator in a car can in fact cause it to speed up. The greater the **acceleration**, the greater the change in velocity over a given time. The formal definition of acceleration is consistent with these notions, but more inclusive.

Note:

Average Acceleration

Average Acceleration is the rate at which velocity changes,

Equation:

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_0}{t_f - t_0},$$

where \bar{a} is average acceleration, v is velocity, and t is time. (The bar over the a means *average* acceleration.)

Because acceleration is velocity in m/s divided by time in s, the SI units for acceleration are m/s^2 , meters per second squared or meters per second per second, which literally means by how many meters per second the velocity changes every second.

Recall that velocity is a vector—it has both magnitude and direction. This means that a change in velocity can be a change in magnitude (or speed), but it can also be a change in *direction*. For example, if a car turns a corner at constant speed, it is accelerating because its direction is changing. The quicker you turn, the greater the acceleration. So there is an acceleration when velocity changes either in magnitude (an increase or decrease in speed) or in direction, or both.

Note:

Acceleration as a Vector

Acceleration is a vector in the same direction as the *change* in velocity, Δv . Since velocity is a vector, it can change either in magnitude or in direction. Acceleration is therefore a change in either speed or direction, or both.

Keep in mind that although acceleration is in the direction of the *change* in velocity, it is not always in the direction of *motion*. When an object's acceleration is in the same direction of its motion, the object will speed up. However, when an object's acceleration is opposite to the direction of its motion, the object will slow down. Speeding up and slowing down should not be confused with a positive and negative acceleration. The next two examples should help to make this distinction clear.

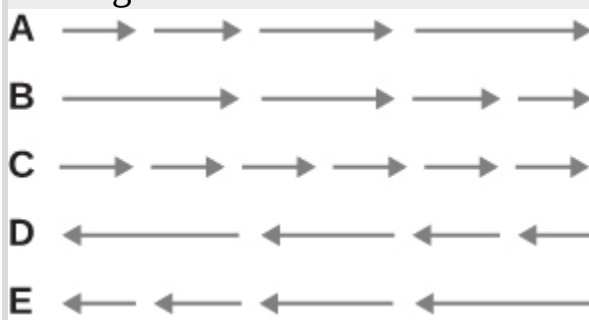


A subway train in Sao Paulo, Brazil, decelerates as it comes

into a station. It is accelerating
in a direction opposite to its
direction of motion. (credit:
Yusuke Kawasaki, Flickr)

Note:

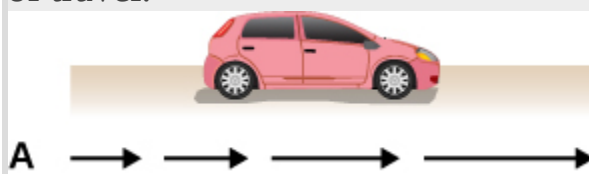
Making Connections: Car Motion



Above are arrows representing
the motion of five cars (A–E).

In all five cases, the positive
direction should be considered
to the right of the page.

Consider the acceleration and velocity of each car in terms of its direction
of travel.



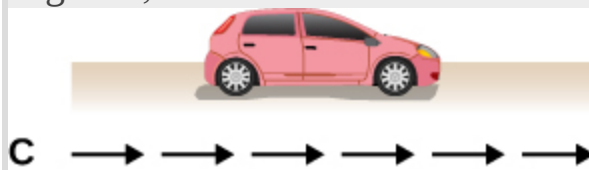
Car A is speeding up.

Because the positive direction is considered to the right of the paper, Car A is moving with a positive velocity. Because it is speeding up while moving with a positive velocity, its acceleration is also considered positive.



Car B is slowing down.

Because the positive direction is considered to the right of the paper, Car B is also moving with a positive velocity. However, because it is slowing down while moving with a positive velocity, its acceleration is considered negative. (This can be viewed in a mathematical manner as well. If the car was originally moving with a velocity of $+25$ m/s, it is finishing with a speed less than that, like $+5$ m/s. Because the change in velocity is negative, the acceleration will be as well.)



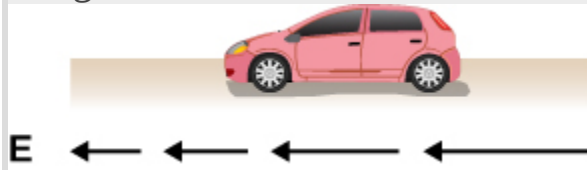
Car C has a constant speed.

Because the positive direction is considered to the right of the paper, Car C is moving with a positive velocity. Because all arrows are of the same length, this car is not changing its speed. As a result, its change in velocity is zero, and its acceleration must be zero as well.



Car D is speeding up in the opposite direction of Cars A, B, C.

Because the car is moving opposite to the positive direction, Car D is moving with a negative velocity. Because it is speeding up while moving in a negative direction, its acceleration is negative as well.



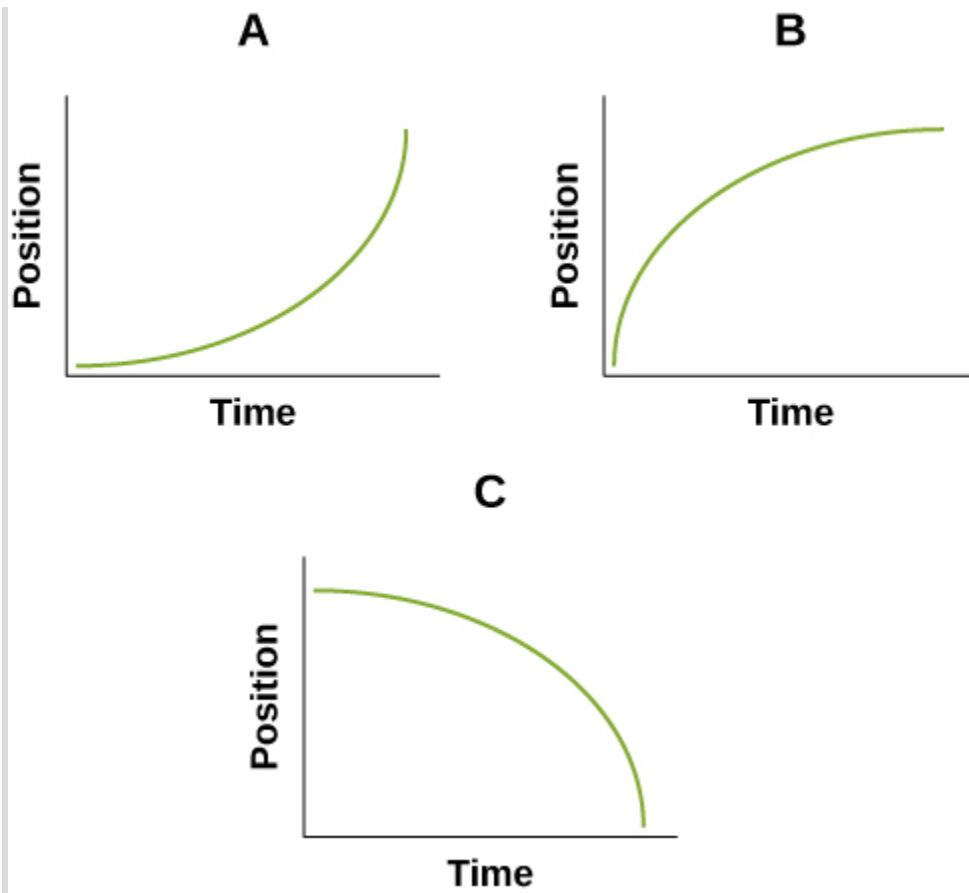
Car E is slowing down in the same direction as Car D and opposite of Cars A, B, C.

Because it is moving opposite to the positive direction, Car E is moving with a negative velocity as well. However, because it is slowing down while moving in a negative direction, its acceleration is actually positive. As in example B, this may be more easily understood in a mathematical sense. The car is originally moving with a large negative velocity (-25 m/s) but slows to a final velocity that is less negative (-5 m/s). This change in velocity, from -25 m/s to -5 m/s, is actually a positive change ($v_f - v_i = -5 \text{ m/s} - -25 \text{ m/s}$ of 20 m/s). Because the change in velocity is positive, the acceleration must also be positive.

Note:

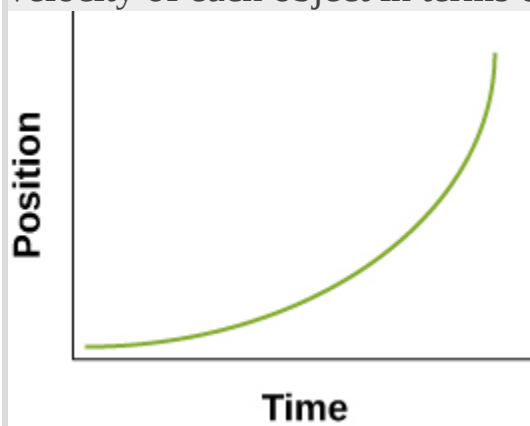
Making Connection - Illustrative Example

The three graphs below are labeled A, B, and C. Each one represents the position of a moving object plotted against time.



Three position and time graphs: A, B, and C.

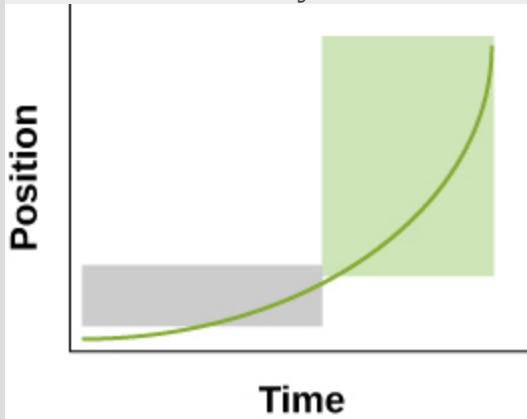
As we did in the previous example, let's consider the acceleration and velocity of each object in terms of its direction of travel.



Graph A of Position (y

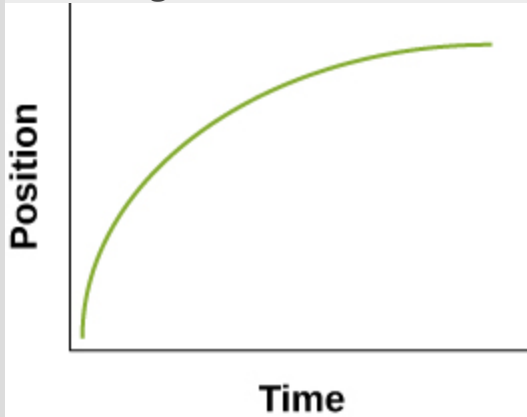
axis) vs. Time (x axis).

Object A is continually increasing its position in the positive direction. As a result, its velocity is considered positive.



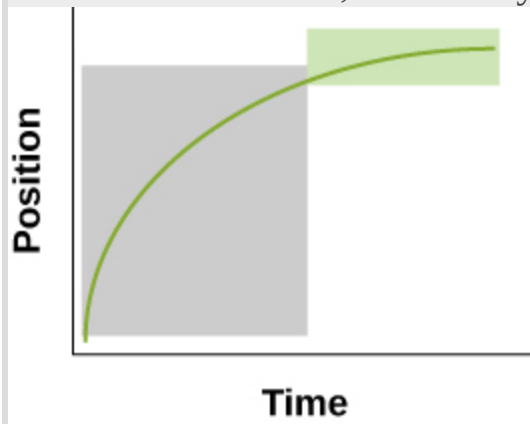
Breakdown of Graph A into two separate sections.

During the first portion of time (shaded grey) the position of the object does not change much, resulting in a small positive velocity. During a later portion of time (shaded green) the position of the object changes more, resulting in a larger positive velocity. Because this positive velocity is increasing over time, the acceleration of the object is considered positive.



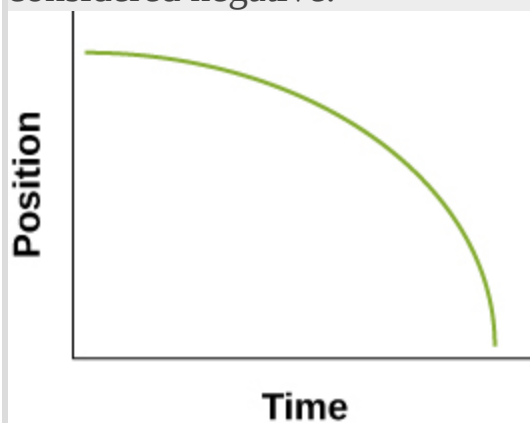
Graph B of Position (y axis) vs. Time (x axis).

As in case A, Object B is continually increasing its position in the positive direction. As a result, its velocity is considered positive.



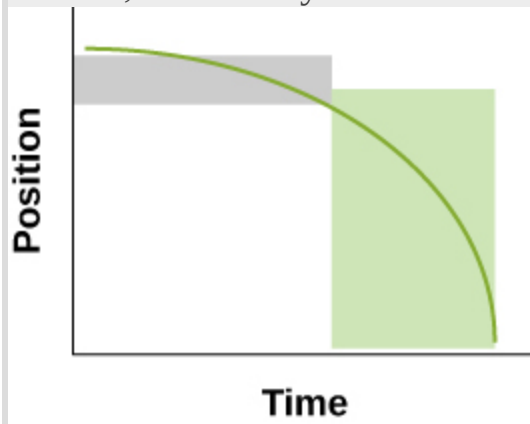
Breakdown of Graph B into two separate sections.

During the first portion of time (shaded grey) the position of the object changes a large amount, resulting in a large positive velocity. During a later portion of time (shaded green) the position of the object does not change as much, resulting in a smaller positive velocity. Because this positive velocity is decreasing over time, the acceleration of the object is considered negative.



Graph C of Position (y axis) vs. Time (x axis).

Object C is continually decreasing its position in the positive direction. As a result, its velocity is considered negative.



Breakdown of Graph C into two separate sections.

During the first portion of time (shaded grey) the position of the object does not change a large amount, resulting in a small negative velocity. During a later portion of time (shaded green) the position of the object changes a much larger amount, resulting in a larger negative velocity. Because the velocity of the object is becoming more negative during the time period, the change in velocity is negative. As a result, the object experiences a negative acceleration.

Example:

Calculating Acceleration: A Racehorse Leaves the Gate

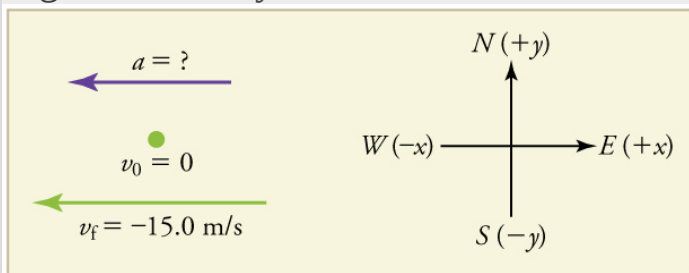
A racehorse coming out of the gate accelerates from rest to a velocity of 15.0 m/s due west in 1.80 s. What is its average acceleration?



(credit: Jon Sullivan, PD
Photo.org)

Strategy

First we draw a sketch and assign a coordinate system to the problem. This is a simple problem, but it always helps to visualize it. Notice that we assign east as positive and west as negative. Thus, in this case, we have negative velocity.



We can solve this problem by identifying Δv and Δt from the given information and then calculating the average acceleration directly from the equation $\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_0}{t_f - t_0}$.

Solution

1. Identify the knowns. $v_0 = 0$, $v_f = -15.0 \text{ m/s}$ (the minus sign indicates direction toward the west), $\Delta t = 1.80 \text{ s}$.
2. Find the change in velocity. Since the horse is going from zero to -15.0 m/s , its change in velocity equals its final velocity:
 $\Delta v = v_f = -15.0 \text{ m/s}$.
3. Plug in the known values (Δv and Δt) and solve for the unknown \bar{a} .

Equation:

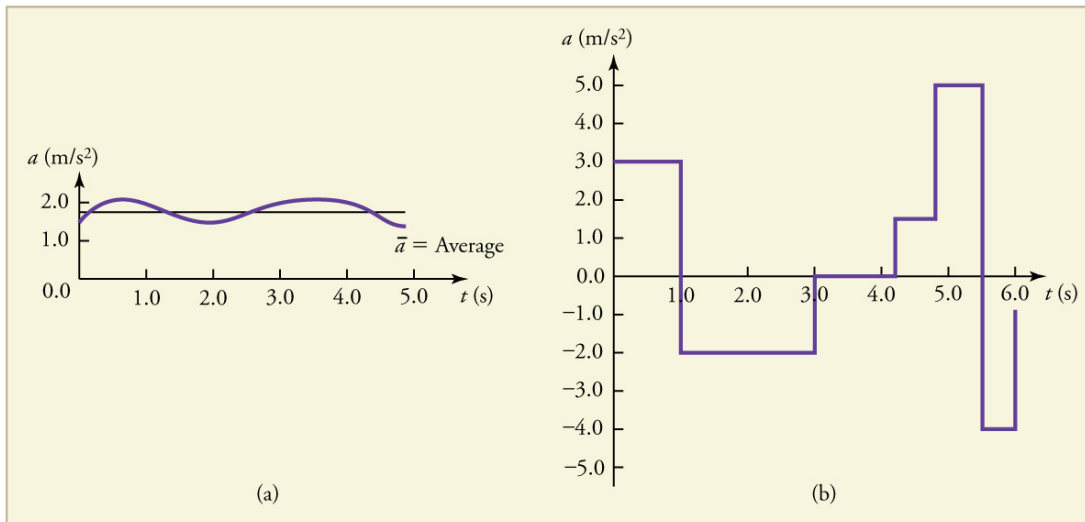
$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{-15.0 \text{ m/s}}{1.80 \text{ s}} = -8.33 \text{ m/s}^2.$$

Discussion

The minus sign for acceleration indicates that acceleration is toward the west. An acceleration of 8.33 m/s^2 due west means that the horse increases its velocity by 8.33 m/s due west each second, that is, 8.33 meters per second per second, which we write as 8.33 m/s^2 . This is truly an average acceleration, because the ride is not smooth. We shall see later that an acceleration of this magnitude would require the rider to hang on with a force nearly equal to his weight.

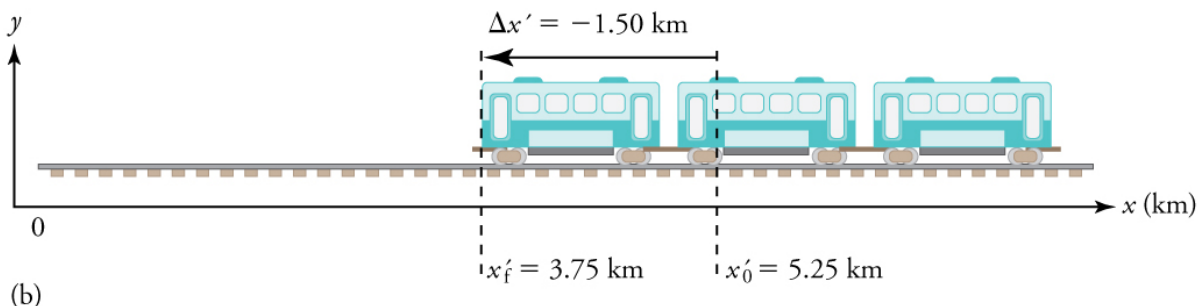
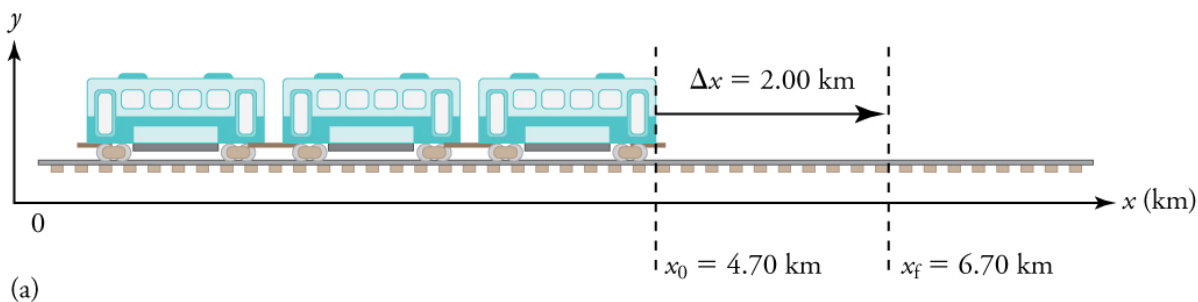
Instantaneous Acceleration

Instantaneous acceleration a , or the *acceleration at a specific instant in time*, is obtained by the same process as discussed for instantaneous velocity in [Time, Velocity, and Speed](#)—that is, by considering an infinitesimally small interval of time. How do we find instantaneous acceleration using only algebra? The answer is that we choose an average acceleration that is representative of the motion. [\[link\]](#) shows graphs of instantaneous acceleration versus time for two very different motions. In [\[link\]\(a\)](#), the acceleration varies slightly and the average over the entire interval is nearly the same as the instantaneous acceleration at any time. In this case, we should treat this motion as if it had a constant acceleration equal to the average (in this case about 1.8 m/s^2). In [\[link\]\(b\)](#), the acceleration varies drastically over time. In such situations it is best to consider smaller time intervals and choose an average acceleration for each. For example, we could consider motion over the time intervals from 0 to 1.0 s and from 1.0 to 3.0 s as separate motions with accelerations of $+3.0 \text{ m/s}^2$ and -2.0 m/s^2 , respectively.



Graphs of instantaneous acceleration versus time for two different one-dimensional motions. (a) Here acceleration varies only slightly and is always in the same direction, since it is positive. The average over the interval is nearly the same as the acceleration at any given time. (b) Here the acceleration varies greatly, perhaps representing a package on a post office conveyor belt that is accelerated forward and backward as it bumps along. It is necessary to consider small time intervals (such as from 0 to 1.0 s) with constant or nearly constant acceleration in such a situation.

The next several examples consider the motion of the subway train shown in [\[link\]](#). In (a) the shuttle moves to the right, and in (b) it moves to the left. The examples are designed to further illustrate aspects of motion and to illustrate some of the reasoning that goes into solving problems.



One-dimensional motion of a subway train considered in [\[link\]](#), [\[link\]](#), [\[link\]](#), [\[link\]](#), and [\[link\]](#). Here we have chosen the x -axis so that $+$ means to the right and $-$ means to the left for displacements, velocities, and accelerations. (a) The subway train moves to the right from x_0 to x_f . Its displacement Δx is $+2.0$ km. (b) The train moves to the left from x'_0 to x'_f . Its displacement $\Delta x'$ is -1.5 km. (Note that the prime symbol ($'$) is used simply to distinguish between displacement in the two different situations. The distances of travel and the size of the cars are on different scales to fit everything into the diagram.)

Example:

Calculating Displacement: A Subway Train

What are the magnitude and sign of displacements for the motions of the subway train shown in parts (a) and (b) of [\[link\]](#)?

Strategy

A drawing with a coordinate system is already provided, so we don't need to make a sketch, but we should analyze it to make sure we understand what it is showing. Pay particular attention to the coordinate system. To find displacement, we use the equation $\Delta x = x_f - x_0$. This is straightforward since the initial and final positions are given.

Solution

1. Identify the knowns. In the figure we see that $x_f = 6.70$ km and $x_0 = 4.70$ km for part (a), and $x'_f = 3.75$ km and $x'_0 = 5.25$ km for part (b).
2. Solve for displacement in part (a).

Equation:

$$\Delta x = x_f - x_0 = 6.70 \text{ km} - 4.70 \text{ km} = +2.00 \text{ km}$$

3. Solve for displacement in part (b).

Equation:

$$\Delta x' = x'_f - x'_0 = 3.75 \text{ km} - 5.25 \text{ km} = -1.50 \text{ km}$$

Discussion

The direction of the motion in (a) is to the right and therefore its displacement has a positive sign, whereas motion in (b) is to the left and thus has a minus sign.

Example:

Comparing Distance Traveled with Displacement: A Subway Train

What are the distances traveled for the motions shown in parts (a) and (b) of the subway train in [\[link\]](#)?

Strategy

To answer this question, think about the definitions of distance and distance traveled, and how they are related to displacement. Distance between two positions is defined to be the magnitude of displacement, which was found in [\[link\]](#). Distance traveled is the total length of the path traveled between the two positions. (See [Displacement](#).) In the case of the subway train shown in [\[link\]](#), the distance traveled is the same as the distance between the initial and final positions of the train.

Solution

1. The displacement for part (a) was $+2.00$ km. Therefore, the distance between the initial and final positions was 2.00 km, and the distance traveled was 2.00 km.
2. The displacement for part (b) was -1.5 km. Therefore, the distance between the initial and final positions was 1.50 km, and the distance traveled was 1.50 km.

Discussion

Distance is a scalar. It has magnitude but no sign to indicate direction.

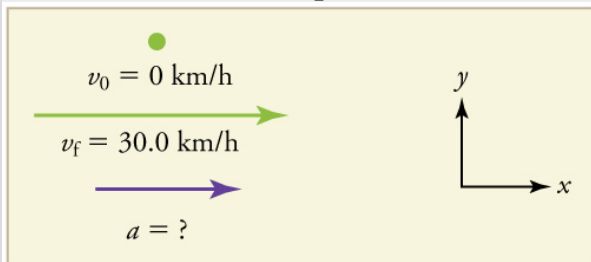
Example:

Calculating Acceleration: A Subway Train Speeding Up

Suppose the train in [\[link\]](#)(a) accelerates from rest to 30.0 km/h in the first 20.0 s of its motion. What is its average acceleration during that time interval?

Strategy

It is worth it at this point to make a simple sketch:



This problem involves three steps. First we must determine the change in velocity, then we must determine the change in time, and finally we use these values to calculate the acceleration.

Solution

1. Identify the knowns. $v_0 = 0$ (the train starts at rest), $v_f = 30.0$ km/h, and $\Delta t = 20.0$ s.
2. Calculate Δv . Since the train starts from rest, its change in velocity is $\Delta v = +30.0$ km/h, where the plus sign means velocity to the right.
3. Plug in known values and solve for the unknown, \bar{a} .

Equation:

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{+30.0 \text{ km/h}}{20.0 \text{ s}}$$

4. Since the units are mixed (we have both hours and seconds for time), we need to convert everything into SI units of meters and seconds. (See [Physical Quantities and Units](#) for more guidance.)

Equation:

$$\bar{a} = \left(\frac{+30 \text{ km/h}}{20.0 \text{ s}} \right) \left(\frac{10^3 \text{ m}}{1 \text{ km}} \right) \left(\frac{1 \text{ h}}{3600 \text{ s}} \right) = 0.417 \text{ m/s}^2$$

Discussion

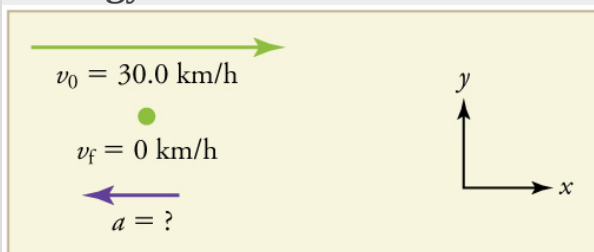
The plus sign means that acceleration is to the right. This is reasonable because the train starts from rest and ends up with a velocity to the right (also positive). So acceleration is in the same direction as the *change* in velocity, as is always the case.

Example:

Calculate Acceleration: A Subway Train Slowing Down

Now suppose that at the end of its trip, the train in [\[link\]](#)(a) slows to a stop from a speed of 30.0 km/h in 8.00 s. What is its average acceleration while stopping?

Strategy



In this case, the train is decelerating and its acceleration is negative because it is toward the left. As in the previous example, we must find the change in velocity and the change in time and then solve for acceleration.

Solution

1. Identify the knowns. $v_0 = 30.0 \text{ km/h}$, $v_f = 0 \text{ km/h}$ (the train is stopped, so its velocity is 0), and $\Delta t = 8.00 \text{ s}$.

2. Solve for the change in velocity, Δv .

Equation:

$$\Delta v = v_f - v_0 = 0 - 30.0 \text{ km/h} = -30.0 \text{ km/h}$$

3. Plug in the knowns, Δv and Δt , and solve for \bar{a} .

Equation:

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{-30.0 \text{ km/h}}{8.00 \text{ s}}$$

4. Convert the units to meters and seconds.

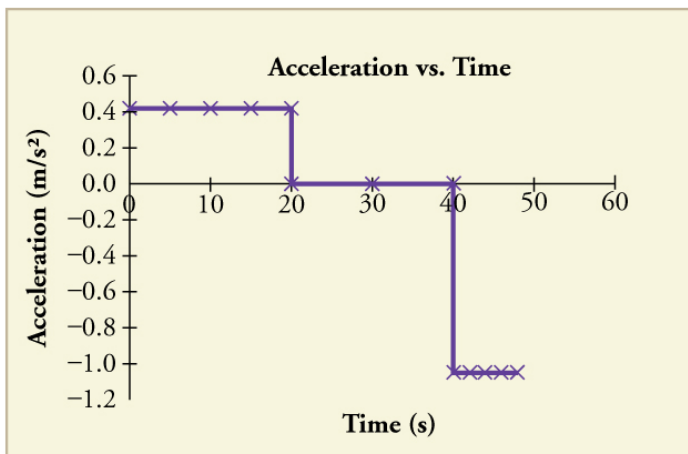
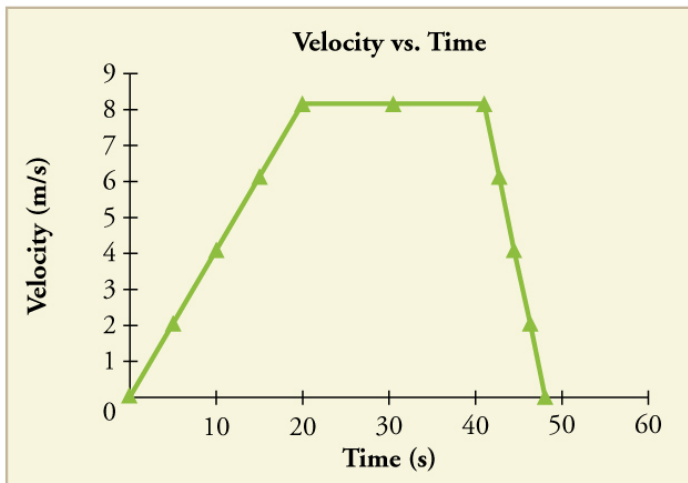
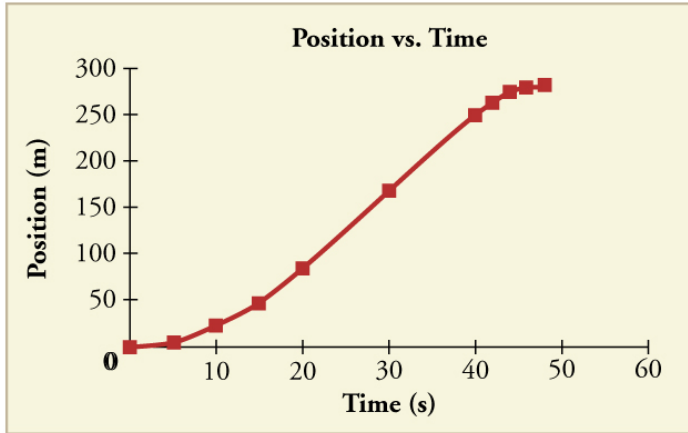
Equation:

$$\bar{a} = \frac{\Delta v}{\Delta t} = \left(\frac{-30.0 \text{ km/h}}{8.00 \text{ s}} \right) \left(\frac{10^3 \text{ m}}{1 \text{ km}} \right) \left(\frac{1 \text{ h}}{3600 \text{ s}} \right) = -1.04 \text{ m/s}^2.$$

Discussion

The minus sign indicates that acceleration is to the left. This sign is reasonable because the train initially has a positive velocity in this problem, and a negative acceleration would oppose the motion. Again, acceleration is in the same direction as the *change* in velocity, which is negative here. This acceleration can be called a deceleration because it has a direction opposite to the velocity.

The graphs of position, velocity, and acceleration vs. time for the trains in [\[link\]](#) and [\[link\]](#) are displayed in [\[link\]](#). (We have taken the velocity to remain constant from 20 to 40 s, after which the train decelerates.)



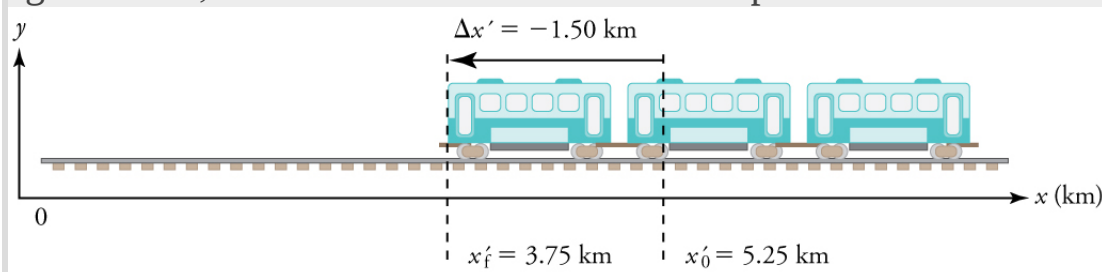
(a) Position of the train over time.
 Notice that the train's position changes slowly at the beginning of the journey, then more and more quickly as it picks up speed. Its position then

changes more slowly as it slows down at the end of the journey. In the middle of the journey, while the velocity remains constant, the position changes at a constant rate. (b) Velocity of the train over time. The train's velocity increases as it accelerates at the beginning of the journey. It remains the same in the middle of the journey (where there is no acceleration). It decreases as the train decelerates at the end of the journey. (c) The acceleration of the train over time. The train has positive acceleration as it speeds up at the beginning of the journey. It has no acceleration as it travels at constant velocity in the middle of the journey. Its acceleration is negative as it slows down at the end of the journey.

Example:

Calculating Average Velocity: The Subway Train

What is the average velocity of the train in part b of [\[link\]](#), and shown again below, if it takes 5.00 min to make its trip?



Strategy

Average velocity is displacement divided by time. It will be negative here, since the train moves to the left and has a negative displacement.

Solution

1. Identify the knowns. $x_f = 3.75 \text{ km}$, $x_0 = 5.25 \text{ km}$, $\Delta t = 5.00 \text{ min}$.
2. Determine displacement, $\Delta x'$. We found $\Delta x'$ to be -1.5 km in [\[link\]](#).
3. Solve for average velocity.

Equation:

$$\bar{v} = \frac{\Delta x'}{\Delta t} = \frac{-1.50 \text{ km}}{5.00 \text{ min}}$$

4. Convert units.

Equation:

$$\bar{v} = \frac{\Delta x'}{\Delta t} = \left(\frac{-1.50 \text{ km}}{5.00 \text{ min}} \right) \left(\frac{60 \text{ min}}{1 \text{ h}} \right) = -18.0 \text{ km/h}$$

Discussion

The negative velocity indicates motion to the left.

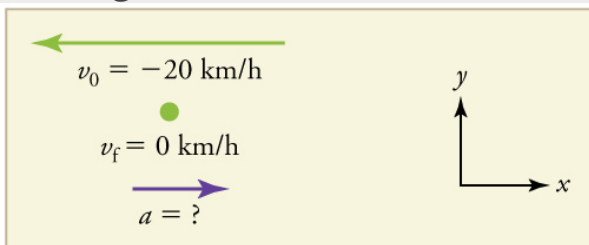
Example:

Calculating Deceleration: The Subway Train

Finally, suppose the train in [\[link\]](#) slows to a stop from a velocity of 20.0 km/h in 10.0 s . What is its average acceleration?

Strategy

Once again, let's draw a sketch:



As before, we must find the change in velocity and the change in time to calculate average acceleration.

Solution

1. Identify the knowns. $v_0 = -20 \text{ km/h}$, $v_f = 0 \text{ km/h}$, $\Delta t = 10.0 \text{ s}$.

2. Calculate Δv . The change in velocity here is actually positive, since
Equation:

$$\Delta v = v_f - v_0 = 0 - (-20 \text{ km/h}) = +20 \text{ km/h}.$$

3. Solve for \bar{a} .

Equation:

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{+20.0 \text{ km/h}}{10.0 \text{ s}}$$

4. Convert units.

Equation:

$$\bar{a} = \left(\frac{+20.0 \text{ km/h}}{10.0 \text{ s}} \right) \left(\frac{10^3 \text{ m}}{1 \text{ km}} \right) \left(\frac{1 \text{ h}}{3600 \text{ s}} \right) = +0.556 \text{ m/s}^2$$

Discussion

The plus sign means that acceleration is to the right. This is reasonable because the train initially has a negative velocity (to the left) in this problem and a positive acceleration opposes the motion (and so it is to the right). Again, acceleration is in the same direction as the *change* in velocity, which is positive here. As in [\[link\]](#), this acceleration can be called a deceleration since it is in the direction opposite to the velocity.

Sign and Direction

Perhaps the most important thing to note about these examples is the signs of the answers. In our chosen coordinate system, plus means the quantity is to the right and minus means it is to the left. This is easy to imagine for displacement and velocity. But it is a little less obvious for acceleration. Most people interpret negative acceleration as the slowing of an object. This was not the case in [\[link\]](#), where a positive acceleration slowed a negative velocity. The crucial distinction was that the acceleration was in the opposite direction from the velocity. In fact, a negative acceleration will

increase a negative velocity. For example, the train moving to the left in [\[link\]](#) is sped up by an acceleration to the left. In that case, both v and a are negative. The plus and minus signs give the directions of the accelerations. If acceleration has the same sign as the velocity, the object is speeding up. If acceleration has the opposite sign as the velocity, the object is slowing down.

Exercise:

Check Your Understanding

Problem:

An airplane lands on a runway traveling east. Describe its acceleration.


Solution:

If we take east to be positive, then the airplane has negative acceleration, as it is accelerating toward the west. It is also decelerating: its acceleration is opposite in direction to its velocity.

Note:

PhET Explorations: Moving Man Simulation

Learn about position, velocity, and acceleration graphs. Move the little man back and forth with the mouse and plot his motion. Set the position, velocity, or acceleration and let the simulation move the man for you.

 Because the positive direction is considered to the right of the paper, Car B is also moving with a positive velocity. However, because it is slowing down while moving with a positive velocity, its acceleration is considered negative. (This can be viewed in a mathematical manner as well. If the car was originally moving with a velocity of +25 m/s, it is finishing with a speed less than that, like +5 m/s. Because the change in velocity is negative, the acceleration will be as well.)

[Moving Man](#)

Section Summary

- Acceleration is the rate at which velocity changes. In symbols, **average acceleration** \bar{a} is

Equation:

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_0}{t_f - t_0}.$$

- The SI unit for acceleration is m/s^2 .
- Acceleration is a vector, and thus has both a magnitude and direction.
- Acceleration can be caused by either a change in the magnitude or the direction of the velocity.
- Instantaneous acceleration a is the acceleration at a specific instant in time.
- Deceleration is an acceleration with a direction opposite to that of the velocity.

Conceptual Questions

Exercise:

Problem:

Is it possible for speed to be constant while acceleration is not zero?
Give an example of such a situation.

Exercise:

Problem:

Is it possible for velocity to be constant while acceleration is not zero?
Explain.

Exercise:

Problem:

Give an example in which velocity is zero yet acceleration is not.

Exercise:**Problem:**

If a subway train is moving to the left (has a negative velocity) and then comes to a stop, what is the direction of its acceleration? Is the acceleration positive or negative?

Exercise:**Problem:**

Plus and minus signs are used in one-dimensional motion to indicate direction. What is the sign of an acceleration that reduces the magnitude of a negative velocity? Of a positive velocity?

Problems & Exercises**Exercise:****Problem:**

A cheetah can accelerate from rest to a speed of 30.0 m/s in 7.00 s. What is its acceleration?

Solution:

$$4.29 \text{ m/s}^2$$

Exercise:**Problem: Professional Application**

Dr. John Paul Stapp was U.S. Air Force officer who studied the effects of extreme deceleration on the human body. On December 10, 1954, Stapp rode a rocket sled, accelerating from rest to a top speed of 282 m/s (1015 km/h) in 5.00 s, and was brought jarringly back to rest in only 1.40 s! Calculate his (a) acceleration and (b) deceleration.

Express each in multiples of g (9.80 m/s^2) by taking its ratio to the acceleration of gravity.

Exercise:

Problem:

A commuter backs her car out of her garage with an acceleration of 1.40 m/s^2 . (a) How long does it take her to reach a speed of 2.00 m/s ? (b) If she then brakes to a stop in 0.800 s , what is her deceleration?

Solution:

(a) 1.43 s

(b) -2.50 m/s^2

Exercise:

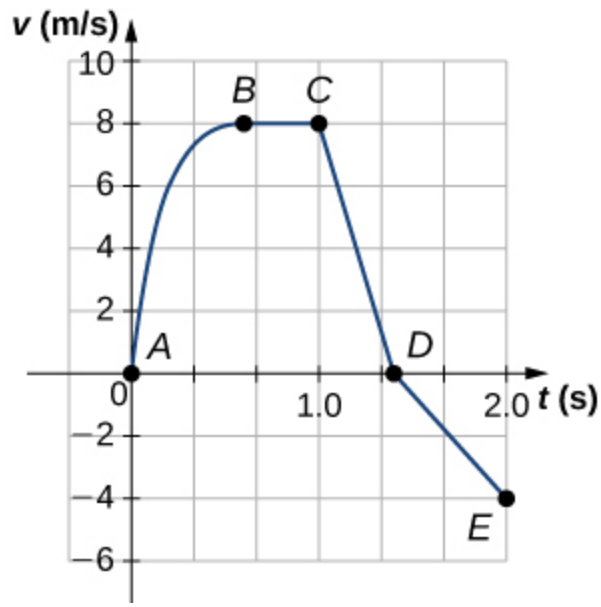
Problem:

Assume that an intercontinental ballistic missile goes from rest to a suborbital speed of 6.50 km/s in 60.0 s (the actual speed and time are classified). What is its average acceleration in m/s^2 and in multiples of g (9.80 m/s^2)?

Test Prep for AP Courses

Exercise:

Problem:



Graph showing Velocity vs.
Time of a cart.

A cart is constrained to move along a straight line. A varying net force along the direction of motion is exerted on the cart. The cart's velocity v as a function of time t is shown in the graph. The five labeled points divide the graph into four sections.

Which of the following correctly ranks the magnitude of the average acceleration of the cart during the four sections of the graph?

- a. $a_{CD} > a_{AB} > a_{BC} > a_{DE}$
- b. $a_{BC} > a_{AB} > a_{CD} > a_{DE}$
- c. $a_{AB} > a_{BC} > a_{DE} > a_{CD}$
- d. $a_{CD} > a_{AB} > a_{DE} > a_{BC}$

Exercise:

Problem: Push a book across a table and observe it slow to a stop.

Draw graphs showing the book's position vs. time and velocity vs. time if the direction of its motion is considered positive.

Draw graphs showing the book's position vs. time and velocity vs. time if the direction of its motion is considered negative.

Solution:

The position vs. time graph should be represented with a positively sloped line whose slope steadily decreases to zero. The y-intercept of the graph may be any value. The line on the velocity vs. time graph should have a positive y-intercept and a negative slope. Because the final velocity of the book is zero, the line should finish on the x-axis.

The position vs. time graph should be represented with a negatively sloped line whose slope steadily decreases to zero. The y-intercept of the graph may be any value. The line on the velocity vs. time graph should have a negative y-intercept and a positive slope. Because the final velocity of the book is zero, the line should finish on the x-axis.]

Glossary

acceleration

the rate of change in velocity; the change in velocity over time

average acceleration

the change in velocity divided by the time over which it changes

instantaneous acceleration

acceleration at a specific point in time

deceleration

acceleration in the direction opposite to velocity; acceleration that results in a decrease in velocity

Motion Equations for Constant Acceleration in One Dimension



Kinematic equations can help us describe and predict the motion of moving objects such as these kayaks racing in Newbury, England. (credit: Barry Skeates, Flickr)

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Calculate displacement of an object that is not accelerating, given initial position and velocity.
- Calculate final velocity of an accelerating object, given initial velocity, acceleration, and time.
- Calculate displacement and final position of an accelerating object, given initial position, initial velocity, time, and acceleration.

The information presented in this section supports the following AP® learning objectives and science practices:

- **3.A.1.1** The student is able to express the motion of an object using narrative, mathematical, or graphical representations. (S.P. 1.5, 2.1, 2.2)
- **3.A.1.3** The student is able to analyze experimental data describing the motion of an object and is able to express the results of the analysis using narrative, mathematical, and graphical representations. (S.P. 5.1)

We might know that the greater the acceleration of, say, a car moving away from a stop sign, the greater the displacement in a given time. But we have not developed a specific equation that relates acceleration and displacement. In this section, we develop some convenient equations for kinematic relationships, starting from the definitions of displacement, velocity, and acceleration already covered.

Notation: t , x , v , a

First, let us make some simplifications in notation. Taking the initial time to be zero, as if time is measured with a stopwatch, is a great simplification. Since elapsed time is $\Delta t = t_f - t_0$, taking $t_0 = 0$ means that $\Delta t = t_f$, the final time on the stopwatch. When initial time is taken to be zero, we use the subscript 0 to denote initial values of position and velocity. That is, x_0 is *the initial position* and v_0 is *the initial velocity*. We put no subscripts on the final values. That is, t is *the final time*, x is *the final position*, and v is *the final velocity*. This gives a simpler expression for elapsed time—now, $\Delta t = t$. It also simplifies the expression for displacement, which is now $\Delta x = x - x_0$. Also, it simplifies the expression for change in velocity, which is now $\Delta v = v - v_0$. To summarize, using the simplified notation, with the initial time taken to be zero,

Equation:

$$\left. \begin{aligned}\Delta t &= t \\ \Delta x &= x - x_0 \\ \Delta v &= v - v_0\end{aligned}\right\}$$

where *the subscript 0 denotes an initial value and the absence of a subscript denotes a final value* in whatever motion is under consideration.

We now make the important assumption that *acceleration is constant*. This assumption allows us to avoid using calculus to find instantaneous acceleration. Since acceleration is constant, the average and instantaneous accelerations are equal. That is,

Equation:

$$\bar{a} = a = \text{constant},$$

so we use the symbol a for acceleration at all times. Assuming acceleration to be constant does not seriously limit the situations we can study nor degrade the accuracy of our treatment. For one thing, acceleration *is* constant in a great number of situations. Furthermore, in many other situations we can accurately describe motion by assuming a constant acceleration equal to the average acceleration for that motion. Finally, in motions where acceleration changes drastically, such as a car accelerating to top speed and then braking to a stop, the motion can be considered in separate parts, each of which has its own constant acceleration.

Note:

Solving for Displacement (Δx) and Final Position (x) from Average Velocity when Acceleration (a) is Constant

To get our first two new equations, we start with the definition of average velocity:

Equation:

$$\bar{v} = \frac{\Delta x}{\Delta t}.$$

Substituting the simplified notation for Δx and Δt yields

Equation:

$$\bar{v} = \frac{x - x_0}{t}.$$

Solving for x yields

Equation:

$$x = x_0 + \bar{v}t,$$

where the average velocity is

Equation:

$$\bar{v} = \frac{v_0 + v}{2} \text{ (constant } a\text{)}.$$

The equation $\bar{v} = \frac{v_0 + v}{2}$ reflects the fact that, when acceleration is constant, v is just the simple average of the initial and final velocities. For example, if you steadily increase your velocity (that is, with constant acceleration) from 30 to 60 km/h, then your average velocity during this steady increase is 45 km/h. Using the equation $\bar{v} = \frac{v_0 + v}{2}$ to check this, we see that

Equation:

$$\bar{v} = \frac{v_0 + v}{2} = \frac{30 \text{ km/h} + 60 \text{ km/h}}{2} = 45 \text{ km/h},$$

which seems logical.

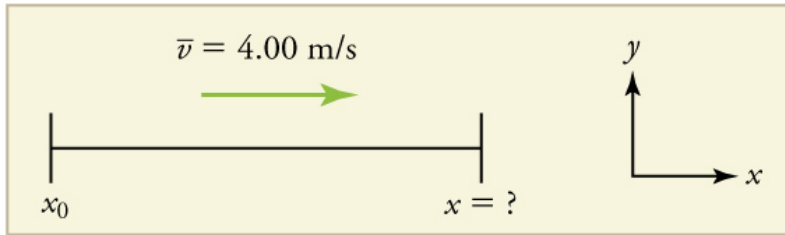
Example:

Calculating Displacement: How Far does the Jogger Run?

A jogger runs down a straight stretch of road with an average velocity of 4.00 m/s for 2.00 min. What is his final position, taking his initial position to be zero?

Strategy

Draw a sketch.



The final position x is given by the equation

Equation:

$$x = x_0 + \bar{v}t.$$

To find x , we identify the values of x_0 , \bar{v} , and t from the statement of the problem and substitute them into the equation.

Solution

1. Identify the knowns. $\bar{v} = 4.00 \text{ m/s}$, $\Delta t = 2.00 \text{ min}$, and $x_0 = 0 \text{ m}$.
2. Enter the known values into the equation.

Equation:

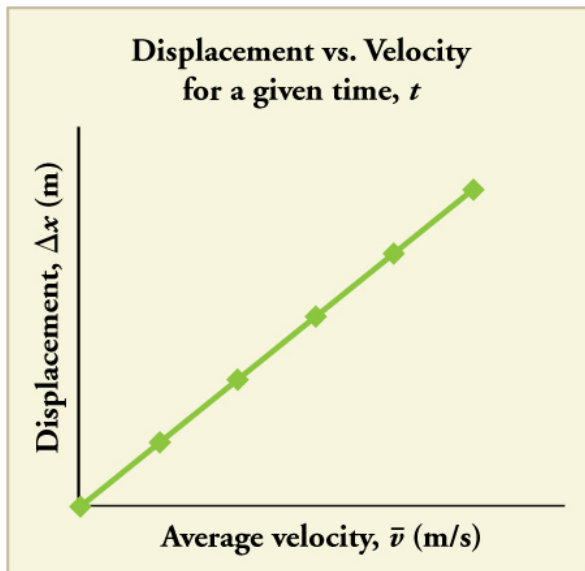
$$x = x_0 + \bar{v}t = 0 + (4.00 \text{ m/s})(120 \text{ s}) = 480 \text{ m}$$

Discussion

Velocity and final displacement are both positive, which means they are in the same direction.

The equation $x = x_0 + \bar{v}t$ gives insight into the relationship between displacement, average velocity, and time. It shows, for example, that displacement is a linear function of average velocity. (By linear function, we mean that displacement depends on \bar{v} rather than on \bar{v} raised to some other power, such as \bar{v}^2 . When graphed, linear functions look like straight

lines with a constant slope.) On a car trip, for example, we will get twice as far in a given time if we average 90 km/h than if we average 45 km/h.



There is a linear relationship between displacement and average velocity. For a given time t , an object moving twice as fast as another object will move twice as far as the other object.

Note:

Solving for Final Velocity

We can derive another useful equation by manipulating the definition of acceleration.

Equation:

$$a = \frac{\Delta v}{\Delta t}$$

Substituting the simplified notation for Δv and Δt gives us

Equation:

$$a = \frac{v - v_0}{t} \text{ (constant } a\text{)}.$$

Solving for v yields

Equation:

$$v = v_0 + at \text{ (constant } a\text{)}.$$

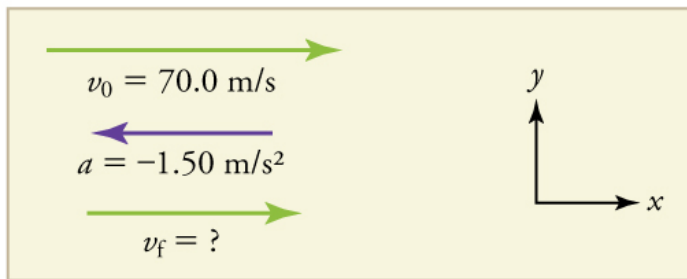
Example:

Calculating Final Velocity: An Airplane Slowing Down after Landing

An airplane lands with an initial velocity of 70.0 m/s and then decelerates at 1.50 m/s^2 for 40.0 s. What is its final velocity?

Strategy

Draw a sketch. We draw the acceleration vector in the direction opposite the velocity vector because the plane is decelerating.



Solution

1. Identify the knowns. $v_0 = 70.0 \text{ m/s}$, $a = -1.50 \text{ m/s}^2$, $t = 40.0 \text{ s}$.
2. Identify the unknown. In this case, it is final velocity, v_f .
3. Determine which equation to use. We can calculate the final velocity using the equation $v = v_0 + at$.
4. Plug in the known values and solve.

Equation:

$$v = v_0 + at = 70.0 \text{ m/s} + (-1.50 \text{ m/s}^2)(40.0 \text{ s}) = 10.0 \text{ m/s}$$

Discussion

The final velocity is much less than the initial velocity, as desired when slowing down, but still positive. With jet engines, reverse thrust could be maintained long enough to stop the plane and start moving it backward. That would be indicated by a negative final velocity, which is not the case here.



The airplane lands with an initial velocity of 70.0 m/s and slows to a final velocity of 10.0 m/s before heading for the terminal. Note that the acceleration is negative because its direction is opposite to its velocity, which is positive.

In addition to being useful in problem solving, the equation $v = v_0 + at$ gives us insight into the relationships among velocity, acceleration, and time. From it we can see, for example, that

- final velocity depends on how large the acceleration is and how long it lasts
- if the acceleration is zero, then the final velocity equals the initial velocity ($v = v_0$), as expected (i.e., velocity is constant)
- if a is negative, then the final velocity is less than the initial velocity

(All of these observations fit our intuition, and it is always useful to examine basic equations in light of our intuition and experiences to check that they do indeed describe nature accurately.)

Note:**Making Connections: Real-World Connection**

The Space Shuttle *Endeavor*
blasts off from the Kennedy
Space Center in February 2010.
(credit: Matthew Simantov,
Flickr)

An intercontinental ballistic missile (ICBM) has a larger average acceleration than the Space Shuttle and achieves a greater velocity in the first minute or two of flight (actual ICBM burn times are classified—short-burn-time missiles are more difficult for an enemy to destroy). But the Space Shuttle obtains a greater final velocity, so that it can orbit the earth rather than come directly back down as an ICBM does. The Space Shuttle does this by accelerating for a longer time.

Note:**Solving for Final Position When Velocity is Not Constant ($a \neq 0$)**

We can combine the equations above to find a third equation that allows us to calculate the final position of an object experiencing constant acceleration. We start with

Equation:

$$v = v_0 + at.$$

Adding v_0 to each side of this equation and dividing by 2 gives

Equation:

$$\frac{v_0 + v}{2} = v_0 + \frac{1}{2}at.$$

Since $\frac{v_0 + v}{2} = \bar{v}$ for constant acceleration, then

Equation:

$$\bar{v} = v_0 + \frac{1}{2}at.$$

Now we substitute this expression for \bar{v} into the equation for displacement,

$x = x_0 + \bar{v}t$, yielding

Equation:

$$x = x_0 + v_0t + \frac{1}{2}at^2 \text{ (constant } a\text{)}.$$

Example:

Calculating Displacement of an Accelerating Object: Dragsters

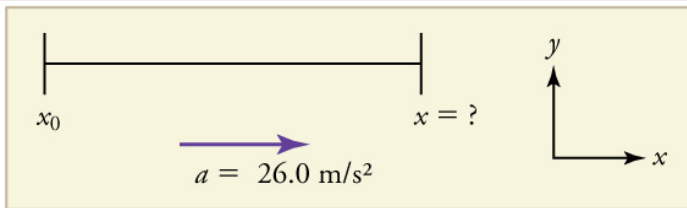
Dragsters can achieve average accelerations of 26.0 m/s^2 . Suppose such a dragster accelerates from rest at this rate for 5.56 s. How far does it travel in this time?



U.S. Army Top Fuel pilot
Tony “The Sarge”
Schumacher begins a race
with a controlled burnout.
(credit: Lt. Col. William
Thurmond. Photo
Courtesy of U.S. Army.)

Strategy

Draw a sketch.



We are asked to find displacement, which is x if we take x_0 to be zero. (Think about it like the starting line of a race. It can be anywhere, but we call it 0 and measure all other positions relative to it.) We can use the equation $x = x_0 + v_0t + \frac{1}{2}at^2$ once we identify v_0 , a , and t from the statement of the problem.

Solution

1. Identify the knowns. Starting from rest means that $v_0 = 0$, a is given as 26.0 m/s^2 and t is given as 5.56 s .
2. Plug the known values into the equation to solve for the unknown x :

Equation:

$$x = x_0 + v_0t + \frac{1}{2}at^2.$$

Since the initial position and velocity are both zero, this simplifies to

Equation:

$$x = \frac{1}{2}at^2.$$

Substituting the identified values of a and t gives

Equation:

$$x = \frac{1}{2} (26.0 \text{ m/s}^2) (5.56 \text{ s})^2,$$

yielding

Equation:

$$x = 402 \text{ m.}$$

Discussion

If we convert 402 m to miles, we find that the distance covered is very close to one quarter of a mile, the standard distance for drag racing. So the answer is reasonable. This is an impressive displacement in only 5.56 s, but top-notch dragsters can do a quarter mile in even less time than this.

What else can we learn by examining the equation $x = x_0 + v_0 t + \frac{1}{2} a t^2$?

We see that:

- displacement depends on the square of the elapsed time when acceleration is not zero. In [\[link\]](#), the dragster covers only one fourth of the total distance in the first half of the elapsed time
- if acceleration is zero, then the initial velocity equals average velocity ($v_0 = \bar{v}$) and $x = x_0 + v_0 t + \frac{1}{2} a t^2$ becomes $x = x_0 + v_0 t$

Note:

Solving for Final Velocity when Velocity Is Not Constant ($a \neq 0$)

A fourth useful equation can be obtained from another algebraic manipulation of previous equations.

If we solve $v = v_0 + at$ for t , we get

Equation:

$$t = \frac{v - v_0}{a}.$$

Substituting this and $\bar{v} = \frac{v_0 + v}{2}$ into $x = x_0 + \bar{v}t$, we get

Equation:

$$v^2 = v_0^2 + 2a(x - x_0) \text{ (constant } a\text{)}.$$

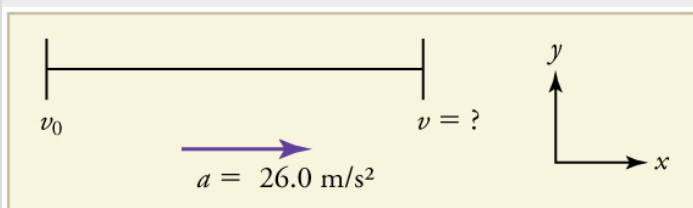
Example:

Calculating Final Velocity: Dragsters

Calculate the final velocity of the dragster in [\[link\]](#) without using information about time.

Strategy

Draw a sketch.



The equation $v^2 = v_0^2 + 2a(x - x_0)$ is ideally suited to this task because it relates velocities, acceleration, and displacement, and no time information is required.

Solution

1. Identify the known values. We know that $v_0 = 0$, since the dragster starts from rest. Then we note that $x - x_0 = 402$ m (this was the answer in [\[link\]](#)). Finally, the average acceleration was given to be $a = 26.0$ m/s².
2. Plug the knowns into the equation $v^2 = v_0^2 + 2a(x - x_0)$ and solve for v .

Equation:

$$v^2 = 0 + 2(26.0 \text{ m/s}^2)(402 \text{ m}).$$

Thus

Equation:

$$v^2 = 2.09 \times 10^4 \text{ m}^2/\text{s}^2.$$

To get v , we take the square root:

Equation:

$$v = \sqrt{2.09 \times 10^4 \text{ m}^2/\text{s}^2} = 145 \text{ m/s}.$$

Discussion

145 m/s is about 522 km/h or about 324 mi/h, but even this breakneck speed is short of the record for the quarter mile. Also, note that a square root has two values; we took the positive value to indicate a velocity in the same direction as the acceleration.

An examination of the equation $v^2 = v_0^2 + 2a(x - x_0)$ can produce further insights into the general relationships among physical quantities:

- The final velocity depends on how large the acceleration is and the distance over which it acts
- For a fixed deceleration, a car that is going twice as fast doesn't simply stop in twice the distance—it takes much further to stop. (This is why we have reduced speed zones near schools.)

Putting Equations Together

In the following examples, we further explore one-dimensional motion, but in situations requiring slightly more algebraic manipulation. The examples also give insight into problem-solving techniques. The box below provides easy reference to the equations needed.

Note:

Summary of Kinematic Equations (constant a)

Equation:

$$x = x_0 + \bar{v}t$$

Equation:

$$\bar{v} = \frac{v_0 + v}{2}$$

Equation:

$$v = v_0 + at$$

Equation:

$$x = x_0 + v_0t + \frac{1}{2}at^2$$

Equation:

$$v^2 = v_0^2 + 2a(x - x_0)$$

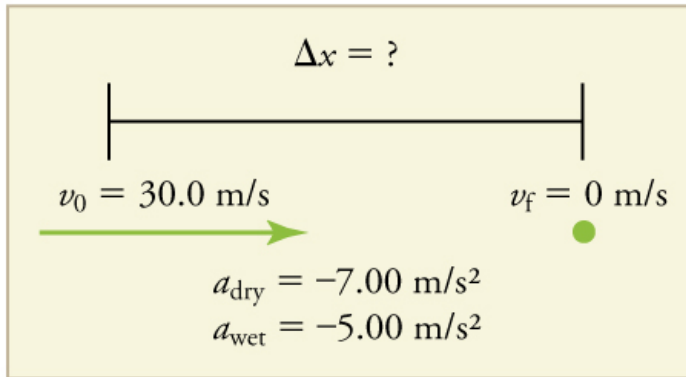
Example:

Calculating Displacement: How Far Does a Car Go When Coming to a Halt?

On dry concrete, a car can decelerate at a rate of 7.00 m/s^2 , whereas on wet concrete it can decelerate at only 5.00 m/s^2 . Find the distances necessary to stop a car moving at 30.0 m/s (about 110 km/h) (a) on dry concrete and (b) on wet concrete. (c) Repeat both calculations, finding the displacement from the point where the driver sees a traffic light turn red, taking into account his reaction time of 0.500 s to get his foot on the brake.

Strategy

Draw a sketch.



In order to determine which equations are best to use, we need to list all of the known values and identify exactly what we need to solve for. We shall do this explicitly in the next several examples, using tables to set them off.

Solution for (a)

1. Identify the knowns and what we want to solve for. We know that $v_0 = 30.0 \text{ m/s}$; $v = 0$; $a = -7.00 \text{ m/s}^2$ (a is negative because it is in a direction opposite to velocity). We take x_0 to be 0. We are looking for displacement Δx , or $x - x_0$.

2. Identify the equation that will help up solve the problem. The best equation to use is

Equation:

$$v^2 = v_0^2 + 2a(x - x_0).$$

This equation is best because it includes only one unknown, x . We know the values of all the other variables in this equation. (There are other equations that would allow us to solve for x , but they require us to know the stopping time, t , which we do not know. We could use them but it would entail additional calculations.)

3. Rearrange the equation to solve for x .

Equation:

$$x - x_0 = \frac{v^2 - v_0^2}{2a}$$

4. Enter known values.

Equation:

$$x - 0 = \frac{0^2 - (30.0 \text{ m/s})^2}{2(-7.00 \text{ m/s}^2)}$$

Thus,

Equation:

$$x = 64.3 \text{ m on dry concrete.}$$

Solution for (b)

This part can be solved in exactly the same manner as Part A. The only difference is that the deceleration is -5.00 m/s^2 . The result is

Equation:

$$x_{\text{wet}} = 90.0 \text{ m on wet concrete.}$$

Solution for (c)

Once the driver reacts, the stopping distance is the same as it is in Parts A and B for dry and wet concrete. So to answer this question, we need to calculate how far the car travels during the reaction time, and then add that to the stopping time. It is reasonable to assume that the velocity remains constant during the driver's reaction time.

1. Identify the knowns and what we want to solve for. We know that $\bar{v} = 30.0 \text{ m/s}$; $t_{\text{reaction}} = 0.500 \text{ s}$; $a_{\text{reaction}} = 0$. We take $x_{0-\text{reaction}}$ to be 0. We are looking for x_{reaction} .
2. Identify the best equation to use.

$x = x_0 + \bar{v}t$ works well because the only unknown value is x , which is what we want to solve for.

3. Plug in the knowns to solve the equation.

Equation:

$$x = 0 + (30.0 \text{ m/s})(0.500 \text{ s}) = 15.0 \text{ m.}$$

This means the car travels 15.0 m while the driver reacts, making the total displacements in the two cases of dry and wet concrete 15.0 m greater than if he reacted instantly.

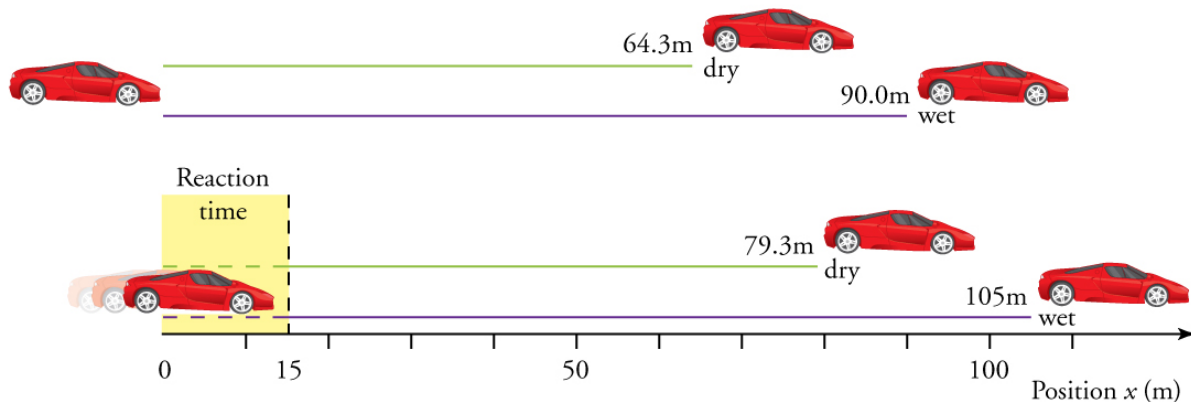
4. Add the displacement during the reaction time to the displacement when braking.

Equation:

$$x_{\text{braking}} + x_{\text{reaction}} = x_{\text{total}}$$

a. $64.3 \text{ m} + 15.0 \text{ m} = 79.3 \text{ m}$ when dry

b. $90.0 \text{ m} + 15.0 \text{ m} = 105 \text{ m}$ when wet



The distance necessary to stop a car varies greatly, depending on road conditions and driver reaction time. Shown here are the braking distances for dry and wet pavement, as calculated in this example, for a car initially traveling at 30.0 m/s . Also shown are the total distances traveled from the point where the driver first sees a light turn red, assuming a 0.500 s reaction time.

Discussion

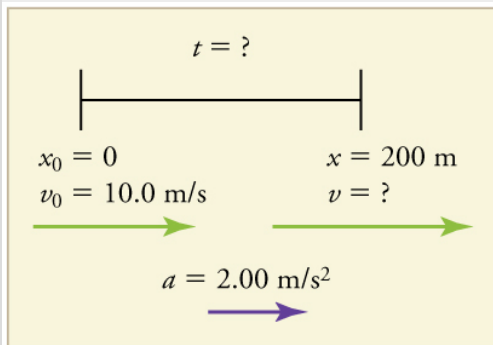
The displacements found in this example seem reasonable for stopping a fast-moving car. It should take longer to stop a car on wet rather than dry pavement. It is interesting that reaction time adds significantly to the displacements. But more important is the general approach to solving problems. We identify the knowns and the quantities to be determined and then find an appropriate equation. There is often more than one way to solve a problem. The various parts of this example can in fact be solved by other methods, but the solutions presented above are the shortest.

Example:**Calculating Time: A Car Merges into Traffic**

Suppose a car merges into freeway traffic on a 200-m-long ramp. If its initial velocity is 10.0 m/s and it accelerates at 2.00 m/s^2 , how long does it take to travel the 200 m up the ramp? (Such information might be useful to a traffic engineer.)

Strategy

Draw a sketch.



We are asked to solve for the time t . As before, we identify the known quantities in order to choose a convenient physical relationship (that is, an equation with one unknown, t).

Solution

1. Identify the knowns and what we want to solve for. We know that $v_0 = 10 \text{ m/s}$; $a = 2.00 \text{ m/s}^2$; and $x = 200 \text{ m}$.
2. We need to solve for t . Choose the best equation. $x = x_0 + v_0 t + \frac{1}{2} a t^2$ works best because the only unknown in the equation is the variable t for which we need to solve.
3. We will need to rearrange the equation to solve for t . In this case, it will be easier to plug in the knowns first.

Equation:

$$200 \text{ m} = 0 \text{ m} + (10.0 \text{ m/s})t + \frac{1}{2} (2.00 \text{ m/s}^2) t^2$$

4. Simplify the equation. The units of meters (m) cancel because they are in each term. We can get the units of seconds (s) to cancel by taking $t = t \text{ s}$, where t is the magnitude of time and s is the unit. Doing so leaves

Equation:

$$200 = 10t + t^2.$$

5. Use the quadratic formula to solve for t .

(a) Rearrange the equation to get 0 on one side of the equation.

Equation:

$$t^2 + 10t - 200 = 0$$

This is a quadratic equation of the form

Equation:

$$at^2 + bt + c = 0,$$

where the constants are $a = 1.00$, $b = 10.0$, and $c = -200$.

(b) Its solutions are given by the quadratic formula:

Equation:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This yields two solutions for t , which are

Equation:

$$t = 10.0 \text{ and } -20.0.$$

In this case, then, the time is $t = t$ in seconds, or

Equation:

$$t = 10.0 \text{ s and } -20.0 \text{ s}.$$

A negative value for time is unreasonable, since it would mean that the event happened 20 s before the motion began. We can discard that solution.

Thus,

Equation:

$$t = 10.0 \text{ s}.$$

Discussion

Whenever an equation contains an unknown squared, there will be two solutions. In some problems both solutions are meaningful, but in others, such as the above, only one solution is reasonable. The 10.0 s answer seems reasonable for a typical freeway on-ramp.

With the basics of kinematics established, we can go on to many other interesting examples and applications. In the process of developing kinematics, we have also glimpsed a general approach to problem solving that produces both correct answers and insights into physical relationships. [Problem-Solving Basics](#) discusses problem-solving basics and outlines an approach that will help you succeed in this invaluable task.

Note:

Making Connections: Take-Home Experiment—Breaking News

We have been using SI units of meters per second squared to describe some examples of acceleration or deceleration of cars, runners, and trains. To achieve a better feel for these numbers, one can measure the braking deceleration of a car doing a slow (and safe) stop. Recall that, for average acceleration, $\bar{a} = \Delta v / \Delta t$. While traveling in a car, slowly apply the brakes as you come up to a stop sign. Have a passenger note the initial speed in miles per hour and the time taken (in seconds) to stop. From this, calculate the deceleration in miles per hour per second. Convert this to meters per second squared and compare with other decelerations mentioned in this chapter. Calculate the distance traveled in braking.

Exercise:

Check Your Understanding

Problem:

A manned rocket accelerates at a rate of 20 m/s^2 during launch. How long does it take the rocket reach a velocity of 400 m/s ?

Solution:

To answer this, choose an equation that allows you to solve for time t , given only a , v_0 , and v .

Equation:

$$v = v_0 + at$$

Rearrange to solve for t .

Equation:

$$t = \frac{v - v_0}{a} = \frac{400 \text{ m/s} - 0 \text{ m/s}}{20 \text{ m/s}^2} = 20 \text{ s}$$

Section Summary

- To simplify calculations we take acceleration to be constant, so that $\bar{a} = a$ at all times.
- We also take initial time to be zero.
- Initial position and velocity are given a subscript 0; final values have no subscript. Thus,

Equation:

$$\left. \begin{aligned} \Delta t &= t \\ \Delta x &= x - x_0 \\ \Delta v &= v - v_0 \end{aligned} \right\}$$

- The following kinematic equations for motion with constant a are useful:

Equation:

$$x = x_0 + \bar{v}t$$

Equation:

$$\bar{v} = \frac{v_0 + v}{2}$$

Equation:

$$v = v_0 + at$$

Equation:

$$x = x_0 + v_0t + \frac{1}{2}at^2$$

Equation:

$$v^2 = v_0^2 + 2a(x - x_0)$$

- In vertical motion, y is substituted for x .

Problems & Exercises

Exercise:

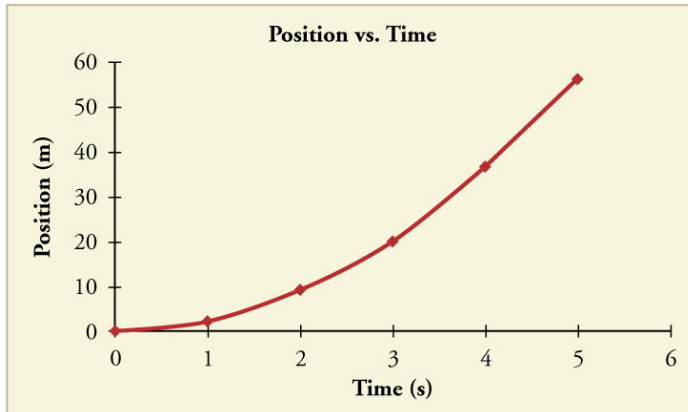
Problem:

An Olympic-class sprinter starts a race with an acceleration of 4.50 m/s^2 . (a) What is her speed 2.40 s later? (b) Sketch a graph of her position vs. time for this period.

Solution:

(a) 10.8 m/s

(b)



Exercise:

Problem:

A well-thrown ball is caught in a well-padded mitt. If the deceleration of the ball is $2.10 \times 10^4 \text{ m/s}^2$, and 1.85 ms ($1 \text{ ms} = 10^{-3} \text{ s}$) elapses from the time the ball first touches the mitt until it stops, what was the initial velocity of the ball?

Solution:

38.9 m/s (about 87 miles per hour)

Exercise:

Problem:

A bullet in a gun is accelerated from the firing chamber to the end of the barrel at an average rate of $6.20 \times 10^5 \text{ m/s}^2$ for $8.10 \times 10^{-4} \text{ s}$. What is its muzzle velocity (that is, its final velocity)?

Exercise:

Problem:

(a) A light-rail commuter train accelerates at a rate of 1.35 m/s^2 . How long does it take to reach its top speed of 80.0 km/h , starting from rest? (b) The same train ordinarily decelerates at a rate of 1.65 m/s^2 . How long does it take to come to a stop from its top speed? (c) In emergencies the train can decelerate more rapidly, coming to rest from 80.0 km/h in 8.30 s . What is its emergency deceleration in m/s^2 ?

Solution:

(a) 16.5 s

(b) 13.5 s

(c) -2.68 m/s^2

Exercise:**Problem:**

While entering a freeway, a car accelerates from rest at a rate of 2.40 m/s^2 for 12.0 s . (a) Draw a sketch of the situation. (b) List the knowns in this problem. (c) How far does the car travel in those 12.0 s ? To solve this part, first identify the unknown, and then discuss how you chose the appropriate equation to solve for it. After choosing the equation, show your steps in solving for the unknown, check your units, and discuss whether the answer is reasonable. (d) What is the car's final velocity? Solve for this unknown in the same manner as in part (c), showing all steps explicitly.

Exercise:**Problem:**

At the end of a race, a runner decelerates from a velocity of 9.00 m/s at a rate of 2.00 m/s^2 . (a) How far does she travel in the next 5.00 s ? (b) What is her final velocity? (c) Evaluate the result. Does it make sense?

Solution:

(a) 20.0 m

(b) -1.00 m/s

(c) This result does not really make sense. If the runner starts at 9.00 m/s and decelerates at 2.00 m/s^2 , then she will have stopped after 4.50 s. If she continues to decelerate, she will be running backwards.

Exercise:**Problem: Professional Application:**

Blood is accelerated from rest to 30.0 cm/s in a distance of 1.80 cm by the left ventricle of the heart. (a) Make a sketch of the situation. (b) List the knowns in this problem. (c) How long does the acceleration take? To solve this part, first identify the unknown, and then discuss how you chose the appropriate equation to solve for it. After choosing the equation, show your steps in solving for the unknown, checking your units. (d) Is the answer reasonable when compared with the time for a heartbeat?

Exercise:**Problem:**

In a slap shot, a hockey player accelerates the puck from a velocity of 8.00 m/s to 40.0 m/s in the same direction. If this shot takes $3.33 \times 10^{-2} \text{ s}$, calculate the distance over which the puck accelerates.

Solution:

0.799 m

Exercise:

Problem:

A powerful motorcycle can accelerate from rest to 26.8 m/s (100 km/h) in only 3.90 s. (a) What is its average acceleration? (b) How far does it travel in that time?

Exercise:**Problem:**

Freight trains can produce only relatively small accelerations and decelerations. (a) What is the final velocity of a freight train that accelerates at a rate of 0.0500 m/s^2 for 8.00 min, starting with an initial velocity of 4.00 m/s? (b) If the train can slow down at a rate of 0.550 m/s^2 , how long will it take to come to a stop from this velocity? (c) How far will it travel in each case?

Solution:

(a) 28.0 m/s

(b) 50.9 s

(c) 7.68 km to accelerate and 713 m to decelerate

Exercise:**Problem:**

A fireworks shell is accelerated from rest to a velocity of 65.0 m/s over a distance of 0.250 m. (a) How long did the acceleration last? (b) Calculate the acceleration.

Exercise:

Problem:

A swan on a lake gets airborne by flapping its wings and running on top of the water. (a) If the swan must reach a velocity of 6.00 m/s to take off and it accelerates from rest at an average rate of 0.350 m/s^2 , how far will it travel before becoming airborne? (b) How long does this take?

Solution:

(a) 51.4 m

(b) 17.1 s

Exercise:**Problem: Professional Application:**

A woodpecker's brain is specially protected from large decelerations by tendon-like attachments inside the skull. While pecking on a tree, the woodpecker's head comes to a stop from an initial velocity of 0.600 m/s in a distance of only 2.00 mm. (a) Find the acceleration in m/s^2 and in multiples of g ($g = 9.80 \text{ m/s}^2$). (b) Calculate the stopping time. (c) The tendons cradling the brain stretch, making its stopping distance 4.50 mm (greater than the head and, hence, less deceleration of the brain). What is the brain's deceleration, expressed in multiples of g ?

Exercise:**Problem:**

An unwary football player collides with a padded goalpost while running at a velocity of 7.50 m/s and comes to a full stop after compressing the padding and his body 0.350 m. (a) What is his deceleration? (b) How long does the collision last?

Solution:

(a) -80.4 m/s^2

(b) $9.33 \times 10^{-2} \text{ s}$

Exercise:

Problem:

In World War II, there were several reported cases of airmen who jumped from their flaming airplanes with no parachute to escape certain death. Some fell about 20,000 feet (6000 m), and some of them survived, with few life-threatening injuries. For these lucky pilots, the tree branches and snow drifts on the ground allowed their deceleration to be relatively small. If we assume that a pilot's speed upon impact was 123 mph (54 m/s), then what was his deceleration? Assume that the trees and snow stopped him over a distance of 3.0 m.

Exercise:

Problem:

Consider a grey squirrel falling out of a tree to the ground. (a) If we ignore air resistance in this case (only for the sake of this problem), determine a squirrel's velocity just before hitting the ground, assuming it fell from a height of 3.0 m. (b) If the squirrel stops in a distance of 2.0 cm through bending its limbs, compare its deceleration with that of the airman in the previous problem.

Solution:

(a) 7.7 m/s

(b) $-15 \times 10^2 \text{ m/s}^2$. This is about 3 times the deceleration of the pilots, who were falling from thousands of meters high!

Exercise:

Problem:

An express train passes through a station. It enters with an initial velocity of 22.0 m/s and decelerates at a rate of 0.150 m/s^2 as it goes through. The station is 210 m long. (a) How long is the nose of the train in the station? (b) How fast is it going when the nose leaves the station? (c) If the train is 130 m long, when does the end of the train leave the station? (d) What is the velocity of the end of the train as it leaves?

Exercise:**Problem:**

Dragsters can actually reach a top speed of 145 m/s in only 4.45 s —considerably less time than given in [\[link\]](#) and [\[link\]](#). (a) Calculate the average acceleration for such a dragster. (b) Find the final velocity of this dragster starting from rest and accelerating at the rate found in (a) for 402 m (a quarter mile) without using any information on time. (c) Why is the final velocity greater than that used to find the average acceleration? *Hint:* Consider whether the assumption of constant acceleration is valid for a dragster. If not, discuss whether the acceleration would be greater at the beginning or end of the run and what effect that would have on the final velocity.

Solution:

(a) 32.6 m/s^2

(b) 162 m/s

(c) $v > v_{\text{max}}$, because the assumption of constant acceleration is not valid for a dragster. A dragster changes gears, and would have a greater acceleration in first gear than second gear than third gear, etc. The acceleration would be greatest at the beginning, so it would not be accelerating at 32.6 m/s^2 during the last few meters, but substantially less, and the final velocity would be less than 162 m/s .

Exercise:**Problem:**

A bicycle racer sprints at the end of a race to clinch a victory. The racer has an initial velocity of 11.5 m/s and accelerates at the rate of 0.500 m/s^2 for 7.00 s . (a) What is his final velocity? (b) The racer continues at this velocity to the finish line. If he was 300 m from the finish line when he started to accelerate, how much time did he save? (c) One other racer was 5.00 m ahead when the winner started to accelerate, but he was unable to accelerate, and traveled at 11.8 m/s until the finish line. How far ahead of him (in meters and in seconds) did the winner finish?

Exercise:**Problem:**

In 1967, New Zealander Burt Munro set the world record for an Indian motorcycle, on the Bonneville Salt Flats in Utah, with a maximum speed of 183.58 mi/h . The one-way course was 5.00 mi long. Acceleration rates are often described by the time it takes to reach 60.0 mi/h from rest. If this time was 4.00 s , and Burt accelerated at this rate until he reached his maximum speed, how long did it take Burt to complete the course?

Solution:

104 s

Exercise:

Problem:

(a) A world record was set for the men's 100-m dash in the 2008 Olympic Games in Beijing by Usain Bolt of Jamaica. Bolt “coasted” across the finish line with a time of 9.69 s. If we assume that Bolt accelerated for 3.00 s to reach his maximum speed, and maintained that speed for the rest of the race, calculate his maximum speed and his acceleration. (b) During the same Olympics, Bolt also set the world record in the 200-m dash with a time of 19.30 s. Using the same assumptions as for the 100-m dash, what was his maximum speed for this race?

Solution:

(a) $v = 12.2 \text{ m/s}$; $a = 4.07 \text{ m/s}^2$

(b) $v = 11.2 \text{ m/s}$

Test Prep for AP Courses**Exercise:****Problem:**

A group of students is attempting to determine the average acceleration of a marble released from the top of a long ramp. Below is a set of data representing the marble's position with respect to time.

Position (cm)	Time (s)
0.0	0.0

Position (cm)	Time (s)
0.3	0.5
1.25	1.0
2.8	1.5
5.0	2.0
7.75	2.5
11.3	3.0

Use the data table above to construct a graph determining the acceleration of the marble. Select a set of data points from the table and plot those points on the graph. Fill in the blank column in the table for any quantities you graph other than the given data. Label the axes and indicate the scale for each. Draw a best-fit line or curve through your data points.

Using the best-fit line, determine the value of the marble's acceleration.

Problem-Solving Basics for One Dimensional Kinematics



Problem-solving skills are essential to your success in Physics. (credit: scui3asteveo, Flickr)

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Apply problem-solving steps and strategies to solve problems of one-dimensional kinematics.
- Apply strategies to determine whether or not the result of a problem is reasonable, and if not, determine the cause.

Problem-solving skills are obviously essential to success in a quantitative course in physics. More importantly, the ability to apply broad physical

principles, usually represented by equations, to specific situations is a very powerful form of knowledge. It is much more powerful than memorizing a list of facts. Analytical skills and problem-solving abilities can be applied to new situations, whereas a list of facts cannot be made long enough to contain every possible circumstance. Such analytical skills are useful both for solving problems in this text and for applying physics in everyday and professional life.

Problem-Solving Steps

While there is no simple step-by-step method that works for every problem, the following general procedures facilitate problem solving and make it more meaningful. A certain amount of creativity and insight is required as well.

Step 1

Examine the situation to determine which physical principles are involved. It often helps to *draw a simple sketch* at the outset. You will also need to decide which direction is positive and note that on your sketch. Once you have identified the physical principles, it is much easier to find and apply the equations representing those principles. Although finding the correct equation is essential, keep in mind that equations represent physical principles, laws of nature, and relationships among physical quantities. Without a conceptual understanding of a problem, a numerical solution is meaningless.

Step 2

Make a list of what is given or can be inferred from the problem as stated (identify the knowns). Many problems are stated very succinctly and require some inspection to determine what is known. A sketch can also be very useful at this point. Formally identifying the knowns is of particular importance in applying physics to real-world situations. Remember,

“stopped” means velocity is zero, and we often can take initial time and position as zero.

Step 3

Identify exactly what needs to be determined in the problem (identify the unknowns). In complex problems, especially, it is not always obvious what needs to be found or in what sequence. Making a list can help.

Step 4

Find an equation or set of equations that can help you solve the problem. Your list of knowns and unknowns can help here. It is easiest if you can find equations that contain only one unknown—that is, all of the other variables are known, so you can easily solve for the unknown. If the equation contains more than one unknown, then an additional equation is needed to solve the problem. In some problems, several unknowns must be determined to get at the one needed most. In such problems it is especially important to keep physical principles in mind to avoid going astray in a sea of equations. You may have to use two (or more) different equations to get the final answer.

Step 5

Substitute the knowns along with their units into the appropriate equation, and obtain numerical solutions complete with units. This step produces the numerical answer; it also provides a check on units that can help you find errors. If the units of the answer are incorrect, then an error has been made. However, be warned that correct units do not guarantee that the numerical part of the answer is also correct.

Step 6

Check the answer to see if it is reasonable: Does it make sense? This final step is extremely important—the goal of physics is to accurately describe nature. To see if the answer is reasonable, check both its magnitude and its sign, in addition to its units. Your judgment will improve as you solve more and more physics problems, and it will become possible for you to make finer and finer judgments regarding whether nature is adequately described by the answer to a problem. This step brings the problem back to its conceptual meaning. If you can judge whether the answer is reasonable, you have a deeper understanding of physics than just being able to mechanically solve a problem.

When solving problems, we often perform these steps in different order, and we also tend to do several steps simultaneously. There is no rigid procedure that will work every time. Creativity and insight grow with experience, and the basics of problem solving become almost automatic. One way to get practice is to work out the text's examples for yourself as you read. Another is to work as many end-of-section problems as possible, starting with the easiest to build confidence and progressing to the more difficult. Once you become involved in physics, you will see it all around you, and you can begin to apply it to situations you encounter outside the classroom, just as is done in many of the applications in this text.

Unreasonable Results

Physics must describe nature accurately. Some problems have results that are unreasonable because one premise is unreasonable or because certain premises are inconsistent with one another. The physical principle applied correctly then produces an unreasonable result. For example, if a person starting a foot race accelerates at 0.40 m/s^2 for 100 s, his final speed will be 40 m/s (about 150 km/h)—clearly unreasonable because the time of 100 s is an unreasonable premise. The physics is correct in a sense, but there is more to describing nature than just manipulating equations correctly. Checking the result of a problem to see if it is reasonable does more than help uncover errors in problem solving—it also builds intuition in judging whether nature is being accurately described.

Use the following strategies to determine whether an answer is reasonable and, if it is not, to determine what is the cause.

Step 1

Solve the problem using strategies as outlined and in the format followed in the worked examples in the text. In the example given in the preceding paragraph, you would identify the givens as the acceleration and time and use the equation below to find the unknown final velocity. That is,

Equation:

$$v = v_0 + at = 0 + (0.40 \text{ m/s}^2)(100 \text{ s}) = 40 \text{ m/s}.$$

Step 2

Check to see if the answer is reasonable. Is it too large or too small, or does it have the wrong sign, improper units, ...? In this case, you may need to convert meters per second into a more familiar unit, such as miles per hour.

Equation:

$$\left(\frac{40 \text{ m}}{\text{s}}\right) \left(\frac{3.28 \text{ ft}}{\text{m}}\right) \left(\frac{1 \text{ mi}}{5280 \text{ ft}}\right) \left(\frac{60 \text{ s}}{\text{min}}\right) \left(\frac{60 \text{ min}}{1 \text{ h}}\right) = 89 \text{ mph}$$

This velocity is about four times greater than a person can run—so it is too large.

Step 3

If the answer is unreasonable, look for what specifically could cause the identified difficulty. In the example of the runner, there are only two

assumptions that are suspect. The acceleration could be too great or the time too long. First look at the acceleration and think about what the number means. If someone accelerates at 0.40 m/s^2 , their velocity is increasing by 0.4 m/s each second. Does this seem reasonable? If so, the time must be too long. It is not possible for someone to accelerate at a constant rate of 0.40 m/s^2 for 100 s (almost two minutes).

Section Summary

- *The six basic problem solving steps for physics are:*

Step 1. Examine the situation to determine which physical principles are involved.

Step 2. Make a list of what is given or can be inferred from the problem as stated (identify the knowns).

Step 3. Identify exactly what needs to be determined in the problem (identify the unknowns).

Step 4. Find an equation or set of equations that can help you solve the problem.

Step 5. Substitute the knowns along with their units into the appropriate equation, and obtain numerical solutions complete with units.

Step 6. Check the answer to see if it is reasonable: Does it make sense?

Conceptual Questions

Exercise:

Problem:

What information do you need in order to choose which equation or equations to use to solve a problem? Explain.

Exercise:

Problem:

What is the last thing you should do when solving a problem? Explain.

Falling Objects

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Describe the effects of gravity on objects in motion.
- Describe the motion of objects that are in free fall.
- Calculate the position and velocity of objects in free fall.

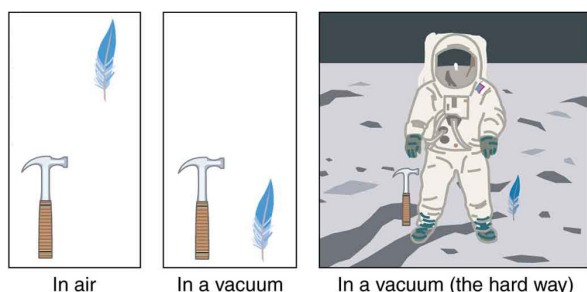
The information presented in this section supports the following AP® learning objectives and science practices:

- **3.A.1.1** The student is able to express the motion of an object using narrative, mathematical, or graphical representations. **(S.P. 1.5, 2.1, 2.2)**
- **3.A.1.2** The student is able to design an experimental investigation of the motion of an object. **(S.P. 4.2)**
- **3.A.1.3** The student is able to analyze experimental data describing the motion of an object and is able to express the results of the analysis using narrative, mathematical, and graphical representations. **(S.P. 5.1)**

Falling objects form an interesting class of motion problems. For example, we can estimate the depth of a vertical mine shaft by dropping a rock into it and listening for the rock to hit the bottom. By applying the kinematics developed so far to falling objects, we can examine some interesting situations and learn much about gravity in the process.

Gravity

The most remarkable and unexpected fact about falling objects is that, if air resistance and friction are negligible, then in a given location all objects fall toward the center of Earth with the *same constant acceleration, independent of their mass*. This experimentally determined fact is unexpected, because we are so accustomed to the effects of air resistance and friction that we expect light objects to fall slower than heavy ones.



A hammer and a feather will fall with the same constant acceleration if air resistance is considered negligible. This is a general characteristic of gravity not unique to Earth, as astronaut David R. Scott demonstrated on the Moon in 1971, where the acceleration due to gravity is only 1.67 m/s^2 .

In the real world, air resistance can cause a lighter object to fall slower than a heavier object of the same size. A tennis ball will reach the ground after a hard baseball dropped at the same time. (It might be difficult to observe the difference if the height is not large.) Air resistance opposes the motion of an object through the air, while friction between objects—such as between clothes and a laundry chute or between a stone and a pool into which it is dropped—also opposes motion between them. For the ideal situations of these first few chapters, an object *falling without air resistance or friction* is defined to be in **free-fall**.

The force of gravity causes objects to fall toward the center of Earth. The acceleration of free-falling objects is therefore called the **acceleration due to gravity**. The acceleration due to gravity is *constant*, which means we can apply the kinematics equations to any falling object where air resistance and friction are negligible. This opens a broad class of interesting situations to us. The acceleration due to gravity is so important that its magnitude is given its own symbol, g . It is constant at any given location on Earth and has the average value

Equation:

$$g = 9.80 \text{ m/s}^2.$$

Although g varies from 9.78 m/s^2 to 9.83 m/s^2 , depending on latitude, altitude, underlying geological formations, and local topography, the average value of 9.80 m/s^2 will be used in this text unless otherwise specified. The direction of the acceleration due to gravity is *downward (towards the center of Earth)*. In fact, its direction *defines* what we call vertical. Note that whether the acceleration a in the kinematic equations has the value $+g$ or $-g$ depends on how we define our coordinate system. If we define the upward direction as positive, then $a = -g = -9.80 \text{ m/s}^2$, and if we define the downward direction as positive, then $a = g = 9.80 \text{ m/s}^2$.

One-Dimensional Motion Involving Gravity

The best way to see the basic features of motion involving gravity is to start with the simplest situations and then progress toward more complex ones. So we start by considering straight up and down motion with no air resistance or friction. These assumptions mean that the velocity (if there is any) is vertical. If the object is dropped, we know the initial velocity is zero. Once the object has left contact with whatever held or threw it, the object is in free-fall. Under these circumstances, the motion is one-dimensional and has constant acceleration of magnitude g . We will also represent vertical displacement with the symbol y and use x for horizontal displacement.

Note:

Kinematic Equations for Objects in Free-Fall where Acceleration = $-g$

Equation:

$$v = v_0 - gt$$

Equation:

$$y = y_0 + v_0t - \frac{1}{2}gt^2$$

Equation:

$$v^2 = v_0^2 - 2g(y - y_0)$$

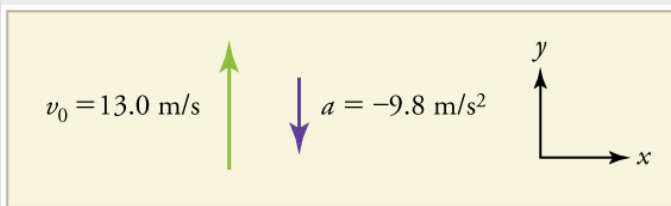
Example:

Calculating Position and Velocity of a Falling Object: A Rock Thrown Upward

A person standing on the edge of a high cliff throws a rock straight up with an initial velocity of 13.0 m/s. The rock misses the edge of the cliff as it falls back to Earth. Calculate the position and velocity of the rock 1.00 s, 2.00 s, and 3.00 s after it is thrown, neglecting the effects of air resistance.

Strategy

Draw a sketch.



We are asked to determine the position y at various times. It is reasonable to take the initial position y_0 to be zero. This problem involves one-dimensional motion in the vertical direction. We use plus and minus signs to indicate direction, with up being positive and down negative. Since up is positive, and the rock is thrown upward, the initial velocity must be positive too. The acceleration due to gravity is downward, so a is negative. It is crucial that the initial velocity and the acceleration due to gravity have opposite signs.

Opposite signs indicate that the acceleration due to gravity opposes the initial motion and will slow and eventually reverse it.

Since we are asked for values of position and velocity at three times, we will refer to these as y_1 and v_1 ; y_2 and v_2 ; and y_3 and v_3 .

Solution for Position y_1

1. Identify the knowns. We know that $y_0 = 0$; $v_0 = 13.0 \text{ m/s}$; $a = -g = -9.80 \text{ m/s}^2$; and $t = 1.00 \text{ s}$.

2. Identify the best equation to use. We will use $y = y_0 + v_0t + \frac{1}{2}at^2$ because it includes only one unknown, y (or y_1 , here), which is the value we want to find.

3. Plug in the known values and solve for y_1 .

Equation:

$$y_1 = 0 + (13.0 \text{ m/s})(1.00 \text{ s}) + \frac{1}{2}(-9.80 \text{ m/s}^2)(1.00 \text{ s})^2 = 8.10 \text{ m}$$

Discussion

The rock is 8.10 m above its starting point at $t = 1.00 \text{ s}$, since $y_1 > y_0$. It could be *moving* up or down; the only way to tell is to calculate v_1 and find out if it is positive or negative.

Solution for Velocity v_1

1. Identify the knowns. We know that $y_0 = 0$; $v_0 = 13.0 \text{ m/s}$;

$a = -g = -9.80 \text{ m/s}^2$; and $t = 1.00 \text{ s}$. We also know from the solution above that $y_1 = 8.10 \text{ m}$.

2. Identify the best equation to use. The most straightforward is $v = v_0 - gt$ (from $v = v_0 + at$, where $a = \text{gravitational acceleration} = -g$).

3. Plug in the knowns and solve.

Equation:

$$v_1 = v_0 - gt = 13.0 \text{ m/s} - (9.80 \text{ m/s}^2)(1.00 \text{ s}) = 3.20 \text{ m/s}$$

Discussion

The positive value for v_1 means that the rock is still heading upward at $t = 1.00 \text{ s}$. However, it has slowed from its original 13.0 m/s, as expected.

Solution for Remaining Times

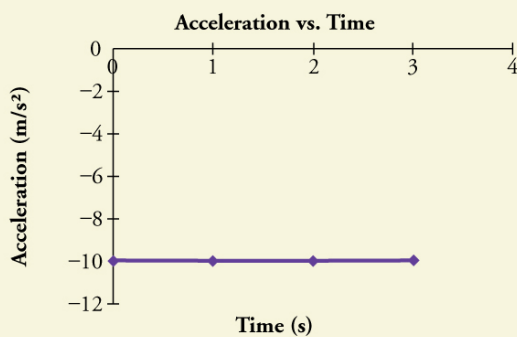
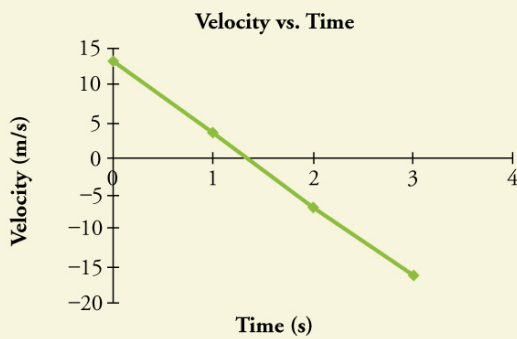
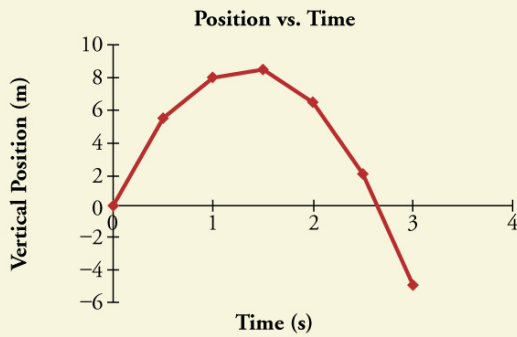
The procedures for calculating the position and velocity at $t = 2.00 \text{ s}$ and 3.00 s are the same as those above. The results are summarized in [\[link\]](#) and

illustrated in [\[link\]](#).

Time, t	Position, y	Velocity, v	Acceleration, a
1.00 s	8.10 m	3.20 m/s	-9.80 m/s^2
2.00 s	6.40 m	-6.60 m/s	-9.80 m/s^2
3.00 s	-5.10 m	-16.4 m/s	-9.80 m/s^2

Results

Graphing the data helps us understand it more clearly.



Vertical position, vertical velocity, and vertical acceleration vs. time for a rock thrown vertically up at the edge of a cliff. Notice that velocity changes linearly with time and that acceleration is constant.

Misconception Alert! Notice that the position vs. time graph shows vertical position only. It is easy to get the impression that the graph shows some

horizontal motion—the shape of the graph looks like the path of a projectile. But this is not the case; the horizontal axis is *time*, not space. The actual path of the rock in space is straight up, and straight down.

Discussion

The interpretation of these results is important. At 1.00 s the rock is above its starting point and heading upward, since y_1 and v_1 are both positive. At 2.00 s, the rock is still above its starting point, but the negative velocity means it is moving downward. At 3.00 s, both y_3 and v_3 are negative, meaning the rock is below its starting point and continuing to move downward. Notice that when the rock is at its highest point (at 1.5 s), its velocity is zero, but its acceleration is still -9.80 m/s^2 . Its acceleration is -9.80 m/s^2 for the whole trip—while it is moving up and while it is moving down. Note that the values for y are the positions (or displacements) of the rock, not the total distances traveled. Finally, note that free-fall applies to upward motion as well as downward. Both have the same acceleration—the acceleration due to gravity, which remains constant the entire time. Astronauts training in the famous Vomit Comet, for example, experience free-fall while arcing up as well as down, as we will discuss in more detail later.

Note:

Making Connections: Take-Home Experiment—Reaction Time

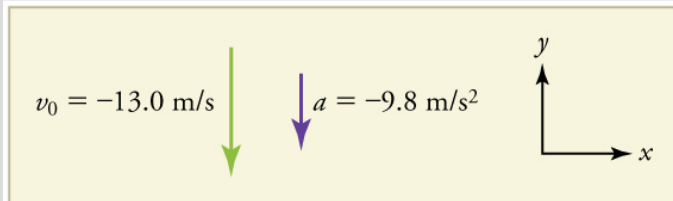
A simple experiment can be done to determine your reaction time. Have a friend hold a ruler between your thumb and index finger, separated by about 1 cm. Note the mark on the ruler that is right between your fingers. Have your friend drop the ruler unexpectedly, and try to catch it between your two fingers. Note the new reading on the ruler. Assuming acceleration is that due to gravity, calculate your reaction time. How far would you travel in a car (moving at 30 m/s) if the time it took your foot to go from the gas pedal to the brake was twice this reaction time?

Example:**Calculating Velocity of a Falling Object: A Rock Thrown Down**

What happens if the person on the cliff throws the rock straight down, instead of straight up? To explore this question, calculate the velocity of the rock when it is 5.10 m below the starting point, and has been thrown downward with an initial speed of 13.0 m/s.

Strategy

Draw a sketch.



Since up is positive, the final position of the rock will be negative because it finishes below the starting point at $y_0 = 0$. Similarly, the initial velocity is downward and therefore negative, as is the acceleration due to gravity. We expect the final velocity to be negative since the rock will continue to move downward.

Solution

1. Identify the knowns. $y_0 = 0$; $y_1 = -5.10$ m; $v_0 = -13.0$ m/s; $a = -g = -9.80$ m/s².
2. Choose the kinematic equation that makes it easiest to solve the problem. The equation $v^2 = v_0^2 + 2a(y - y_0)$ works well because the only unknown in it is v . (We will plug y_1 in for y .)
3. Enter the known values

Equation:

$$v^2 = (-13.0 \text{ m/s})^2 + 2(-9.80 \text{ m/s}^2)(-5.10 \text{ m} - 0 \text{ m}) = 268.96 \text{ m}^2/\text{s}^2,$$

where we have retained extra significant figures because this is an intermediate result.

Taking the square root, and noting that a square root can be positive or negative, gives

Equation:

$$v = \pm 16.4 \text{ m/s}.$$

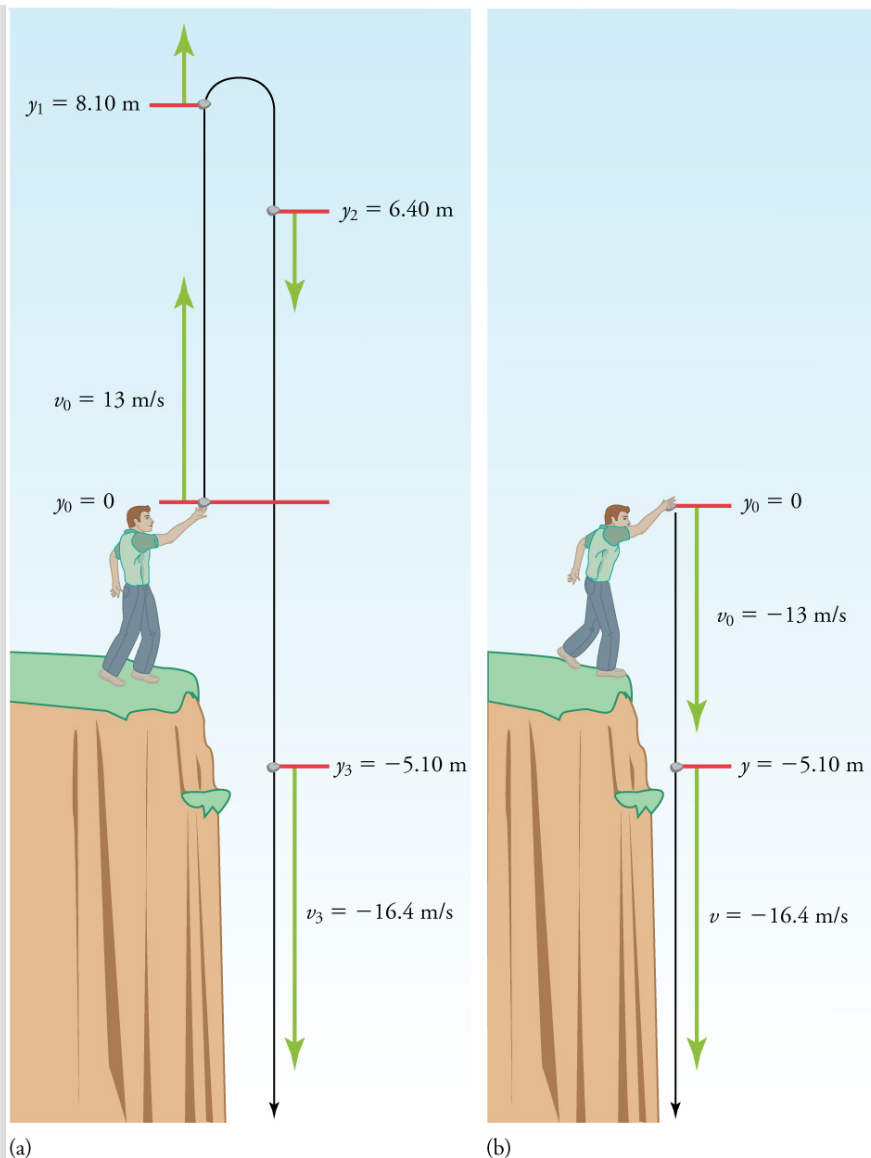
The negative root is chosen to indicate that the rock is still heading down. Thus,

Equation:

$$v = -16.4 \text{ m/s.}$$

Discussion

Note that *this is exactly the same velocity the rock had at this position when it was thrown straight upward with the same initial speed.* (See [\[link\]](#) and [\[link\]](#)(a).) This is not a coincidental result. Because we only consider the acceleration due to gravity in this problem, the *speed* of a falling object depends only on its initial speed and its vertical position relative to the starting point. For example, if the velocity of the rock is calculated at a height of 8.10 m above the starting point (using the method from [\[link\]](#)) when the initial velocity is 13.0 m/s straight up, a result of $\pm 3.20 \text{ m/s}$ is obtained. Here both signs are meaningful; the positive value occurs when the rock is at 8.10 m and heading up, and the negative value occurs when the rock is at 8.10 m and heading back down. It has the same *speed* but the opposite direction.



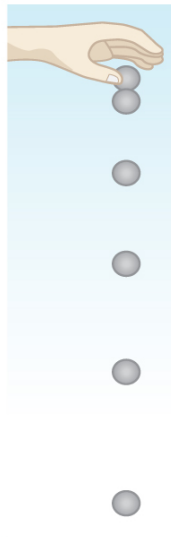
(a) A person throws a rock straight up, as explored in [\[link\]](#). The arrows are velocity vectors at 0, 1.00, 2.00, and 3.00 s. (b) A person throws a rock straight down from a cliff with the same initial speed as before, as in [\[link\]](#). Note that at the same distance below the point of release, the rock has the same velocity in both cases.

Another way to look at it is this: In [\[link\]](#), the rock is thrown up with an initial velocity of 13.0 m/s . It rises and then falls back down. When its

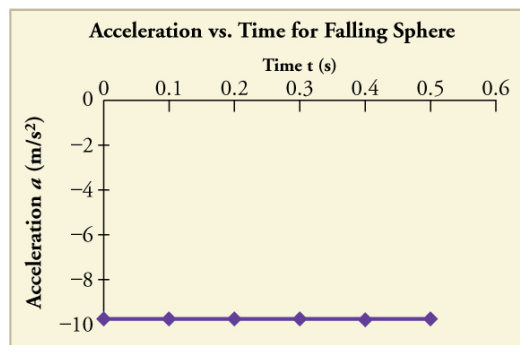
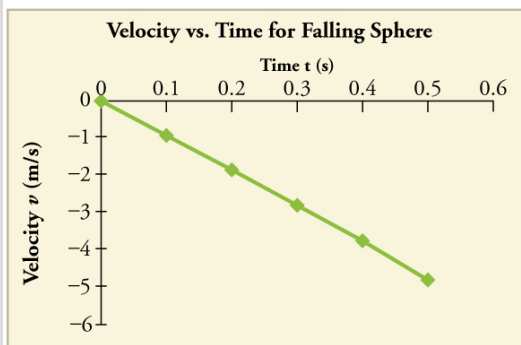
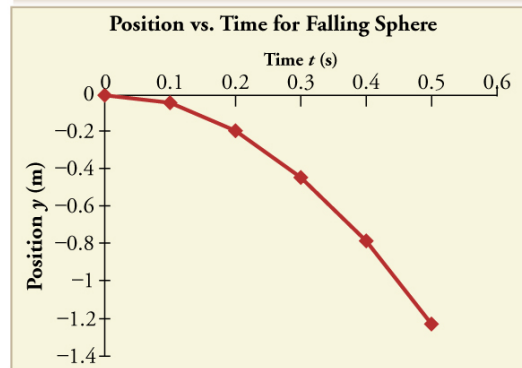
position is $y = 0$ on its way back down, its velocity is -13.0 m/s . That is, it has the same speed on its way down as on its way up. We would then expect its velocity at a position of $y = -5.10 \text{ m}$ to be the same whether we have thrown it upwards at $+13.0 \text{ m/s}$ or thrown it downwards at -13.0 m/s . The velocity of the rock on its way down from $y = 0$ is the same whether we have thrown it up or down to start with, as long as the speed with which it was initially thrown is the same.

Example:**Find g from Data on a Falling Object**

The acceleration due to gravity on Earth differs slightly from place to place, depending on topography (e.g., whether you are on a hill or in a valley) and subsurface geology (whether there is dense rock like iron ore as opposed to light rock like salt beneath you.) The precise acceleration due to gravity can be calculated from data taken in an introductory physics laboratory course. An object, usually a metal ball for which air resistance is negligible, is dropped and the time it takes to fall a known distance is measured. See, for example, [\[link\]](#). Very precise results can be produced with this method if sufficient care is taken in measuring the distance fallen and the elapsed time.



y (m)	v (m/s)	t (s)
0	0	0
-0.049	-0.98	0.1
-0.196	-1.96	0.2
-0.441	-2.94	0.3
-0.784	-3.92	0.4
-1.225	-4.90	0.5



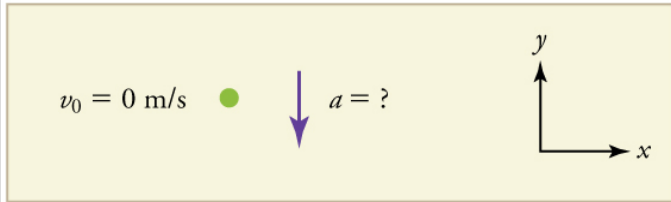
Positions and velocities of a metal ball released from rest when air resistance is negligible. Velocity is seen to increase linearly with time while displacement increases with time squared.

Acceleration is a constant and is equal to gravitational acceleration.

Suppose the ball falls 1.0000 m in 0.45173 s. Assuming the ball is not affected by air resistance, what is the precise acceleration due to gravity at this location?

Strategy

Draw a sketch.



We need to solve for acceleration a . Note that in this case, displacement is downward and therefore negative, as is acceleration.

Solution

1. Identify the knowns. $y_0 = 0$; $y = -1.0000$ m; $t = 0.45173$; $v_0 = 0$.
2. Choose the equation that allows you to solve for a using the known values.

Equation:

$$y = y_0 + v_0 t + \frac{1}{2} a t^2$$

3. Substitute 0 for v_0 and rearrange the equation to solve for a . Substituting 0 for v_0 yields

Equation:

$$y = y_0 + \frac{1}{2} a t^2.$$

Solving for a gives

Equation:

$$a = \frac{2(y - y_0)}{t^2}.$$

4. Substitute known values yields

Equation:

$$a = \frac{2(-1.0000 \text{ m} - 0)}{(0.45173 \text{ s})^2} = -9.8010 \text{ m/s}^2,$$

so, because $a = -g$ with the directions we have chosen,

Equation:

$$g = 9.8010 \text{ m/s}^2.$$

Discussion

The negative value for a indicates that the gravitational acceleration is downward, as expected. We expect the value to be somewhere around the average value of 9.80 m/s^2 , so 9.8010 m/s^2 makes sense. Since the data going into the calculation are relatively precise, this value for g is more precise than the average value of 9.80 m/s^2 ; it represents the local value for the acceleration due to gravity.

Note:

Applying the Science Practices: Finding Acceleration Due to Gravity

While it is well established that the acceleration due to gravity is quite nearly 9.8 m/s^2 at all locations on Earth, you can verify this for yourself with some basic materials.

Your task is to find the acceleration due to gravity at your location.

Achieving an acceleration of precisely 9.8 m/s^2 will be difficult. However, with good preparation and attention to detail, you should be able to get close. Before you begin working, consider the following questions.

What measurements will you need to take in order to find the acceleration due to gravity?

What relationships and equations found in this chapter may be useful in calculating the acceleration?

What variables will you need to hold constant?

What materials will you use to record your measurements?

Upon completing these four questions, record your procedure. Once recorded, you may carry out the experiment. If you find that your experiment cannot be carried out, you may revise your procedure.

Once you have found your experimental acceleration, compare it to the assumed value of 9.8 m/s^2 . If error exists, what were the likely sources of this error? How could you change your procedure in order to improve the accuracy of your findings?

Exercise:**Check Your Understanding****Problem:**

A chunk of ice breaks off a glacier and falls 30.0 meters before it hits the water. Assuming it falls freely (there is no air resistance), how long does it take to hit the water?

Solution:

We know that initial position $y_0 = 0$, final position $y = -30.0$ m, and $a = -g = -9.80$ m/s². We can then use the equation $y = y_0 + v_0t + \frac{1}{2}at^2$ to solve for t . Inserting $a = -g$, we obtain

Equation:

$$y = 0 + 0 - \frac{1}{2}gt^2$$

$$t^2 = \frac{2y}{-g}$$

$$t = \pm \sqrt{\frac{2y}{-g}} = \pm \sqrt{\frac{2(-30.0 \text{ m})}{-9.80 \text{ m/s}^2}} = \pm \sqrt{6.12 \text{ s}^2} = 2.47 \text{ s} \approx 2.5 \text{ s}$$

where we take the positive value as the physically relevant answer. Thus, it takes about 2.5 seconds for the piece of ice to hit the water.

Note:

PhET Explorations: Equation Grapher

Learn about graphing polynomials. The shape of the curve changes as the constants are adjusted. View the curves for the individual terms (e.g. $y = bx$) to see how they add to generate the polynomial curve.

[Equatio
n
Grapher](#)

Section Summary

- An object in free-fall experiences constant acceleration if air resistance is negligible.
- On Earth, all free-falling objects have an acceleration due to gravity g , which averages

Equation:

$$g = 9.80 \text{ m/s}^2.$$

- Whether the acceleration a should be taken as $+g$ or $-g$ is determined by your choice of coordinate system. If you choose the upward direction as positive, $a = -g = -9.80 \text{ m/s}^2$ is negative. In the opposite case, $a = +g = 9.80 \text{ m/s}^2$ is positive. Since acceleration is constant, the kinematic equations above can be applied with the appropriate $+g$ or $-g$ substituted for a .
- For objects in free-fall, up is normally taken as positive for displacement, velocity, and acceleration.

Conceptual Questions

Exercise:

Problem:

What is the acceleration of a rock thrown straight upward on the way up? At the top of its flight? On the way down?

Exercise:

Problem:

An object that is thrown straight up falls back to Earth. This is one-dimensional motion. (a) When is its velocity zero? (b) Does its velocity change direction? (c) Does the acceleration due to gravity have the same sign on the way up as on the way down?

Exercise:**Problem:**

Suppose you throw a rock nearly straight up at a coconut in a palm tree, and the rock misses on the way up but hits the coconut on the way down. Neglecting air resistance, how does the speed of the rock when it hits the coconut on the way down compare with what it would have been if it had hit the coconut on the way up? Is it more likely to dislodge the coconut on the way up or down? Explain.

Exercise:**Problem:**

If an object is thrown straight up and air resistance is negligible, then its speed when it returns to the starting point is the same as when it was released. If air resistance were not negligible, how would its speed upon return compare with its initial speed? How would the maximum height to which it rises be affected?

Exercise:**Problem:**

The severity of a fall depends on your speed when you strike the ground. All factors but the acceleration due to gravity being the same, how many times higher could a safe fall on the Moon be than on Earth (gravitational acceleration on the Moon is about $1/6$ that of the Earth)?

Exercise:

Problem:

How many times higher could an astronaut jump on the Moon than on Earth if his takeoff speed is the same in both locations (gravitational acceleration on the Moon is about $1/6$ of g on Earth)?

Problems & Exercises

Assume air resistance is negligible unless otherwise stated.

Exercise:**Problem:**

Calculate the displacement and velocity at times of (a) 0.500, (b) 1.00, (c) 1.50, and (d) 2.00 s for a ball thrown straight up with an initial velocity of 15.0 m/s. Take the point of release to be $y_0 = 0$.

Solution:

(a) $y_1 = 6.28 \text{ m}$; $v_1 = 10.1 \text{ m/s}$

(b) $y_2 = 10.1 \text{ m}$; $v_2 = 5.20 \text{ m/s}$

(c) $y_3 = 11.5 \text{ m}$; $v_3 = 0.300 \text{ m/s}$

(d) $y_4 = 10.4 \text{ m}$; $v_4 = -4.60 \text{ m/s}$

Exercise:**Problem:**

Calculate the displacement and velocity at times of (a) 0.500, (b) 1.00, (c) 1.50, (d) 2.00, and (e) 2.50 s for a rock thrown straight down with an initial velocity of 14.0 m/s from the Verrazano Narrows Bridge in New York City. The roadway of this bridge is 70.0 m above the water.

Exercise:

Problem:

A basketball referee tosses the ball straight up for the starting tip-off. At what velocity must a basketball player leave the ground to rise 1.25 m above the floor in an attempt to get the ball?

Solution:

$$v_0 = 4.95 \text{ m/s}$$

Exercise:**Problem:**

A rescue helicopter is hovering over a person whose boat has sunk. One of the rescuers throws a life preserver straight down to the victim with an initial velocity of 1.40 m/s and observes that it takes 1.8 s to reach the water. (a) List the knowns in this problem. (b) How high above the water was the preserver released? Note that the downdraft of the helicopter reduces the effects of air resistance on the falling life preserver, so that an acceleration equal to that of gravity is reasonable.

Exercise:**Problem:**

A dolphin in an aquatic show jumps straight up out of the water at a velocity of 13.0 m/s. (a) List the knowns in this problem. (b) How high does his body rise above the water? To solve this part, first note that the final velocity is now a known and identify its value. Then identify the unknown, and discuss how you chose the appropriate equation to solve for it. After choosing the equation, show your steps in solving for the unknown, checking units, and discuss whether the answer is reasonable. (c) How long is the dolphin in the air? Neglect any effects due to his size or orientation.

Solution:

$$(a) a = -9.80 \text{ m/s}^2; v_0 = 13.0 \text{ m/s}; y_0 = 0 \text{ m}$$

(b) $v = 0\text{ m/s}$. Unknown is distance y to top of trajectory, where velocity is zero. Use equation $v^2 = v_0^2 + 2a(y - y_0)$ because it contains all known values except for y , so we can solve for y . Solving for y gives

Equation:

$$v^2 - v_0^2 = 2a(y - y_0)$$

$$\frac{v^2 - v_0^2}{2a} = y - y_0$$

$$y = y_0 + \frac{v^2 - v_0^2}{2a} = 0\text{ m} + \frac{(0\text{ m/s})^2 - (13.0\text{ m/s})^2}{2(-9.80\text{ m/s}^2)} = 8.62\text{ m}$$

Dolphins measure about 2 meters long and can jump several times their length out of the water, so this is a reasonable result.

(c) 2.65 s

Exercise:

Problem:

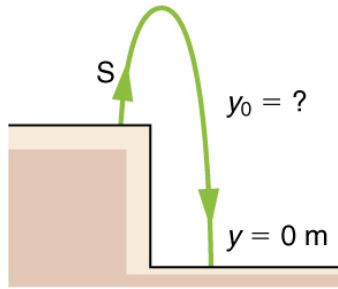
A swimmer bounces straight up from a diving board and falls feet first into a pool. She starts with a velocity of 4.00 m/s, and her takeoff point is 1.80 m above the pool. (a) How long are her feet in the air? (b) What is her highest point above the board? (c) What is her velocity when her feet hit the water?

Exercise:

Problem:

(a) Calculate the height of a cliff if it takes 2.35 s for a rock to hit the ground when it is thrown straight up from the cliff with an initial velocity of 8.00 m/s. (b) How long would it take to reach the ground if it is thrown straight down with the same speed?

Solution:



(a) 8.26 m

(b) 0.717 s

Exercise:

Problem:

A very strong, but inept, shot putter puts the shot straight up vertically with an initial velocity of 11.0 m/s. How long does he have to get out of the way if the shot was released at a height of 2.20 m, and he is 1.80 m tall?

Exercise:

Problem:

You throw a ball straight up with an initial velocity of 15.0 m/s. It passes a tree branch on the way up at a height of 7.00 m. How much additional time will pass before the ball passes the tree branch on the way back down?

Solution:

1.91 s

Exercise:

Problem:

A kangaroo can jump over an object 2.50 m high. (a) Calculate its vertical speed when it leaves the ground. (b) How long is it in the air?

Exercise:

Problem:

Standing at the base of one of the cliffs of Mt. Arapiles in Victoria, Australia, a hiker hears a rock break loose from a height of 105 m. He can't see the rock right away but then does, 1.50 s later. (a) How far above the hiker is the rock when he can see it? (b) How much time does he have to move before the rock hits his head?

Solution:

(a) 94.0 m

(b) 3.13 s

Exercise:**Problem:**

An object is dropped from a height of 75.0 m above ground level. (a) Determine the distance traveled during the first second. (b) Determine the final velocity at which the object hits the ground. (c) Determine the distance traveled during the last second of motion before hitting the ground.

Exercise:**Problem:**

There is a 250-m-high cliff at Half Dome in Yosemite National Park in California. Suppose a boulder breaks loose from the top of this cliff. (a) How fast will it be going when it strikes the ground? (b) Assuming a reaction time of 0.300 s, how long will a tourist at the bottom have to get out of the way after hearing the sound of the rock breaking loose (neglecting the height of the tourist, which would become negligible anyway if hit)? The speed of sound is 335 m/s on this day.

Solution:

(a) -70.0 m/s (downward)

(b) 6.10 s

Exercise:**Problem:**

A ball is thrown straight up. It passes a 2.00-m-high window 7.50 m off the ground on its path up and takes 1.30 s to go past the window. What was the ball's initial velocity?

Exercise:**Problem:**

Suppose you drop a rock into a dark well and, using precision equipment, you measure the time for the sound of a splash to return. (a) Neglecting the time required for sound to travel up the well, calculate the distance to the water if the sound returns in 2.0000 s. (b) Now calculate the distance taking into account the time for sound to travel up the well. The speed of sound is 332.00 m/s in this well.

Solution:

(a) 19.6 m

(b) 18.5 m

Exercise:**Problem:**

A steel ball is dropped onto a hard floor from a height of 1.50 m and rebounds to a height of 1.45 m. (a) Calculate its velocity just before it strikes the floor. (b) Calculate its velocity just after it leaves the floor on its way back up. (c) Calculate its acceleration during contact with the floor if that contact lasts 0.0800 ms (8.00×10^{-5} s). (d) How much did the ball compress during its collision with the floor, assuming the floor is absolutely rigid?

Exercise:

Problem:

A coin is dropped from a hot-air balloon that is 300 m above the ground and rising at 10.0 m/s upward. For the coin, find (a) the maximum height reached, (b) its position and velocity 4.00 s after being released, and (c) the time before it hits the ground.

Solution:

(a) 305 m

(b) 262 m, -29.2 m/s

(c) 8.91 s

Exercise:**Problem:**

A soft tennis ball is dropped onto a hard floor from a height of 1.50 m and rebounds to a height of 1.10 m. (a) Calculate its velocity just before it strikes the floor. (b) Calculate its velocity just after it leaves the floor on its way back up. (c) Calculate its acceleration during contact with the floor if that contact lasts 3.50 ms (3.50×10^{-3} s). (d) How much did the ball compress during its collision with the floor, assuming the floor is absolutely rigid?

Test Prep for AP Courses**Exercise:****Problem:**

Observing a spacecraft land on a distant asteroid, scientists notice that the craft is falling at a rate of 5 m/s. When it is 100 m closer to the surface of the asteroid, the craft reports a velocity of 8 m/s. According to their data, what is the approximate gravitational acceleration on this asteroid?

- a. 0 m/s^2
- b. 0.03 m/s^2
- c. 0.20 m/s^2
- d. 0.65 m/s^2
- e. 33 m/s^2

Solution:

(c)

Glossary

free-fall

the state of movement that results from gravitational force only

acceleration due to gravity

acceleration of an object as a result of gravity

Graphical Analysis of One Dimensional Motion

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Describe a straight-line graph in terms of its slope and y-intercept.
- Determine average velocity or instantaneous velocity from a graph of position vs. time.
- Determine average or instantaneous acceleration from a graph of velocity vs. time.
- Derive a graph of velocity vs. time from a graph of position vs. time.
- Derive a graph of acceleration vs. time from a graph of velocity vs. time.

A graph, like a picture, is worth a thousand words. Graphs not only contain numerical information; they also reveal relationships between physical quantities. This section uses graphs of position, velocity, and acceleration versus time to illustrate one-dimensional kinematics.

Slopes and General Relationships

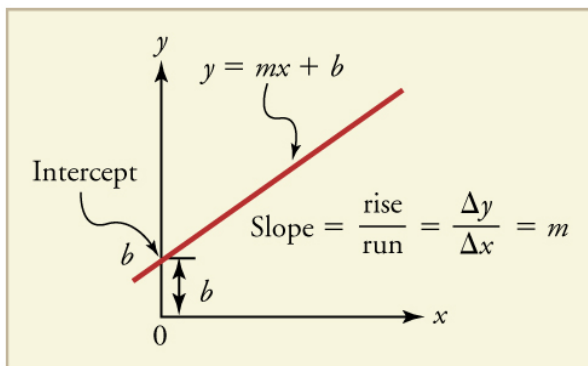
First note that graphs in this text have perpendicular axes, one horizontal and the other vertical. When two physical quantities are plotted against one another in such a graph, the horizontal axis is usually considered to be an **independent variable** and the vertical axis a **dependent variable**. If we

call the horizontal axis the x -axis and the vertical axis the y -axis, as in [\[link\]](#), a straight-line graph has the general form

Equation:

$$y = mx + b.$$

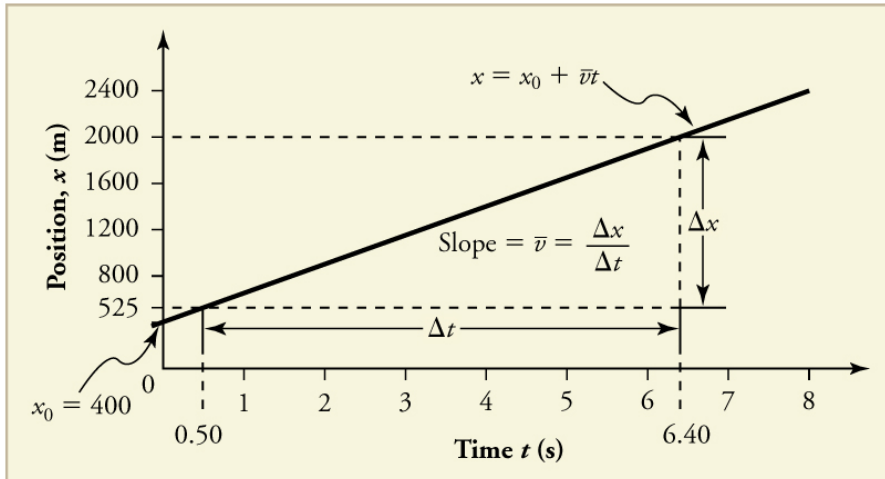
Here m is the **slope**, defined to be the rise divided by the run (as seen in the figure) of the straight line. The letter b is used for the **y-intercept**, which is the point at which the line crosses the vertical axis.



A straight-line graph. The equation for a straight line is $y = mx + b$.

Graph of Position vs. Time ($a = 0$, so v is constant)

Time is usually an independent variable that other quantities, such as position, depend upon. A graph of position versus time would, thus, have x on the vertical axis and t on the horizontal axis. [\[link\]](#) is just such a straight-line graph. It shows a graph of position versus time for a jet-powered car on a very flat dry lake bed in Nevada.



Graph of position versus time for a jet-powered car on the Bonneville Salt Flats.

Using the relationship between dependent and independent variables, we see that the slope in the graph above is average velocity \bar{v} and the intercept is position at time zero—that is, x_0 . Substituting these symbols into $y = mx + b$ gives

Equation:

$$x = \bar{v}t + x_0$$

or

Equation:

$$x = x_0 + \bar{v}t.$$

Thus a graph of position versus time gives a general relationship among position, velocity, and time, as well as giving detailed numerical information about a specific situation.

Note:

The Slope of x vs. t

The slope of the graph of position x vs. time t is velocity v .

Equation:

$$\text{slope} = \frac{\Delta x}{\Delta t} = v$$

Notice that this equation is the same as that derived algebraically from other motion equations in [Motion Equations for Constant Acceleration in One Dimension](#).

From the figure we can see that the car has a position of 400 m at time 0.650 s at $t = 1.0$ s, and so on. Its position at times other than those listed in the table can be read from the graph; furthermore, information about its velocity and acceleration can also be obtained from the graph.

Example:**Determining Average Velocity from a Graph of Position versus Time: Jet Car**

Find the average velocity of the car whose position is graphed in [\[link\]](#).

Strategy

The slope of a graph of x vs. t is average velocity, since slope equals rise over run. In this case, rise = change in position and run = change in time, so that

Equation:

$$\text{slope} = \frac{\Delta x}{\Delta t} = \bar{v}.$$

Since the slope is constant here, any two points on the graph can be used to find the slope. (Generally speaking, it is most accurate to use two widely separated points on the straight line. This is because any error in reading data from the graph is proportionally smaller if the interval is larger.)

Solution

1. Choose two points on the line. In this case, we choose the points labeled on the graph: (6.4 s, 2000 m) and (0.50 s, 525 m). (Note, however, that you could choose any two points.)
2. Substitute the x and t values of the chosen points into the equation. Remember in calculating change (Δ) we always use final value minus initial value.

Equation:

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{2000 \text{ m} - 525 \text{ m}}{6.4 \text{ s} - 0.50 \text{ s}},$$

yielding

Equation:

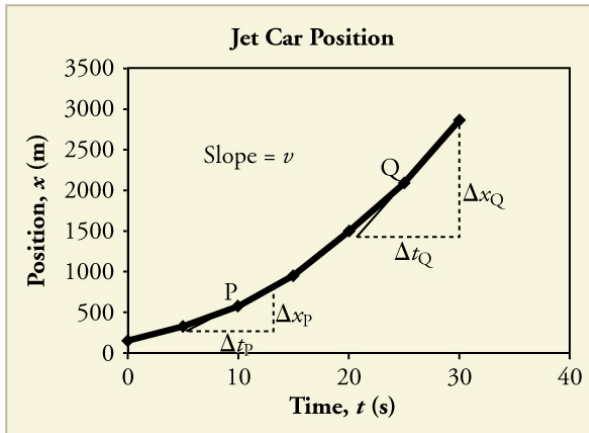
$$\bar{v} = 250 \text{ m/s}.$$

Discussion

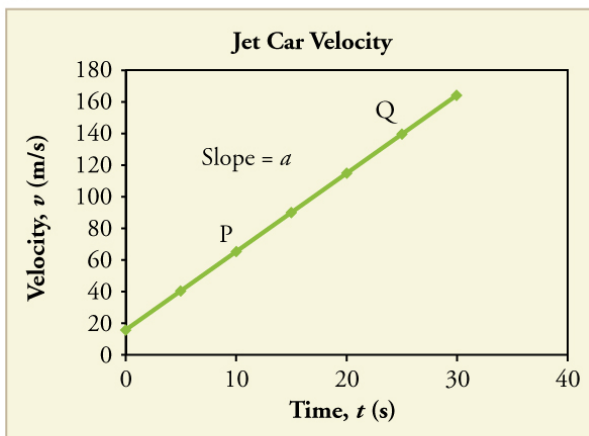
This is an impressively large land speed (900 km/h, or about 560 mi/h): much greater than the typical highway speed limit of 60 mi/h (27 m/s or 96 km/h), but considerably shy of the record of 343 m/s (1234 km/h or 766 mi/h) set in 1997.

Graphs of Motion when a is constant but $a \neq 0$

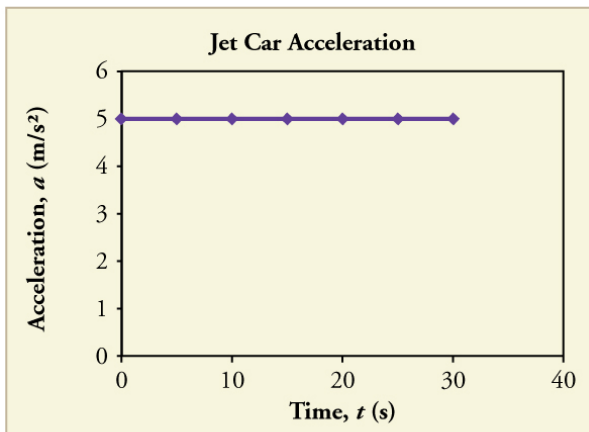
The graphs in [\[link\]](#) below represent the motion of the jet-powered car as it accelerates toward its top speed, but only during the time when its acceleration is constant. Time starts at zero for this motion (as if measured with a stopwatch), and the position and velocity are initially 200 m and 15 m/s, respectively.



(a)



(b)



(c)

Graphs of motion of a jet-powered car during the time span when its acceleration is constant. (a) The slope of an x vs. t graph is velocity. This is

shown at two points, and the instantaneous velocities obtained are plotted in the next graph. Instantaneous velocity at any point is the slope of the tangent at that point. (b) The slope of the v vs. t graph is constant for this part of the motion, indicating constant acceleration. (c) Acceleration has the constant value of 5.0 m/s^2 over the time interval plotted.



A U.S. Air Force jet car speeds down a track. (credit: Matt Trostle, Flickr)

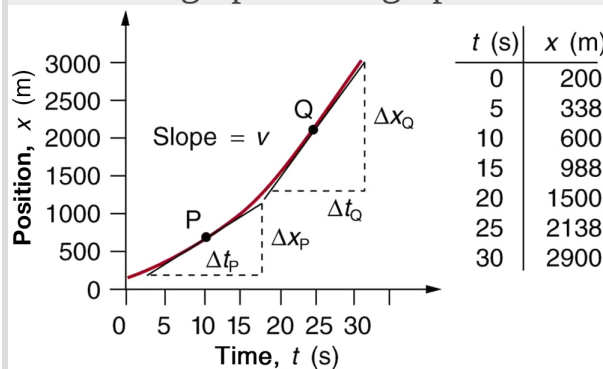
The graph of position versus time in [\[link\]](#)(a) is a curve rather than a straight line. The slope of the curve becomes steeper as time progresses,

showing that the velocity is increasing over time. The slope at any point on a position-versus-time graph is the instantaneous velocity at that point. It is found by drawing a straight line tangent to the curve at the point of interest and taking the slope of this straight line. Tangent lines are shown for two points in [\[link\]\(a\)](#). If this is done at every point on the curve and the values are plotted against time, then the graph of velocity versus time shown in [\[link\]\(b\)](#) is obtained. Furthermore, the slope of the graph of velocity versus time is acceleration, which is shown in [\[link\]\(c\)](#).

Example:

Determining Instantaneous Velocity from the Slope at a Point: Jet Car

Calculate the velocity of the jet car at a time of 25 s by finding the slope of the x vs. t graph in the graph below.



The slope of an x vs. t graph is velocity. This is shown at two points. Instantaneous velocity at any point is the slope of the tangent at that point.

Strategy

The slope of a curve at a point is equal to the slope of a straight line tangent to the curve at that point. This principle is illustrated in [\[link\]](#), where Q is the point at $t = 25$ s.

Solution

1. Find the tangent line to the curve at $t = 25$ s.

2. Determine the endpoints of the tangent. These correspond to a position of 1300 m at time 19 s and a position of 3120 m at time 32 s.
3. Plug these endpoints into the equation to solve for the slope, v .

Equation:

$$\text{slope} = v_Q = \frac{\Delta x_Q}{\Delta t_Q} = \frac{(3120 \text{ m} - 1300 \text{ m})}{(32 \text{ s} - 19 \text{ s})}$$

Thus,

Equation:

$$v_Q = \frac{1820 \text{ m}}{13 \text{ s}} = 140 \text{ m/s}.$$

Discussion

This is the value given in this figure's table for v at $t = 25 \text{ s}$. The value of 140 m/s for v_Q is plotted in [\[link\]](#). The entire graph of v vs. t can be obtained in this fashion.

Carrying this one step further, we note that the slope of a velocity versus time graph is acceleration. Slope is rise divided by run; on a v vs. t graph, rise = change in velocity Δv and run = change in time Δt .

Note:

The Slope of v vs. t

The slope of a graph of velocity v vs. time t is acceleration a .

Equation:

$$\text{slope} = \frac{\Delta v}{\Delta t} = a$$

Since the velocity versus time graph in [\[link\]](#)(b) is a straight line, its slope is the same everywhere, implying that acceleration is constant. Acceleration versus time is graphed in [\[link\]](#)(c).

Additional general information can be obtained from [\[link\]](#) and the expression for a straight line, $y = mx + b$.

In this case, the vertical axis y is V , the intercept b is v_0 , the slope m is a , and the horizontal axis x is t . Substituting these symbols yields

Equation:

$$v = v_0 + at.$$

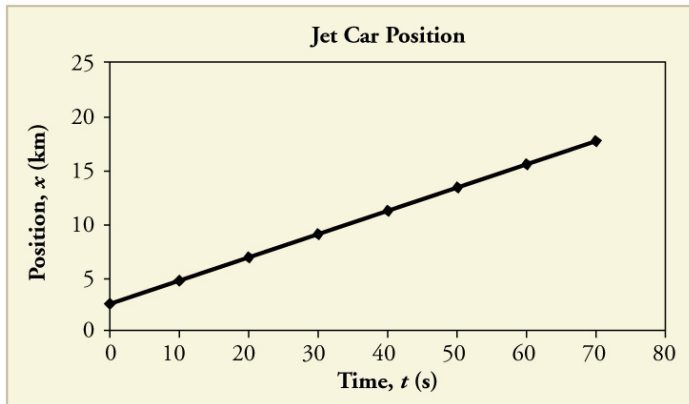
A general relationship for velocity, acceleration, and time has again been obtained from a graph. Notice that this equation was also derived algebraically from other motion equations in [Motion Equations for Constant Acceleration in One Dimension](#).

It is not accidental that the same equations are obtained by graphical analysis as by algebraic techniques. In fact, an important way to *discover* physical relationships is to measure various physical quantities and then make graphs of one quantity against another to see if they are correlated in any way. Correlations imply physical relationships and might be shown by smooth graphs such as those above. From such graphs, mathematical relationships can sometimes be postulated. Further experiments are then performed to determine the validity of the hypothesized relationships.

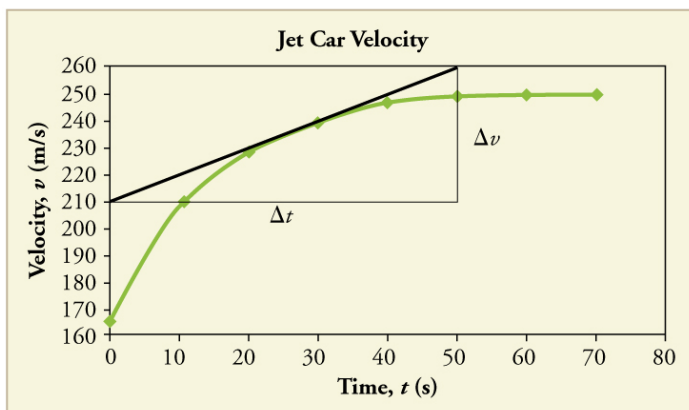
Graphs of Motion Where Acceleration is Not Constant

Now consider the motion of the jet car as it goes from 165 m/s to its top velocity of 250 m/s, graphed in [\[link\]](#). Time again starts at zero, and the initial position and velocity are 2900 m and 165 m/s, respectively. (These were the final position and velocity of the car in the motion graphed in [\[link\]](#).) Acceleration gradually decreases from 5.0 m/s^2 to zero when the car hits 250 m/s. The slope of the x vs. t graph increases until $t = 55 \text{ s}$, after which time the slope is constant. Similarly, velocity increases until 55

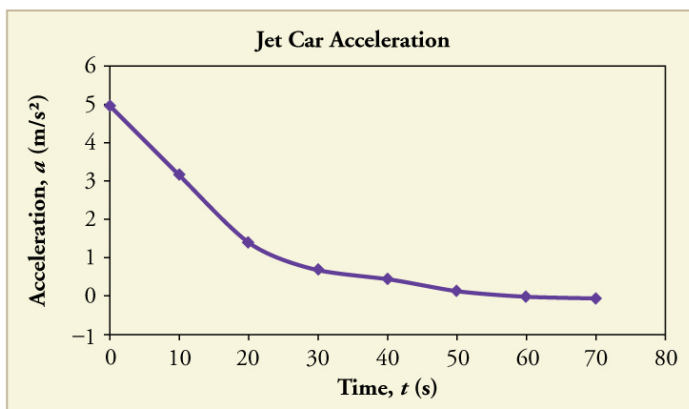
s and then becomes constant, since acceleration decreases to zero at 55 s and remains zero afterward.



(a)



(b)



(c)

Graphs of motion of a jet-powered car as it reaches its top velocity. This motion begins where the motion in

[\[link\]](#) ends. (a) The slope of this graph is velocity; it is plotted in the next graph. (b) The velocity gradually approaches its top value. The slope of this graph is acceleration; it is plotted in the final graph. (c) Acceleration gradually declines to zero when velocity becomes constant.

Example:**Calculating Acceleration from a Graph of Velocity versus Time**

Calculate the acceleration of the jet car at a time of 25 s by finding the slope of the v vs. t graph in [\[link\]](#)(b).

Strategy

The slope of the curve at $t = 25$ s is equal to the slope of the line tangent at that point, as illustrated in [\[link\]](#)(b).

Solution

Determine endpoints of the tangent line from the figure, and then plug them into the equation to solve for slope, a .

Equation:

$$\text{slope} = \frac{\Delta v}{\Delta t} = \frac{(260 \text{ m/s} - 210 \text{ m/s})}{(51 \text{ s} - 1.0 \text{ s})}$$

Equation:

$$a = \frac{50 \text{ m/s}}{50 \text{ s}} = 1.0 \text{ m/s}^2.$$

Discussion

Note that this value for a is consistent with the value plotted in [\[link\]](#)(c) at $t = 25$ s.

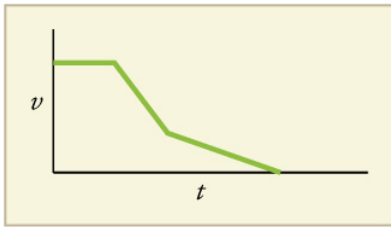
A graph of position versus time can be used to generate a graph of velocity versus time, and a graph of velocity versus time can be used to generate a graph of acceleration versus time. We do this by finding the slope of the graphs at every point. If the graph is linear (i.e., a line with a constant slope), it is easy to find the slope at any point and you have the slope for every point. Graphical analysis of motion can be used to describe both specific and general characteristics of kinematics. Graphs can also be used for other topics in physics. An important aspect of exploring physical relationships is to graph them and look for underlying relationships.

Exercise:

Check Your Understanding

Problem:

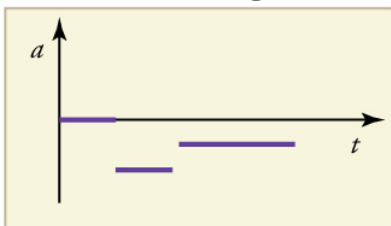
A graph of velocity vs. time of a ship coming into a harbor is shown below. (a) Describe the motion of the ship based on the graph. (b) What would a graph of the ship's acceleration look like?



Solution:

(a) The ship moves at constant velocity and then begins to decelerate at a constant rate. At some point, its deceleration rate decreases. It maintains this lower deceleration rate until it stops moving.

(b) A graph of acceleration vs. time would show zero acceleration in the first leg, large and constant negative acceleration in the second leg, and constant negative acceleration.



Section Summary

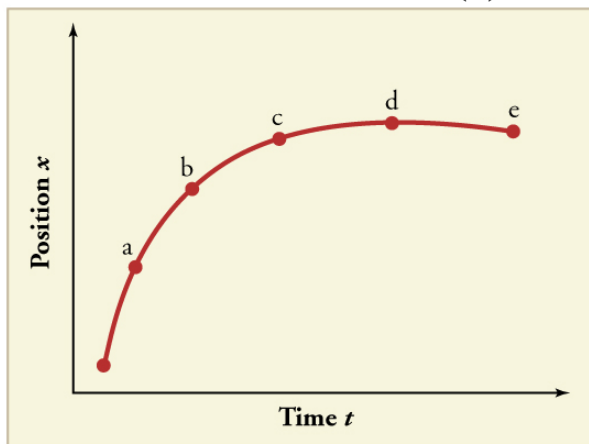
- Graphs of motion can be used to analyze motion.
- Graphical solutions yield identical solutions to mathematical methods for deriving motion equations.
- The slope of a graph of position x vs. time t is velocity v .
- The slope of a graph of velocity v vs. time t graph is acceleration a .
- Average velocity, instantaneous velocity, and acceleration can all be obtained by analyzing graphs.

Conceptual Questions

Exercise:

Problem:

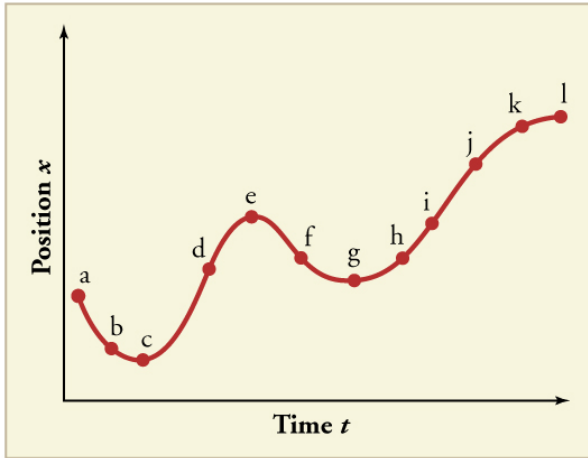
(a) Explain how you can use the graph of position versus time in [\[link\]](#) to describe the change in velocity over time. Identify (b) the time (t_a , t_b , t_c , t_d , or t_e) at which the instantaneous velocity is greatest, (c) the time at which it is zero, and (d) the time at which it is negative.



Exercise:

Problem:

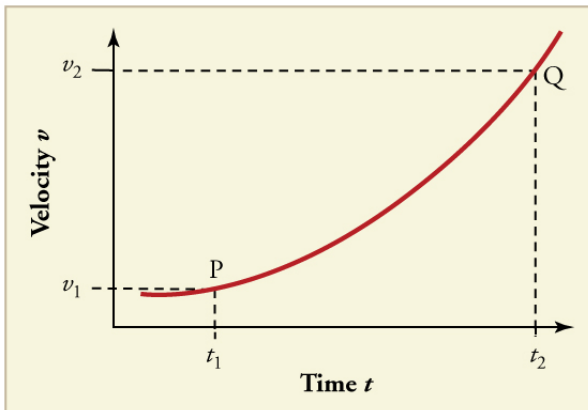
(a) Sketch a graph of velocity versus time corresponding to the graph of position versus time given in [\[link\]](#). (b) Identify the time or times (t_a , t_b , t_c , etc.) at which the instantaneous velocity is greatest. (c) At which times is it zero? (d) At which times is it negative?



Exercise:

Problem:

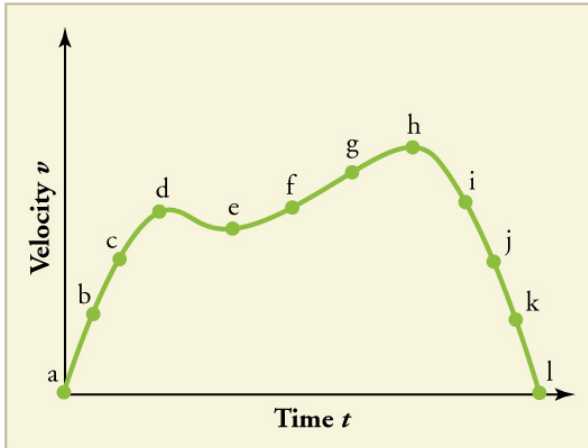
(a) Explain how you can determine the acceleration over time from a velocity versus time graph such as the one in [\[link\]](#). (b) Based on the graph, how does acceleration change over time?



Exercise:

Problem:

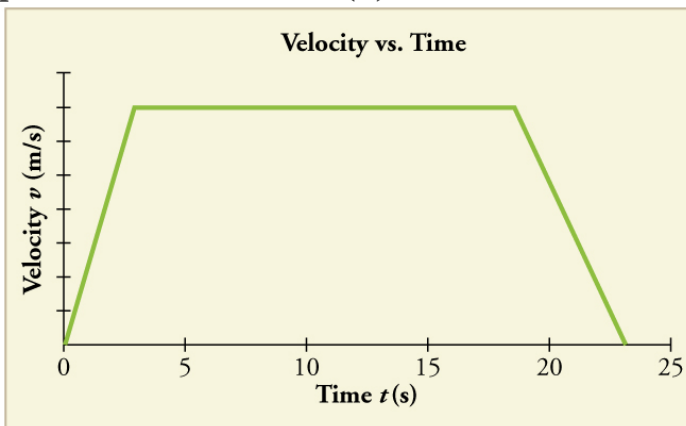
(a) Sketch a graph of acceleration versus time corresponding to the graph of velocity versus time given in [\[link\]](#). (b) Identify the time or times (t_a , t_b , t_c , etc.) at which the acceleration is greatest. (c) At which times is it zero? (d) At which times is it negative?



Exercise:

Problem:

Consider the velocity vs. time graph of a person in an elevator shown in [\[link\]](#). Suppose the elevator is initially at rest. It then accelerates for 3 seconds, maintains that velocity for 15 seconds, then decelerates for 5 seconds until it stops. The acceleration for the entire trip is not constant so we cannot use the equations of motion from [Motion Equations for Constant Acceleration in One Dimension](#) for the complete trip. (We could, however, use them in the three individual sections where acceleration is a constant.) Sketch graphs of (a) position vs. time and (b) acceleration vs. time for this trip.



Exercise:

Problem:

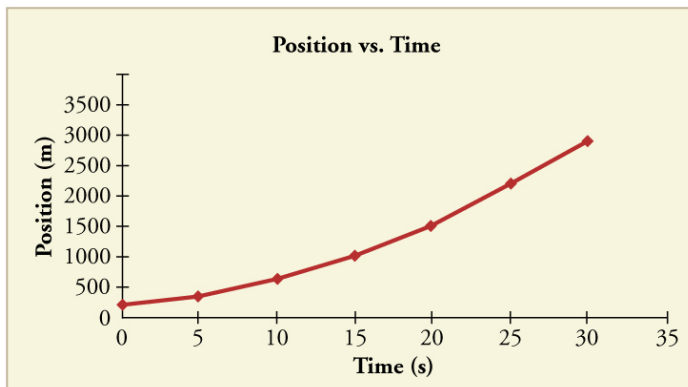
A cylinder is given a push and then rolls up an inclined plane. If the origin is the starting point, sketch the position, velocity, and acceleration of the cylinder vs. time as it goes up and then down the plane.

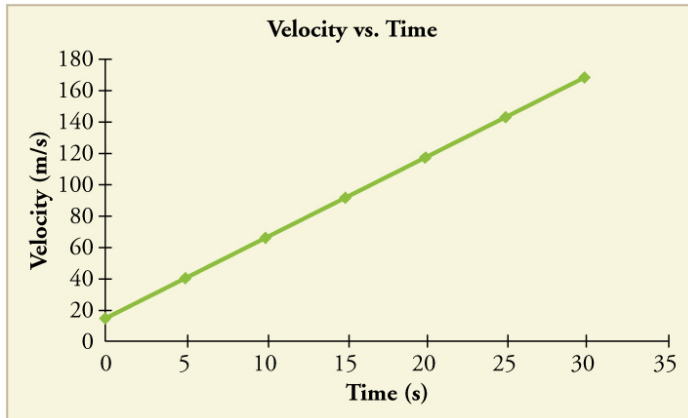
Problems & Exercises

Note: There is always uncertainty in numbers taken from graphs. If your answers differ from expected values, examine them to see if they are within data extraction uncertainties estimated by you.

Exercise:**Problem:**

(a) By taking the slope of the curve in [\[link\]](#), verify that the velocity of the jet car is 115 m/s at $t = 20$ s. (b) By taking the slope of the curve at any point in [\[link\]](#), verify that the jet car's acceleration is 5.0 m/s^2 .





Solution:

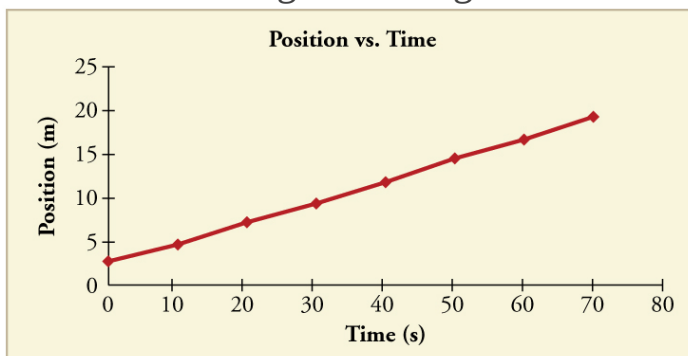
(a) 115 m/s

(b) 5.0 m/s²

Exercise:

Problem:

Using approximate values, calculate the slope of the curve in [\[link\]](#) to verify that the velocity at $t = 10.0$ s is 0.208 m/s. Assume all values are known to 3 significant figures.



Exercise:

Problem:

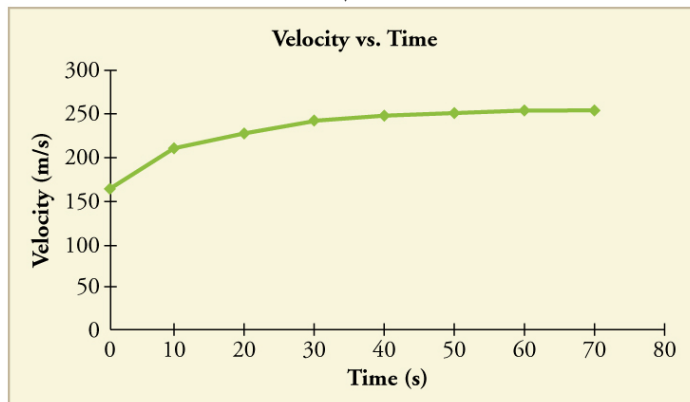
Using approximate values, calculate the slope of the curve in [\[link\]](#) to verify that the velocity at $t = 30.0$ s is 0.238 m/s. Assume all values are known to 3 significant figures.

Solution:

$$v = \frac{(11.7 - 6.95) \times 10^3 \text{ m}}{(40.0 - 20.0) \text{ s}} = 238 \text{ m/s}$$

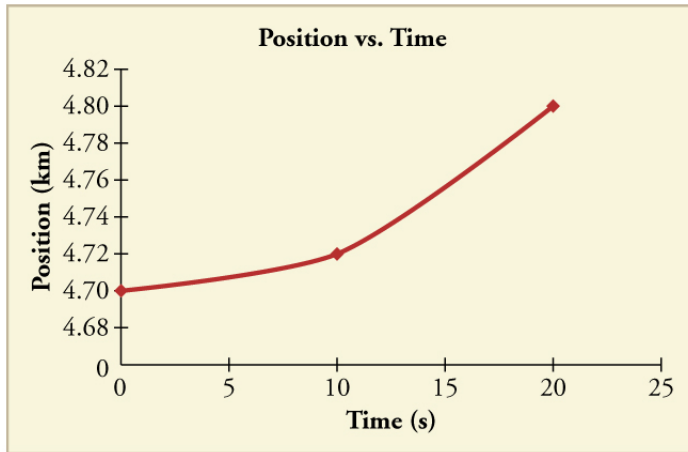
Exercise:**Problem:**

By taking the slope of the curve in [\[link\]](#), verify that the acceleration is approximately 3.2 m/s^2 at $t = 10 \text{ s}$.

**Exercise:****Problem:**

Construct the position graph for the subway shuttle train as shown in [\[link\]](#)(a). Your graph should show the position of the train, in kilometers, from $t = 0$ to 20 s . You will need to use the information on acceleration and velocity given in the examples for this figure.

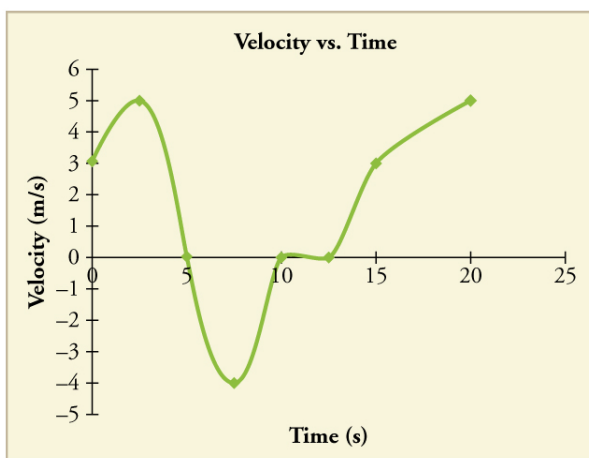
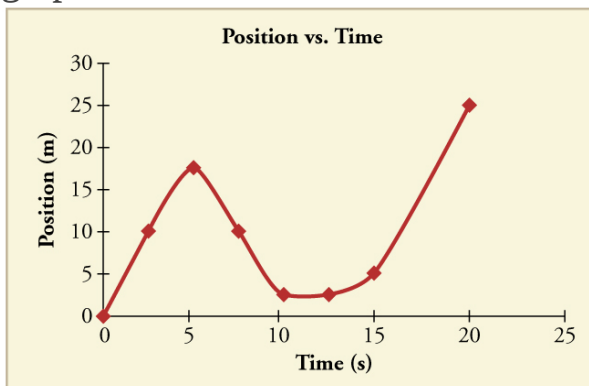
Solution:

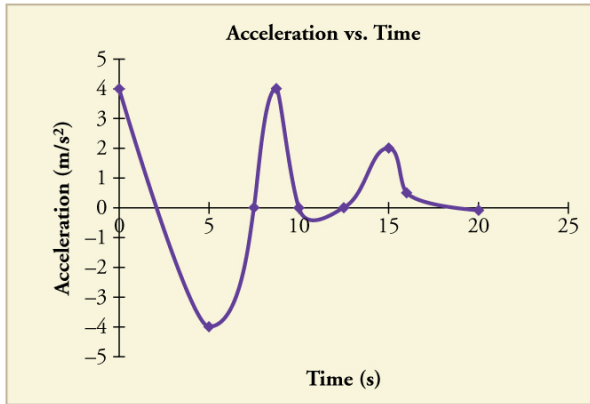


Exercise:

Problem:

(a) Take the slope of the curve in [\[link\]](#) to find the jogger's velocity at $t = 2.5$ s. (b) Repeat at 7.5 s. These values must be consistent with the graph in [\[link\]](#).

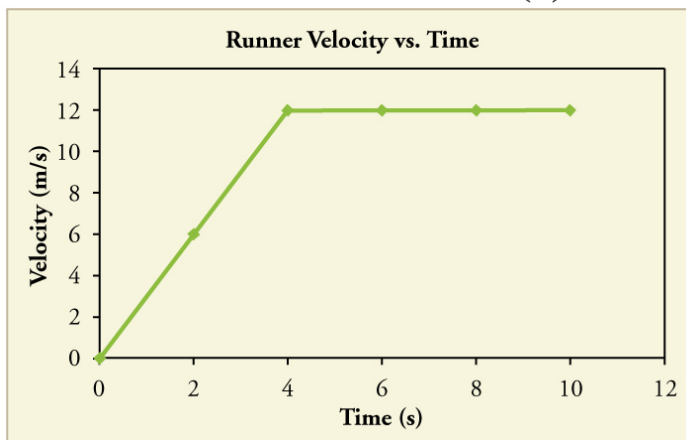




Exercise:

Problem:

A graph of $v(t)$ is shown for a world-class track sprinter in a 100-m race. (See [link](#)). (a) What is his average velocity for the first 4 s? (b) What is his instantaneous velocity at $t = 5$ s? (c) What is his average acceleration between 0 and 4 s? (d) What is his time for the race?



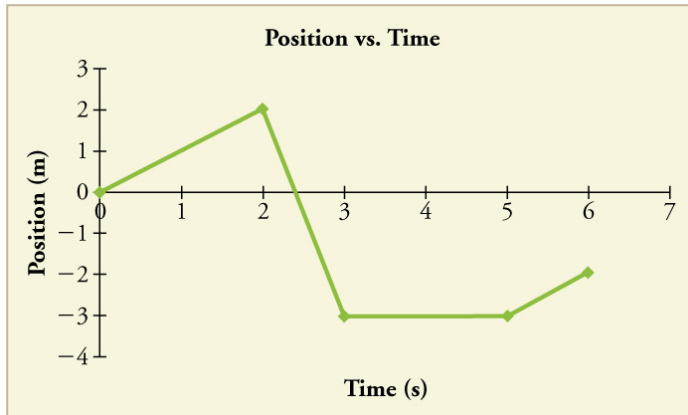
Solution:

- (a) 6 m/s
- (b) 12 m/s
- (c) 3 m/s^2
- (d) 10 s

Exercise:

Problem:

[\[link\]](#) shows the position graph for a particle for 5 s. Draw the corresponding velocity and acceleration graphs.



Glossary

independent variable

the variable that the dependent variable is measured with respect to;
usually plotted along the x -axis

dependent variable

the variable that is being measured; usually plotted along the y -axis

slope

the difference in y -value (the rise) divided by the difference in x -value (the run) of two points on a straight line

y -intercept

the y -value when $x = 0$, or when the graph crosses the y -axis

Connection for AP® Courses

class="introduction"

Everyday motion that we experience is, thankfully, rarely as tortuous as a rollercoaster ride like this—the Dragon Khan in Spain's Universal Port Aventura Amusement Park.

However, most motion is in curved, rather than straight-line, paths.

Motion along a curved path is two- or three-dimensional motion, and can be described in a similar fashion to one-dimensional motion. (credit: Boris23/Wikimedia Commons)



Most instances of motion in everyday life involve changes in displacement and velocity that occur in more than one direction. For example, when you take a long road trip, you drive on different roads in different directions for different amounts of time at different speeds. How can these motions all be combined to determine information about the trip such as the total displacement and average velocity? If you kick a ball from ground level at some angle above the horizontal, how can you describe its motion? To what maximum height does the object rise above the ground? How long is the object in the air? How much horizontal distance is covered before the ball lands? To answer questions such as these, we need to describe motion in two dimensions.

Examining two-dimensional motion requires an understanding of both the scalar and the vector quantities associated with the motion. You will learn how to combine vectors to incorporate both the magnitude and direction of vectors into your analysis. You will learn strategies for simplifying the

calculations involved by choosing the appropriate reference frame and by treating each dimension of the motion separately as a one-dimensional problem, but you will also see that the motion itself occurs in the same way regardless of your chosen reference frame (Essential Knowledge 3.A.1).

This chapter lays a necessary foundation for examining interactions of objects described by forces (Big Idea 3). Changes in direction result from acceleration, which necessitates force on an object. In this chapter, you will concentrate on describing motion that involves changes in direction. In later chapters, you will apply this understanding as you learn about how forces cause these motions (Enduring Understanding 3.A). The concepts in this chapter support:

Big Idea 3 The interactions of an object with other objects can be described by forces.

Enduring Understanding 3.A All forces share certain common characteristics when considered by observers in inertial reference frames.

Essential Knowledge 3.A.1 An observer in a particular reference frame can describe the motion of an object using such quantities as position, displacement, distance, velocity, speed, and acceleration.

Kinematics in Two Dimensions: An Introduction

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Observe that motion in two dimensions consists of horizontal and vertical components.
- Understand the independence of horizontal and vertical vectors in two-dimensional motion.

The information presented in this section supports the following AP® learning objectives and science practices:

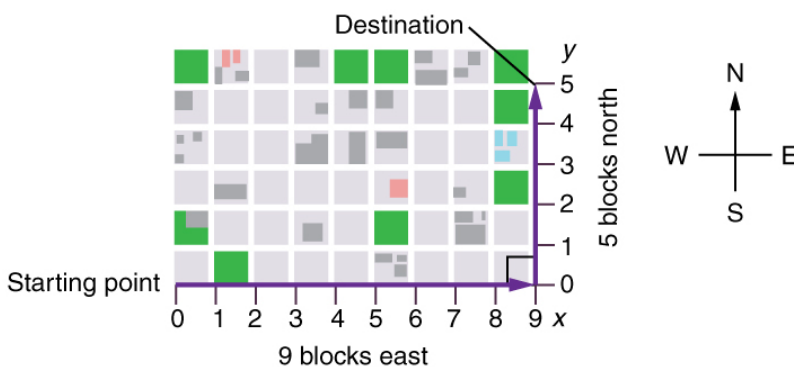
- **3.A.1.1** The student is able to express the motion of an object using narrative, mathematical, and graphical representations. **(S.P. 1.5, 2.1, 2.2)**
- **3.A.1.2** The student is able to design an experimental investigation of the motion of an object. **(S.P. 4.2)**
- **3.A.1.3** The student is able to analyze experimental data describing the motion of an object and is able to express the results of the analysis using narrative, mathematical, and graphical representations. **(S.P. 5.1)**



Walkers and drivers in a city like New York are rarely able to travel in straight lines to reach their destinations. Instead, they must follow roads and sidewalks, making two-dimensional, zigzagged paths. (credit: Margaret W. Carruthers)

Two-Dimensional Motion: Walking in a City

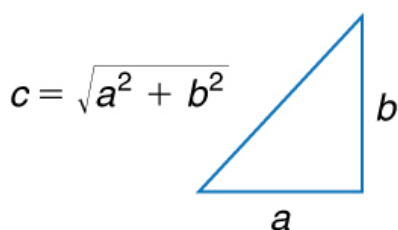
Suppose you want to walk from one point to another in a city with uniform square blocks, as pictured in [\[link\]](#).



A pedestrian walks a two-dimensional path between two points in a city. In this scene, all blocks are square and are the same size.

The straight-line path that a helicopter might fly is blocked to you as a pedestrian, and so you are forced to take a two-dimensional path, such as the one shown. You walk 14 blocks in all, 9 east followed by 5 north. What is the straight-line distance?

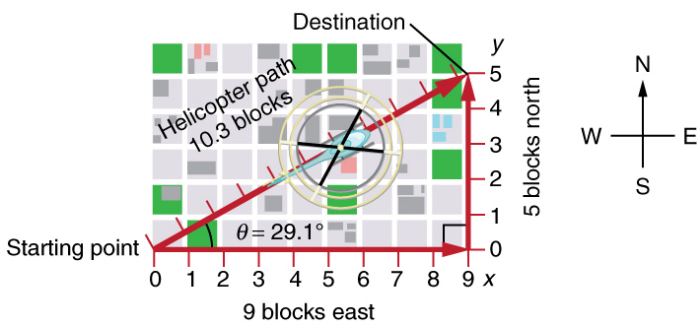
An old adage states that the shortest distance between two points is a straight line. The two legs of the trip and the straight-line path form a right triangle, and so the Pythagorean theorem, $a^2 + b^2 = c^2$, can be used to find the straight-line distance.



The Pythagorean theorem relates the length of the legs of a right triangle, labeled a and b , with the hypotenuse, labeled c . The relationship is given by:
 $a^2 + b^2 = c^2$. This can be rewritten, solving for c :
 $c = \sqrt{a^2 + b^2}$.

The hypotenuse of the triangle is the straight-line path, and so in this case its length in units of city blocks is

$\sqrt{(9 \text{ blocks})^2 + (5 \text{ blocks})^2} = 10.3 \text{ blocks}$, considerably shorter than the 14 blocks you walked. (Note that we are using three significant figures in the answer. Although it appears that “9” and “5” have only one significant digit, they are discrete numbers. In this case “9 blocks” is the same as “9.0 or 9.00 blocks.” We have decided to use three significant figures in the answer in order to show the result more precisely.)



The straight-line path followed by a helicopter between the two points is shorter than the 14 blocks walked by the pedestrian. All blocks are square and the same size.

The fact that the straight-line distance (10.3 blocks) in [\[link\]](#) is less than the total distance walked (14 blocks) is one example of a general characteristic of vectors. (Recall that **vectors** are quantities that have both magnitude and direction.)

As for one-dimensional kinematics, we use arrows to represent vectors. The length of the arrow is proportional to the vector's magnitude. The arrow's length is indicated by hash marks in [\[link\]](#) and [\[link\]](#). The arrow points in

the same direction as the vector. For two-dimensional motion, the path of an object can be represented with three vectors: one vector shows the straight-line path between the initial and final points of the motion, one vector shows the horizontal component of the motion, and one vector shows the vertical component of the motion. The horizontal and vertical components of the motion add together to give the straight-line path. For example, observe the three vectors in [\[link\]](#). The first represents a 9-block displacement east. The second represents a 5-block displacement north. These vectors are added to give the third vector, with a 10.3-block total displacement. The third vector is the straight-line path between the two points. Note that in this example, the vectors that we are adding are perpendicular to each other and thus form a right triangle. This means that we can use the Pythagorean theorem to calculate the magnitude of the total displacement. (Note that we cannot use the Pythagorean theorem to add vectors that are not perpendicular. We will develop techniques for adding vectors having any direction, not just those perpendicular to one another, in [Vector Addition and Subtraction: Graphical Methods](#) and [Vector Addition and Subtraction: Analytical Methods](#).)

The Independence of Perpendicular Motions

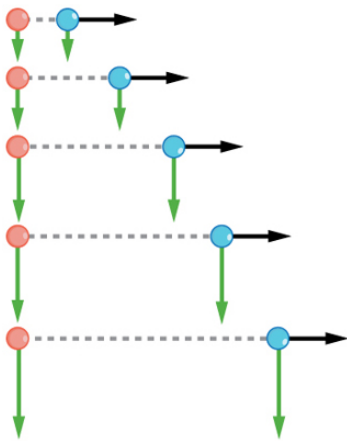
The person taking the path shown in [\[link\]](#) walks east and then north (two perpendicular directions). How far he or she walks east is only affected by his or her motion eastward. Similarly, how far he or she walks north is only affected by his or her motion northward.

Note:

Independence of Motion

The horizontal and vertical components of two-dimensional motion are independent of each other. Any motion in the horizontal direction does not affect motion in the vertical direction, and vice versa.

This is true in a simple scenario like that of walking in one direction first, followed by another. It is also true of more complicated motion involving movement in two directions at once. For example, let's compare the motions of two baseballs. One baseball is dropped from rest. At the same instant, another is thrown horizontally from the same height and follows a curved path. A stroboscope has captured the positions of the balls at fixed time intervals as they fall.



This shows the motions of two identical balls—one falls from rest, the other has an initial horizontal velocity. Each subsequent position is an equal time interval. Arrows represent horizontal and vertical velocities at each position.

The ball on the right has an initial horizontal velocity, while the ball on the left has no horizontal velocity. Despite the difference in horizontal velocities, the vertical velocities and positions are identical for both balls. This shows that the vertical and horizontal motions are independent.

Note:

Applying the Science Practices: Independence of Horizontal and Vertical Motion or Maximum Height and Flight Time

Choose one of the following experiments to design:

Design an experiment to confirm what is shown in [Figure 3.6](#), that the vertical motion of the two balls is independent of the horizontal motion. As you think about your experiment, consider the following questions:

- How will you measure the horizontal and vertical positions of each ball over time? What equipment will this require?

- How will you measure the time interval between each of your position measurements? What equipment will this require?
- If you were to create separate graphs of the horizontal velocity for each ball versus time, what do you predict it would look like? Explain.
- If you were to compare graphs of the vertical velocity for each ball versus time, what do you predict it would look like? Explain.
- If there is a significant amount of air resistance, how will that affect each of your graphs?

Design a two-dimensional ballistic motion experiment that demonstrates the relationship between the maximum height reached by an object and the object's time of flight. As you think about your experiment, consider the following questions:

- How will you measure the maximum height reached by your object?
- How can you take advantage of the symmetry of an object in ballistic motion launched from ground level, reaching maximum height, and returning to ground level?
- Will it make a difference if your object has no horizontal component to its velocity? Explain.
- Will you need to measure the time at multiple different positions? Why or why not?
- Predict what a graph of travel time versus maximum height will look like. Will it be linear? Parabolic? Horizontal? Explain the shape of your predicted graph qualitatively or quantitatively.
- If there is a significant amount of air resistance, how will that affect your measurements and your results?

It is remarkable that for each flash of the strobe, the vertical positions of the two balls are the same. This similarity implies that the vertical motion is independent of whether or not the ball is moving horizontally. (Assuming no air resistance, the vertical motion of a falling object is influenced by gravity only, and not by any horizontal forces.) Careful examination of the ball thrown horizontally shows that it travels the same horizontal distance

between flashes. This is due to the fact that there are no additional forces on the ball in the horizontal direction after it is thrown. This result means that the horizontal velocity is constant, and affected neither by vertical motion nor by gravity (which is vertical). Note that this case is true only for ideal conditions. In the real world, air resistance will affect the speed of the balls in both directions.

The two-dimensional curved path of the horizontally thrown ball is composed of two independent one-dimensional motions (horizontal and vertical). The key to analyzing such motion, called *projectile motion*, is to *resolve* (break) it into motions along perpendicular directions. Resolving two-dimensional motion into perpendicular components is possible because the components are independent. We shall see how to resolve vectors in [Vector Addition and Subtraction: Graphical Methods](#) and [Vector Addition and Subtraction: Analytical Methods](#). We will find such techniques to be useful in many areas of physics.

Note:

PhET Explorations: Ladybug Motion 2D

Learn about position, velocity and acceleration vectors. Move the ladybug by setting the position, velocity or acceleration, and see how the vectors change. Choose linear, circular or elliptical motion, and record and playback the motion to analyze the behavior.

[Ladybug
Motion
2D](#)

Test Prep for AP Courses

Exercise:**Problem:**

A ball is thrown at an angle of 45 degrees above the horizontal. Which of the following best describes the acceleration of the ball from the instant after it leaves the thrower's hand until the time it hits the ground?

- a. Always in the same direction as the motion, initially positive and gradually dropping to zero by the time it hits the ground
 - b. Initially positive in the upward direction, then zero at maximum height, then negative from there until it hits the ground
 - c. Always in the opposite direction as the motion, initially positive and gradually dropping to zero by the time it hits the ground
 - d. Always in the downward direction with the same constant value
-

Solution:

(d)

Exercise:**Problem:**

In an experiment, a student launches a ball with an initial horizontal velocity at an elevation 2 meters above ground. The ball follows a parabolic trajectory until it hits the ground. Which of the following accurately describes the graph of the ball's vertical acceleration versus time (taking the downward direction to be negative)?

- a. A negative value that does not change with time
- b. A gradually increasing negative value (straight line)
- c. An increasing rate of negative values over time (parabolic curve)
- d. Zero at all times since the initial motion is horizontal

Exercise:

Problem:

A student wishes to design an experiment to show that the acceleration of an object is independent of the object's velocity. To do this, ball A is launched horizontally with some initial speed at an elevation 1.5 meters above the ground, ball B is dropped from rest 1.5 meters above the ground, and ball C is launched vertically with some initial speed at an elevation 1.5 meters above the ground. What information would the student need to collect about each ball in order to test the hypothesis?

Solution:

We would need to know the horizontal and vertical positions of each ball at several times. From that data, we could deduce the velocities over several time intervals and also the accelerations (both horizontal and vertical) for each ball over several time intervals.

Summary

- The shortest path between any two points is a straight line. In two dimensions, this path can be represented by a vector with horizontal and vertical components.
- The horizontal and vertical components of a vector are independent of one another. Motion in the horizontal direction does not affect motion in the vertical direction, and vice versa.

Glossary**vector**

a quantity that has both magnitude and direction; an arrow used to represent quantities with both magnitude and direction

Vector Addition and Subtraction: Graphical Methods

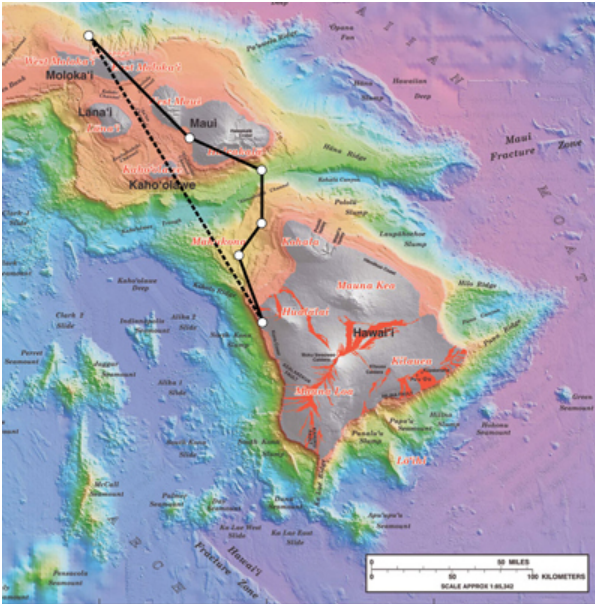
LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Understand the rules of vector addition, subtraction, and multiplication.
- Apply graphical methods of vector addition and subtraction to determine the displacement of moving objects.

The information presented in this section supports the following AP[®] learning objectives and science practices:

- **3.A.1.1** The student is able to express the motion of an object using narrative, mathematical, and graphical representations. **(S.P. 1.5, 2.1, 2.2)**
- **3.A.1.3** The student is able to analyze experimental data describing the motion of an object and is able to express the results of the analysis using narrative, mathematical, and graphical representations. **(S.P. 5.1)**



Displacement can be determined graphically using a scale map, such as this one of the Hawaiian Islands. A journey from Hawai'i to Moloka'i has a number of legs, or journey segments. These segments can be added graphically with a ruler to determine the total two-dimensional displacement of the journey. (credit: US Geological Survey)

Vectors in Two Dimensions

A **vector** is a quantity that has magnitude and direction. Displacement, velocity, acceleration, and force, for example, are all vectors. In one-dimensional, or straight-line, motion, the direction of a vector can be given simply by a plus or minus sign. In two dimensions (2-d), however, we specify the direction of a vector relative to some reference frame (i.e.,

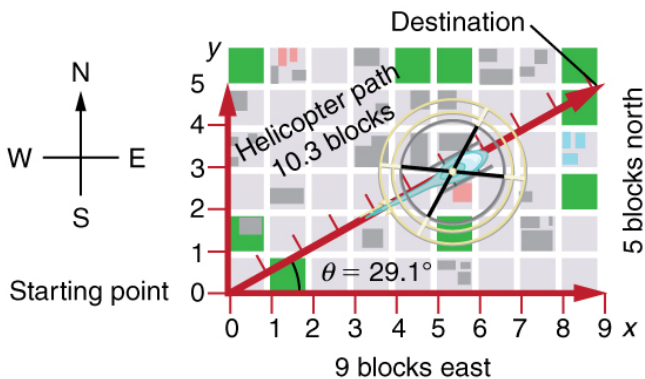
coordinate system), using an arrow having length proportional to the vector's magnitude and pointing in the direction of the vector.

[\[link\]](#) shows such a *graphical representation of a vector*, using as an example the total displacement for the person walking in a city considered in [Kinematics in Two Dimensions: An Introduction](#). We shall use the notation that a boldface symbol, such as \mathbf{D} , stands for a vector. Its magnitude is represented by the symbol in italics, D , and its direction by θ .

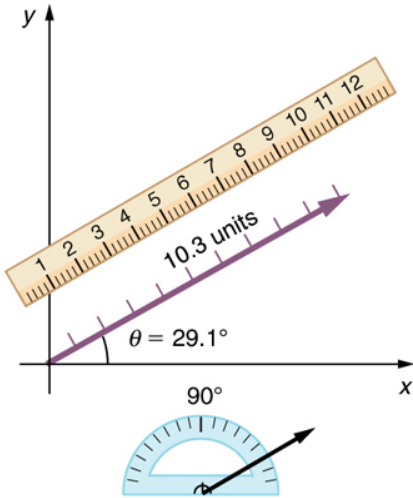
Note:

Vectors in this Text

In this text, we will represent a vector with a boldface variable. For example, we will represent the quantity force with the vector \mathbf{F} , which has both magnitude and direction. The magnitude of the vector will be represented by a variable in italics, such as F , and the direction of the variable will be given by an angle θ .



A person walks 9 blocks east and 5 blocks north. The displacement is 10.3 blocks at an angle 29.1° north of east.

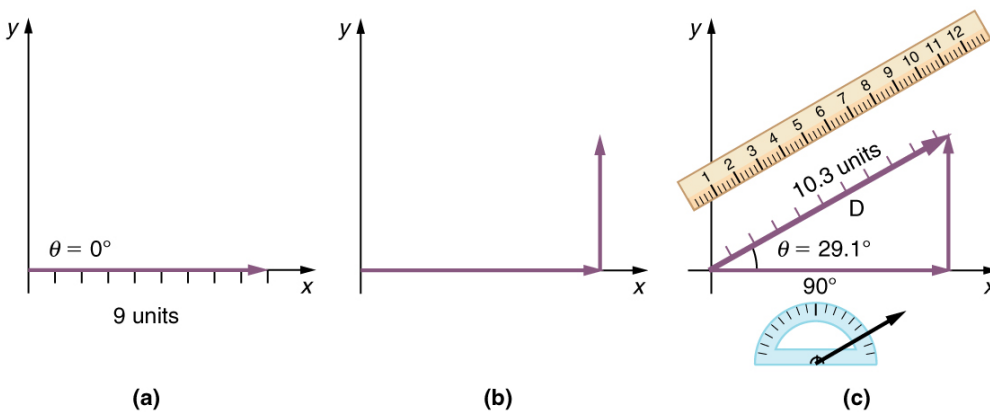


To describe the resultant vector for the person walking in a city considered in [\[link\]](#) graphically, draw an arrow to represent the total displacement vector

D. Using a protractor, draw a line at an angle θ relative to the east-west axis. The length D of the arrow is proportional to the vector's magnitude and is measured along the line with a ruler. In this example, the magnitude D of the vector is 10.3 units, and the direction θ is 29.1° north of east.

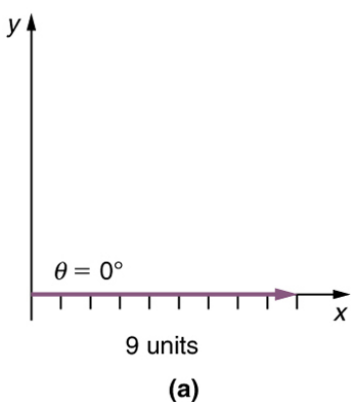
Vector Addition: Head-to-Tail Method

The **head-to-tail method** is a graphical way to add vectors, described in [\[link\]](#) below and in the steps following. The **tail** of the vector is the starting point of the vector, and the **head** (or tip) of a vector is the final, pointed end of the arrow.

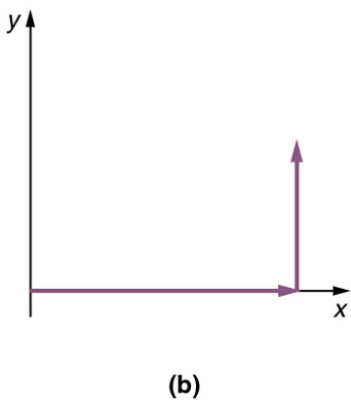


Head-to-Tail Method: The head-to-tail method of graphically adding vectors is illustrated for the two displacements of the person walking in a city considered in [\[link\]](#). (a) Draw a vector representing the displacement to the east. (b) Draw a vector representing the displacement to the north. The tail of this vector should originate from the head of the first, east-pointing vector. (c) Draw a line from the tail of the east-pointing vector to the head of the north-pointing vector to form the sum or **resultant vector** D. The length of the arrow D is proportional to the vector's magnitude and is measured to be 10.3 units . Its direction, described as the angle with respect to the east (or horizontal axis) θ is measured with a protractor to be 29.1° .

Step 1. Draw an arrow to represent the first vector (9 blocks to the east) using a ruler and protractor.

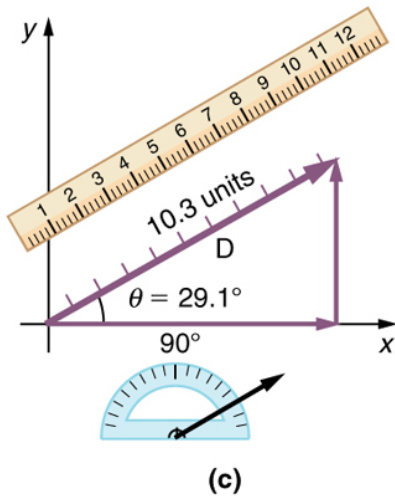


Step 2. Now draw an arrow to represent the second vector (5 blocks to the north). Place the tail of the second vector at the head of the first vector.



Step 3. If there are more than two vectors, continue this process for each vector to be added. Note that in our example, we have only two vectors, so we have finished placing arrows tip to tail.

Step 4. Draw an arrow from the tail of the first vector to the head of the last vector. This is the **resultant**, or the sum, of the other vectors.



Step 5. To get the **magnitude** of the resultant, *measure its length with a ruler.* (Note that in most calculations, we will use the Pythagorean theorem to determine this length.)

Step 6. To get the **direction** of the resultant, *measure the angle it makes with the reference frame using a protractor.* (Note that in most calculations, we will use trigonometric relationships to determine this angle.)

The graphical addition of vectors is limited in accuracy only by the precision with which the drawings can be made and the precision of the measuring tools. It is valid for any number of vectors.

Example:

Adding Vectors Graphically Using the Head-to-Tail Method: A Woman Takes a Walk

Use the graphical technique for adding vectors to find the total displacement of a person who walks the following three paths (displacements) on a flat field. First, she walks 25.0 m in a direction 49.0° north of east. Then, she walks 23.0 m heading 15.0° north of east. Finally, she turns and walks 32.0 m in a direction 68.0° south of east.

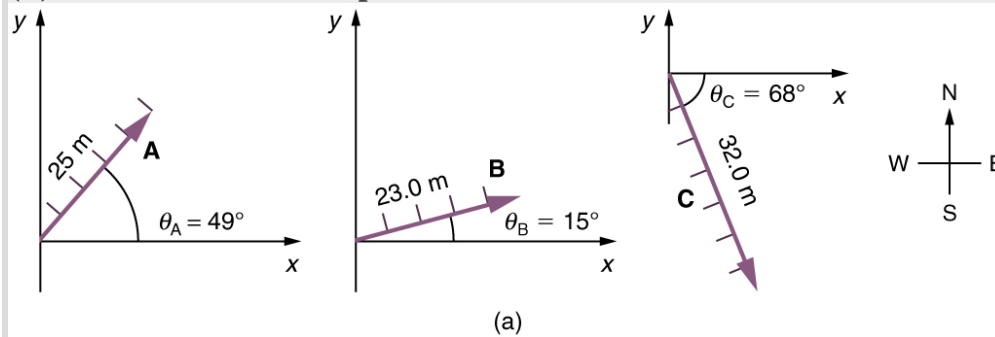
Strategy

Represent each displacement vector graphically with an arrow, labeling the first A, the second B, and the third C, making the lengths proportional to the distance and the directions as specified relative to an east-west line.

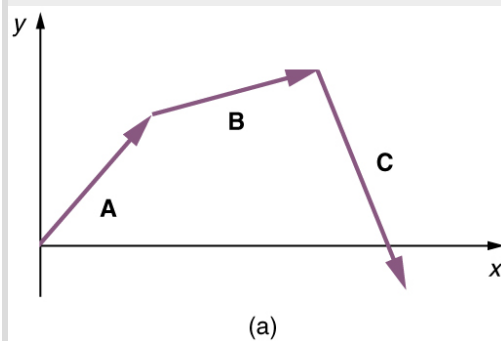
The head-to-tail method outlined above will give a way to determine the magnitude and direction of the resultant displacement, denoted **R**.

Solution

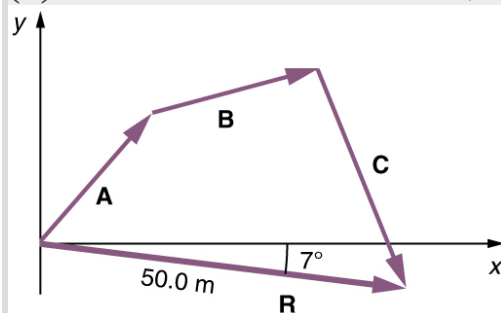
(1) Draw the three displacement vectors.



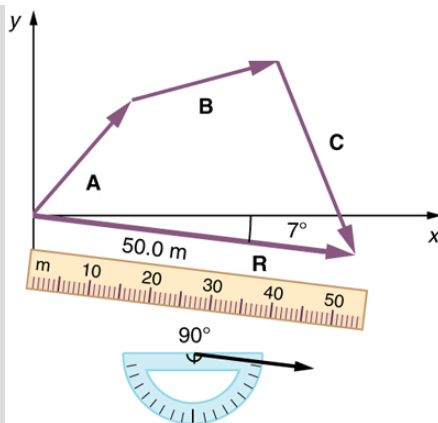
(2) Place the vectors head to tail retaining both their initial magnitude and direction.



(3) Draw the resultant vector, **R**.



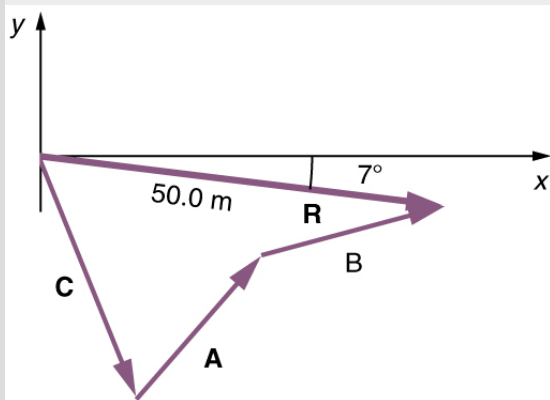
(4) Use a ruler to measure the magnitude of **R**, and a protractor to measure the direction of **R**. While the direction of the vector can be specified in many ways, the easiest way is to measure the angle between the vector and the nearest horizontal or vertical axis. Since the resultant vector is south of the eastward pointing axis, we flip the protractor upside down and measure the angle between the eastward axis and the vector.



In this case, the total displacement **R** is seen to have a magnitude of 50.0 m and to lie in a direction 7.0° south of east. By using its magnitude and direction, this vector can be expressed as $R = 50.0 \text{ m}$ and $\theta = 7.0^\circ$ south of east.

Discussion

The head-to-tail graphical method of vector addition works for any number of vectors. It is also important to note that the resultant is independent of the order in which the vectors are added. Therefore, we could add the vectors in any order as illustrated in [\[link\]](#) and we will still get the same solution.



Here, we see that when the same vectors are added in a different order, the result is the same. This characteristic is true in every case and is an important characteristic of vectors. Vector addition is **commutative**. Vectors can be added in any order.

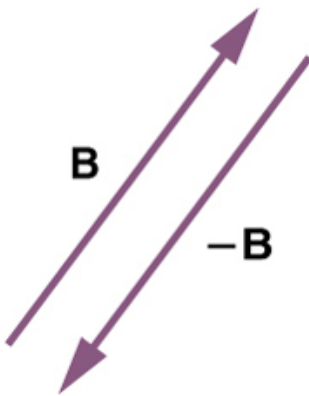
Equation:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}.$$

(This is true for the addition of ordinary numbers as well—you get the same result whether you add $2 + 3$ or $3 + 2$, for example).

Vector Subtraction

Vector subtraction is a straightforward extension of vector addition. To define subtraction (say we want to subtract \mathbf{B} from \mathbf{A} , written $\mathbf{A} - \mathbf{B}$, we must first define what we mean by subtraction. The *negative* of a vector \mathbf{B} is defined to be $-\mathbf{B}$; that is, graphically *the negative of any vector has the same magnitude but the opposite direction*, as shown in [\[link\]](#). In other words, \mathbf{B} has the same length as $-\mathbf{B}$, but points in the opposite direction. Essentially, we just flip the vector so it points in the opposite direction.



The negative
of a vector is
just another
vector of the
same
magnitude but
pointing in the
opposite
direction. So
B is the
negative of

$-\mathbf{B}$; it has the same length but opposite direction.

The *subtraction* of vector \mathbf{B} from vector \mathbf{A} is then simply defined to be the addition of $-\mathbf{B}$ to \mathbf{A} . Note that vector subtraction is the addition of a negative vector. The order of subtraction does not affect the results.

Equation:

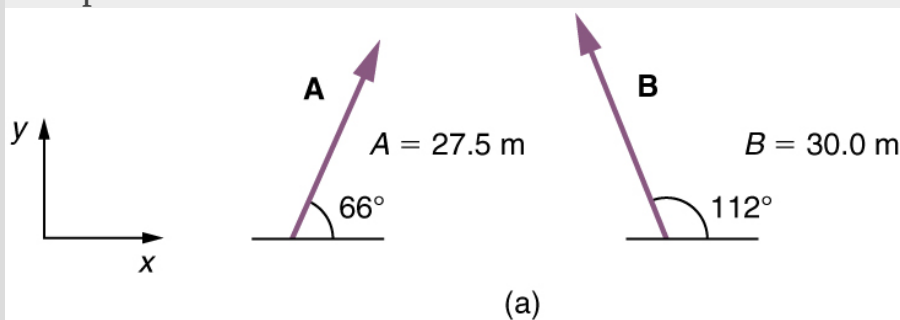
$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}).$$

This is analogous to the subtraction of scalars (where, for example, $5 - 2 = 5 + (-2)$). Again, the result is independent of the order in which the subtraction is made. When vectors are subtracted graphically, the techniques outlined above are used, as the following example illustrates.

Example:

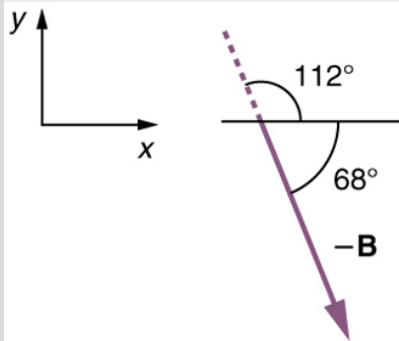
Subtracting Vectors Graphically: A Woman Sailing a Boat

A woman sailing a boat at night is following directions to a dock. The instructions read to first sail 27.5 m in a direction 66.0° north of east from her current location, and then travel 30.0 m in a direction 112° north of east (or 22.0° west of north). If the woman makes a mistake and travels in the *opposite* direction for the second leg of the trip, where will she end up? Compare this location with the location of the dock.



Strategy

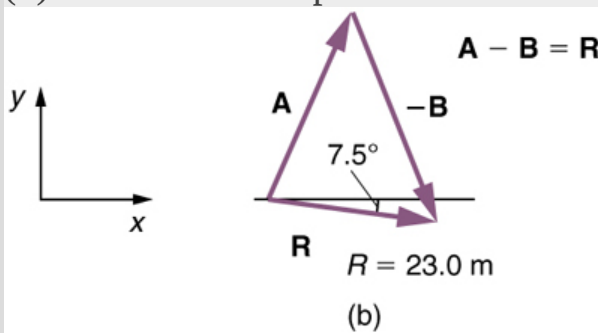
We can represent the first leg of the trip with a vector \mathbf{A} , and the second leg of the trip with a vector \mathbf{B} . The dock is located at a location $\mathbf{A} + \mathbf{B}$. If the woman mistakenly travels in the *opposite* direction for the second leg of the journey, she will travel a distance B (30.0 m) in the direction $180^\circ - 112^\circ = 68^\circ$ south of east. We represent this as $-\mathbf{B}$, as shown below. The vector $-\mathbf{B}$ has the same magnitude as \mathbf{B} but is in the opposite direction. Thus, she will end up at a location $\mathbf{A} + (-\mathbf{B})$, or $\mathbf{A} - \mathbf{B}$.



We will perform vector addition to compare the location of the dock, $\mathbf{A} + \mathbf{B}$, with the location at which the woman mistakenly arrives, $\mathbf{A} + (-\mathbf{B})$.

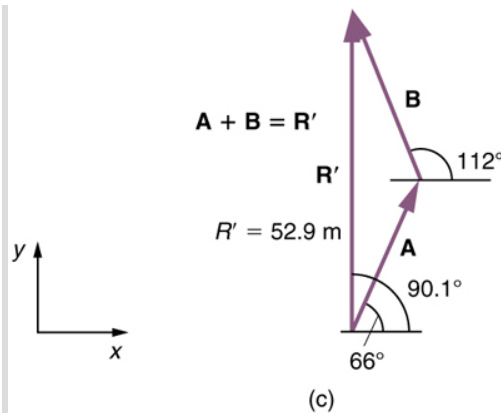
Solution

- (1) To determine the location at which the woman arrives by accident, draw vectors \mathbf{A} and $-\mathbf{B}$.
- (2) Place the vectors head to tail.
- (3) Draw the resultant vector \mathbf{R} .
- (4) Use a ruler and protractor to measure the magnitude and direction of \mathbf{R} .



In this case, $R = 23.0$ m and $\theta = 7.5^\circ$ south of east.

- (5) To determine the location of the dock, we repeat this method to add vectors \mathbf{A} and \mathbf{B} . We obtain the resultant vector \mathbf{R} :



In this case $R = 52.9 \text{ m}$ and $\theta = 90.1^\circ$ north of east.

We can see that the woman will end up a significant distance from the dock if she travels in the opposite direction for the second leg of the trip.

Discussion

Because subtraction of a vector is the same as addition of a vector with the opposite direction, the graphical method of subtracting vectors works the same as for addition.

Multiplication of Vectors and Scalars

If we decided to walk three times as far on the first leg of the trip considered in the preceding example, then we would walk $3 \times 27.5 \text{ m}$, or 82.5 m , in a direction 66.0° north of east. This is an example of multiplying a vector by a positive **scalar**. Notice that the magnitude changes, but the direction stays the same.

If the scalar is negative, then multiplying a vector by it changes the vector's magnitude and gives the new vector the *opposite* direction. For example, if you multiply by -2 , the magnitude doubles but the direction changes. We can summarize these rules in the following way: When vector \mathbf{A} is multiplied by a scalar c ,

- the magnitude of the vector becomes the absolute value of cA ,
- if c is positive, the direction of the vector does not change,
- if c is negative, the direction is reversed.

In our case, $c = 3$ and $A = 27.5$ m. Vectors are multiplied by scalars in many situations. Note that division is the inverse of multiplication. For example, dividing by 2 is the same as multiplying by the value $(1/2)$. The rules for multiplication of vectors by scalars are the same for division; simply treat the divisor as a scalar between 0 and 1.

Resolving a Vector into Components

In the examples above, we have been adding vectors to determine the resultant vector. In many cases, however, we will need to do the opposite. We will need to take a single vector and find what other vectors added together produce it. In most cases, this involves determining the perpendicular **components** of a single vector, for example the x - and y -components, or the north-south and east-west components.

For example, we may know that the total displacement of a person walking in a city is 10.3 blocks in a direction 29.0° north of east and want to find out how many blocks east and north had to be walked. This method is called *finding the components (or parts)* of the displacement in the east and north directions, and it is the inverse of the process followed to find the total displacement. It is one example of finding the components of a vector. There are many applications in physics where this is a useful thing to do. We will see this soon in [Projectile Motion](#), and much more when we cover **forces** in [Dynamics: Newton's Laws of Motion](#). Most of these involve finding components along perpendicular axes (such as north and east), so that right triangles are involved. The analytical techniques presented in [Vector Addition and Subtraction: Analytical Methods](#) are ideal for finding vector components.

Note:

PhET Explorations: Maze Game

Learn about position, velocity, and acceleration in the "Arena of Pain". Use the green arrow to move the ball. Add more walls to the arena to make the game more difficult. Try to make a goal as fast as you can.

Test Prep for AP Courses

Exercise:

Problem:

A ball is launched vertically upward. The vertical position of the ball is recorded at various points in time in the table shown.

Height (m)	Time (sec)
0.490	0.1
0.882	0.2
1.176	0.3
1.372	0.4
1.470	0.5
1.470	0.6
1.372	0.7

Which of the following correctly describes the graph of the ball's vertical velocity versus time?

- a. Always positive, steadily decreasing
- b. Always positive, constant
- c. Initially positive, steadily decreasing, becoming negative at the end
- d. Initially zero, steadily getting more and more negative

Exercise:

Problem:

Height (m)	Time (sec)
0.490	0.1
0.882	0.2
1.176	0.3
1.372	0.4
1.470	0.5
1.470	0.6
1.372	0.7

A ball is launched at an angle of 60 degrees above the horizontal, and the vertical position of the ball is recorded at various points in time in the table shown, assuming the ball was at a height of 0 at time $t = 0$.

- Draw a graph of the ball's vertical velocity versus time.
- Describe the graph of the ball's horizontal velocity.
- Draw a graph of the ball's vertical acceleration versus time.

Solution:

The graph of the ball's vertical velocity over time should begin at 4.90 m/s during the time interval 0 - 0.1 sec (there should be a data point at $t = 0.05$ sec, $v = 4.90$ m/s). It should then have a slope of -9.8 m/s^2 , crossing through $v = 0$ at $t = 0.55$ sec and ending at $v = -0.98$ m/s at $t = 0.65$ sec.

The graph of the ball's horizontal velocity would be a constant positive value, a flat horizontal line at some positive velocity from $t = 0$ until $t = 0.7$ sec.

Summary

- The **graphical method of adding vectors \mathbf{A} and \mathbf{B}** involves drawing vectors on a graph and adding them using the head-to-tail method. The resultant vector \mathbf{R} is defined such that $\mathbf{A} + \mathbf{B} = \mathbf{R}$. The magnitude and direction of \mathbf{R} are then determined with a ruler and protractor, respectively.
- The **graphical method of subtracting vector \mathbf{B} from \mathbf{A}** involves adding the opposite of vector \mathbf{B} , which is defined as $-\mathbf{B}$. In this case, $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}) = \mathbf{R}$. Then, the head-to-tail method of addition is followed in the usual way to obtain the resultant vector \mathbf{R} .
- Addition of vectors is **commutative** such that $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.
- The **head-to-tail method** of adding vectors involves drawing the first vector on a graph and then placing the tail of each subsequent vector at

the head of the previous vector. The resultant vector is then drawn from the tail of the first vector to the head of the final vector.

- If a vector \mathbf{A} is multiplied by a scalar quantity c , the magnitude of the product is given by cA . If c is positive, the direction of the product points in the same direction as \mathbf{A} ; if c is negative, the direction of the product points in the opposite direction as \mathbf{A} .

Conceptual Questions

Exercise:

Problem:

Which of the following is a vector: a person's height, the altitude on Mt. Everest, the age of the Earth, the boiling point of water, the cost of this book, the Earth's population, the acceleration of gravity?

Exercise:

Problem:

Give a specific example of a vector, stating its magnitude, units, and direction.

Exercise:

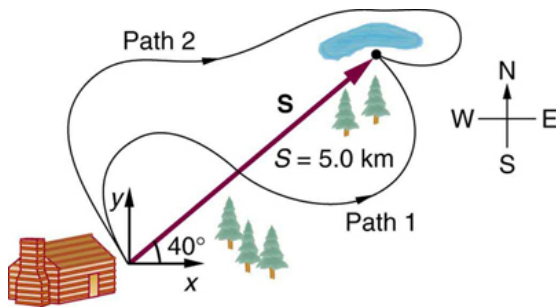
Problem:

What do vectors and scalars have in common? How do they differ?

Exercise:

Problem:

Two campers in a national park hike from their cabin to the same spot on a lake, each taking a different path, as illustrated below. The total distance traveled along Path 1 is 7.5 km, and that along Path 2 is 8.2 km. What is the final displacement of each camper?



Exercise:

Problem:

If an airplane pilot is told to fly 123 km in a straight line to get from San Francisco to Sacramento, explain why he could end up anywhere on the circle shown in [\[link\]](#). What other information would he need to get to Sacramento?



Exercise:

Problem:

Suppose you take two steps **A** and **B** (that is, two nonzero displacements). Under what circumstances can you end up at your starting point? More generally, under what circumstances can two nonzero vectors add to give zero? Is the maximum distance you can end up from the starting point $\mathbf{A} + \mathbf{B}$ the sum of the lengths of the two steps?

Exercise:

Problem: Explain why it is not possible to add a scalar to a vector.

Exercise:

Problem:

If you take two steps of different sizes, can you end up at your starting point? More generally, can two vectors with different magnitudes ever add to zero? Can three or more?

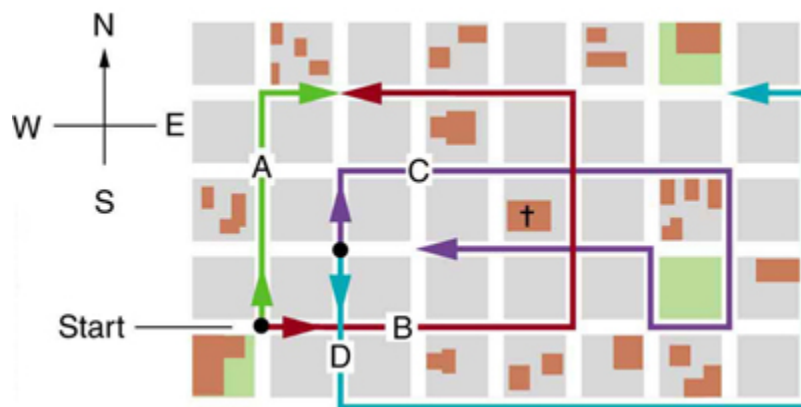
Problems & Exercises

Use graphical methods to solve these problems. You may assume data taken from graphs is accurate to three digits.

Exercise:

Problem:

Find the following for path A in [\[link\]](#): (a) the total distance traveled, and (b) the magnitude and direction of the displacement from start to finish.



The various lines represent paths taken by different people walking in a city. All blocks are 120 m on a side.

Solution:

(a) 480 m

(b) 379 m, 18.4° east of north

Exercise:**Problem:**

Find the following for path B in [\[link\]](#): (a) the total distance traveled, and (b) the magnitude and direction of the displacement from start to finish.

Exercise:**Problem:**

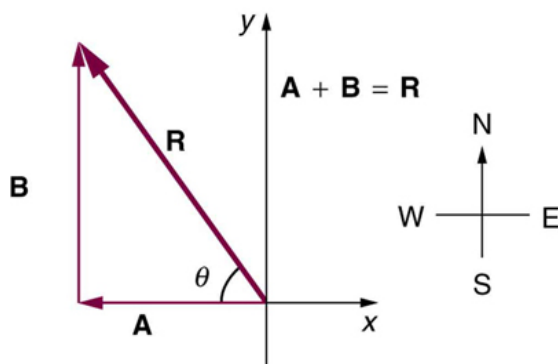
Find the north and east components of the displacement for the hikers shown in [\[link\]](#).

Solution:

north component 3.21 km, east component 3.83 km

Exercise:**Problem:**

Suppose you walk 18.0 m straight west and then 25.0 m straight north. How far are you from your starting point, and what is the compass direction of a line connecting your starting point to your final position? (If you represent the two legs of the walk as vector displacements **A** and **B**, as in [\[link\]](#), then this problem asks you to find their sum $\mathbf{R} = \mathbf{A} + \mathbf{B}$.)

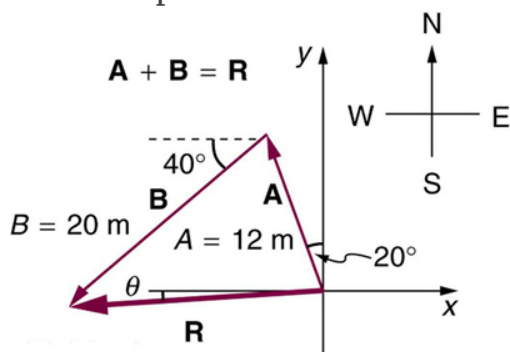


The two displacements **A** and **B** add to give a total displacement **R** having magnitude R and direction θ .

Exercise:

Problem:

Suppose you first walk 12.0 m in a direction 20° west of north and then 20.0 m in a direction 40.0° south of west. How far are you from your starting point, and what is the compass direction of a line connecting your starting point to your final position? (If you represent the two legs of the walk as vector displacements **A** and **B**, as in [\[link\]](#), then this problem finds their sum $\mathbf{R} = \mathbf{A} + \mathbf{B}$.)



Solution:

19.5 m, 4.65° south of west

Exercise:**Problem:**

Repeat the problem above, but reverse the order of the two legs of the walk; show that you get the same final result. That is, you first walk leg **B**, which is 20.0 m in a direction exactly 40° south of west, and then leg **A**, which is 12.0 m in a direction exactly 20° west of north. (This problem shows that $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.)

Exercise:**Problem:**

(a) Repeat the problem two problems prior, but for the second leg you walk 20.0 m in a direction 40.0° north of east (which is equivalent to subtracting **B** from **A**—that is, to finding $\mathbf{R} = \mathbf{A} - \mathbf{B}$). (b) Repeat the problem two problems prior, but now you first walk 20.0 m in a direction 40.0° south of west and then 12.0 m in a direction 20.0° east of south (which is equivalent to subtracting **A** from **B**—that is, to finding $\mathbf{R} = \mathbf{B} - \mathbf{A} = -\mathbf{R}$). Show that this is the case.

Solution:

(a) 26.6 m, 65.1° north of east

(b) 26.6 m, 65.1° south of west

Exercise:**Problem:**

Show that the *order* of addition of three vectors does not affect their sum. Show this property by choosing any three vectors **A**, **B**, and **C**, all having different lengths and directions. Find the sum $\mathbf{A} + \mathbf{B} + \mathbf{C}$ then find their sum when added in a different order and show the result is the same. (There are five other orders in which **A**, **B**, and **C** can be added; choose only one.)

Exercise:

Problem:

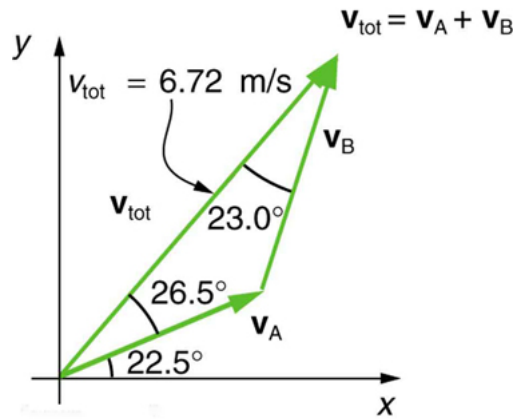
Show that the sum of the vectors discussed in [\[link\]](#) gives the result shown in [\[link\]](#).

Solution:

52.9 m, 90.1° with respect to the x-axis.

Exercise:

Problem: Find the magnitudes of velocities v_A and v_B in [\[link\]](#)



The two velocities \mathbf{v}_A
and \mathbf{v}_B add to give a total
 \mathbf{v}_{tot} .

Exercise:**Problem:**

Find the components of v_{tot} along the x- and y-axes in [\[link\]](#).

Solution:

x-component 4.41 m/s

y-component 5.07 m/s

Exercise:

Problem:

Find the components of v_{tot} along a set of perpendicular axes rotated 30° counterclockwise relative to those in [\[link\]](#).

Glossary

component (of a 2-d vector)

a piece of a vector that points in either the vertical or the horizontal direction; every 2-d vector can be expressed as a sum of two vertical and horizontal vector components

commutative

refers to the interchangeability of order in a function; vector addition is commutative because the order in which vectors are added together does not affect the final sum

direction (of a vector)

the orientation of a vector in space

head (of a vector)

the end point of a vector; the location of the tip of the vector's arrowhead; also referred to as the “tip”

head-to-tail method

a method of adding vectors in which the tail of each vector is placed at the head of the previous vector

magnitude (of a vector)

the length or size of a vector; magnitude is a scalar quantity

resultant

the sum of two or more vectors

resultant vector

the vector sum of two or more vectors

scalar

a quantity with magnitude but no direction

tail

the start point of a vector; opposite to the head or tip of the arrow

Vector Addition and Subtraction: Analytical Methods

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Understand the rules of vector addition and subtraction using analytical methods.
- Apply analytical methods to determine vertical and horizontal component vectors.
- Apply analytical methods to determine the magnitude and direction of a resultant vector.

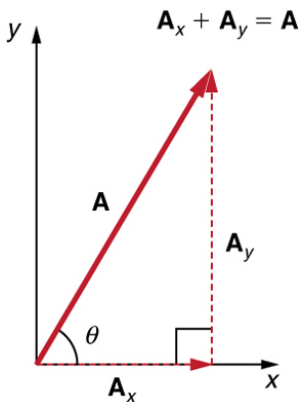
The information presented in this section supports the following AP® learning objectives and science practices:

- **3.A.1.1** The student is able to express the motion of an object using narrative, mathematical, and graphical representations. (S.P. 1.5, 2.1, 2.2)

Analytical methods of vector addition and subtraction employ geometry and simple trigonometry rather than the ruler and protractor of graphical methods. Part of the graphical technique is retained, because vectors are still represented by arrows for easy visualization. However, analytical methods are more concise, accurate, and precise than graphical methods, which are limited by the accuracy with which a drawing can be made. Analytical methods are limited only by the accuracy and precision with which physical quantities are known.

Resolving a Vector into Perpendicular Components

Analytical techniques and right triangles go hand-in-hand in physics because (among other things) motions along perpendicular directions are independent. We very often need to separate a vector into perpendicular components. For example, given a vector like \mathbf{A} in [\[link\]](#), we may wish to find which two perpendicular vectors, \mathbf{A}_x and \mathbf{A}_y , add to produce it.



The vector \mathbf{A} , with its tail at the origin of an x, y -coordinate system, is shown together with its x - and y -components, \mathbf{A}_x and \mathbf{A}_y . These vectors form a right triangle. The analytical relationships among these

vectors are
summarized
below.

\mathbf{A}_x and \mathbf{A}_y are defined to be the components of \mathbf{A} along the x - and y -axes. The three vectors \mathbf{A} , \mathbf{A}_x , and \mathbf{A}_y form a right triangle:

Equation:

$$\mathbf{A}_x + \mathbf{A}_y = \mathbf{A}.$$

Note that this relationship between vector components and the resultant vector holds only for vector quantities (which include both magnitude and direction). The relationship does not apply for the magnitudes alone. For example, if $\mathbf{A}_x = 3$ m east, $\mathbf{A}_y = 4$ m north, and $\mathbf{A} = 5$ m north-east, then it is true that the vectors $\mathbf{A}_x + \mathbf{A}_y = \mathbf{A}$. However, it is *not* true that the sum of the magnitudes of the vectors is also equal. That is,

Equation:

$$3 \text{ m} + 4 \text{ m} \neq 5 \text{ m}$$

Thus,

Equation:

$$A_x + A_y \neq A$$

If the vector \mathbf{A} is known, then its magnitude A (its length) and its angle θ (its direction) are known. To find A_x and A_y , its x - and y -components, we use the following relationships for a right triangle.

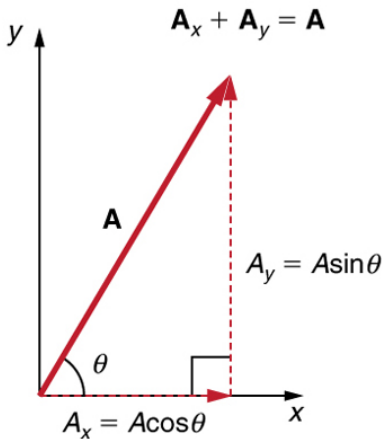
Equation:

$$A_x = A \cos \theta$$

and

Equation:

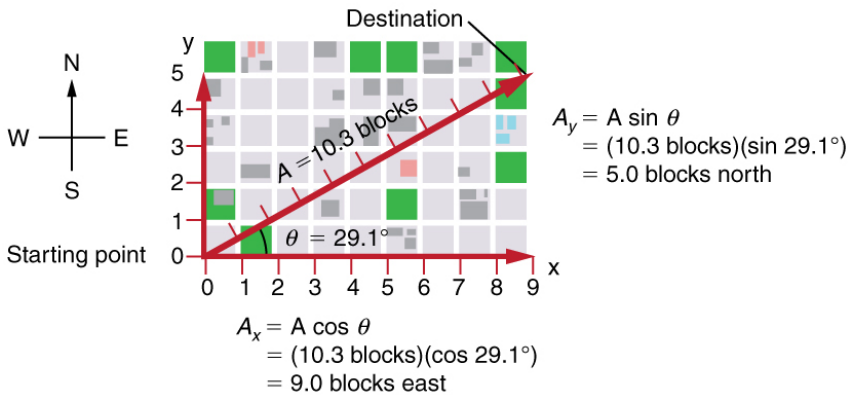
$$A_y = A \sin \theta.$$



The magnitudes of the vector components \mathbf{A}_x and \mathbf{A}_y can be related to the resultant vector \mathbf{A} and the angle θ with trigonometric identities. Here we see that

$$A_x = A \cos \theta \text{ and } A_y = A \sin \theta.$$

Suppose, for example, that \mathbf{A} is the vector representing the total displacement of the person walking in a city considered in [Kinematics in Two Dimensions: An Introduction](#) and [Vector Addition and Subtraction: Graphical Methods](#).



We can use the relationships $A_x = A \cos \theta$ and $A_y = A \sin \theta$ to determine the magnitude of the horizontal and vertical component vectors in this example.

Then $A = 10.3$ blocks and $\theta = 29.1^\circ$, so that

Equation:

$$A_x = A \cos \theta = (10.3 \text{ blocks})(\cos 29.1^\circ) = 9.0 \text{ blocks}$$

Equation:

$$A_y = A \sin \theta = (10.3 \text{ blocks})(\sin 29.1^\circ) = 5.0 \text{ blocks.}$$

Calculating a Resultant Vector

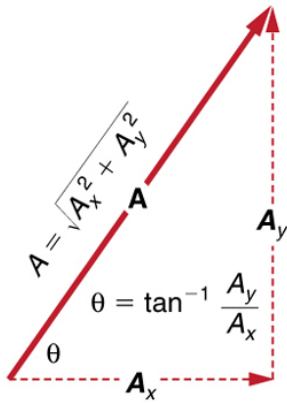
If the perpendicular components \mathbf{A}_x and \mathbf{A}_y of a vector \mathbf{A} are known, then \mathbf{A} can also be found analytically. To find the magnitude A and direction θ of a vector from its perpendicular components \mathbf{A}_x and \mathbf{A}_y , we use the following relationships:

Equation:

$$A = \sqrt{A_x^2 + A_y^2}$$

Equation:

$$\theta = \tan^{-1}(A_y/A_x).$$



The
magnitude
and direction
of the
resultant
vector can
be
determined
once the
horizontal
and vertical
components
 A_x and A_y
have been
determined.

Note that the equation $A = \sqrt{A_x^2 + A_y^2}$ is just the Pythagorean theorem relating the legs of a right triangle to the length of the hypotenuse. For

example, if A_x and A_y are 9 and 5 blocks, respectively, then

$A = \sqrt{9^2 + 5^2} = 10.3$ blocks, again consistent with the example of the person walking in a city. Finally, the direction is $\theta = \tan^{-1}(5/9) = 29.1^\circ$, as before.

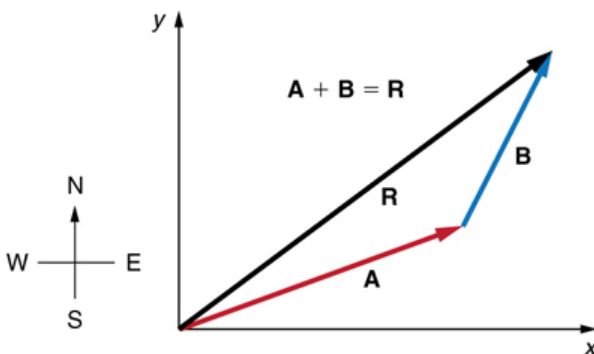
Note:

Determining Vectors and Vector Components with Analytical Methods

Equations $A_x = A \cos \theta$ and $A_y = A \sin \theta$ are used to find the perpendicular components of a vector—that is, to go from A and θ to A_x and A_y . Equations $A = \sqrt{A_x^2 + A_y^2}$ and $\theta = \tan^{-1}(A_y/A_x)$ are used to find a vector from its perpendicular components—that is, to go from A_x and A_y to A and θ . Both processes are crucial to analytical methods of vector addition and subtraction.

Adding Vectors Using Analytical Methods

To see how to add vectors using perpendicular components, consider [\[link\]](#), in which the vectors **A** and **B** are added to produce the resultant **R**.

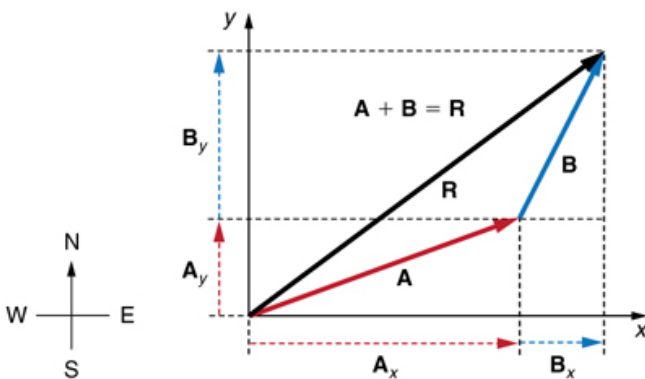


Vectors **A** and **B** are two legs of a walk, and **R** is the resultant or total displacement. You can

use analytical methods to determine the magnitude and direction of \mathbf{R} .

If \mathbf{A} and \mathbf{B} represent two legs of a walk (two displacements), then \mathbf{R} is the total displacement. The person taking the walk ends up at the tip of \mathbf{R} . There are many ways to arrive at the same point. In particular, the person could have walked first in the x -direction and then in the y -direction. Those paths are the x - and y -components of the resultant, \mathbf{R}_x and \mathbf{R}_y . If we know \mathbf{R}_x and \mathbf{R}_y , we can find R and θ using the equations $A = \sqrt{A_x^2 + A_y^2}$ and $\theta = \tan^{-1}(A_y/A_x)$. When you use the analytical method of vector addition, you can determine the components or the magnitude and direction of a vector.

Step 1. Identify the x - and y -axes that will be used in the problem. Then, find the components of each vector to be added along the chosen perpendicular axes. Use the equations $A_x = A \cos \theta$ and $A_y = A \sin \theta$ to find the components. In [\[link\]](#), these components are A_x , A_y , B_x , and B_y . The angles that vectors \mathbf{A} and \mathbf{B} make with the x -axis are θ_A and θ_B , respectively.



To add vectors \mathbf{A} and \mathbf{B} , first determine the horizontal and

vertical components of each vector.

These are the dotted vectors \mathbf{A}_x , \mathbf{A}_y , \mathbf{B}_x and \mathbf{B}_y shown in the image.

Step 2. Find the components of the resultant along each axis by adding the components of the individual vectors along that axis. That is, as shown in [\[link\]](#),

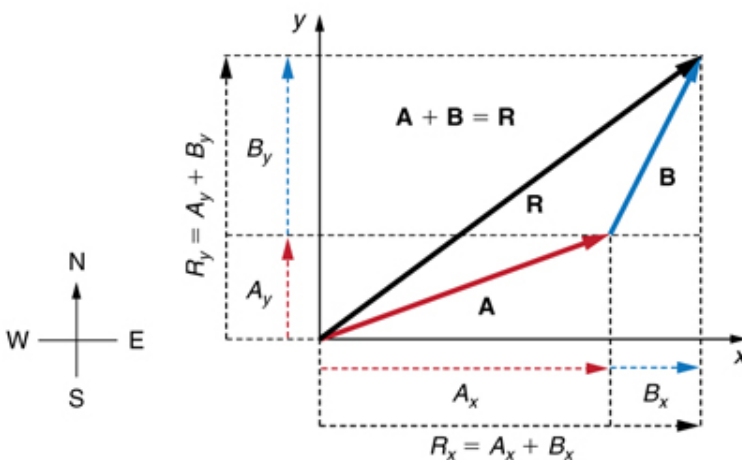
Equation:

$$R_x = A_x + B_x$$

and

Equation:

$$R_y = A_y + B_y.$$



The magnitude of the vectors \mathbf{A}_x and \mathbf{B}_x add to give the magnitude R_x of the resultant vector in the horizontal direction. Similarly, the magnitudes of

the vectors \mathbf{A}_y and \mathbf{B}_y add to give the magnitude R_y of the resultant vector in the vertical direction.

Components along the same axis, say the x -axis, are vectors along the same line and, thus, can be added to one another like ordinary numbers. The same is true for components along the y -axis. (For example, a 9-block eastward walk could be taken in two legs, the first 3 blocks east and the second 6 blocks east, for a total of 9, because they are along the same direction.) So resolving vectors into components along common axes makes it easier to add them. Now that the components of \mathbf{R} are known, its magnitude and direction can be found.

Step 3. *To get the magnitude R of the resultant, use the Pythagorean theorem:*

Equation:

$$R = \sqrt{R_x^2 + R_y^2}.$$

Step 4. *To get the direction of the resultant:*

Equation:

$$\theta = \tan^{-1}(R_y/R_x).$$

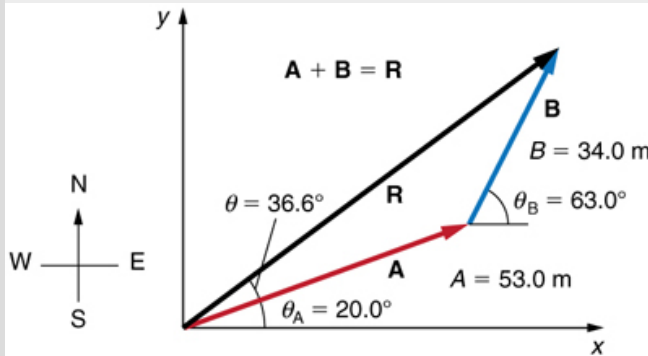
The following example illustrates this technique for adding vectors using perpendicular components.

Example:

Adding Vectors Using Analytical Methods

Add the vector \mathbf{A} to the vector \mathbf{B} shown in [\[link\]](#), using perpendicular components along the x - and y -axes. The x - and y -axes are along the east–

west and north–south directions, respectively. Vector **A** represents the first leg of a walk in which a person walks 53.0 m in a direction 20.0° north of east. Vector **B** represents the second leg, a displacement of 34.0 m in a direction 63.0° north of east.



Vector **A** has magnitude 53.0 m and direction 20.0° north of the x -axis. Vector **B** has magnitude 34.0 m and direction 63.0° north of the x -axis. You can use analytical methods to determine the magnitude and direction of **R**.

Strategy

The components of **A** and **B** along the x - and y -axes represent walking due east and due north to get to the same ending point. Once found, they are combined to produce the resultant.

Solution

Following the method outlined above, we first find the components of **A** and **B** along the x - and y -axes. Note that $A = 53.0$ m, $\theta_A = 20.0^\circ$, $B = 34.0$ m, and $\theta_B = 63.0^\circ$. We find the x -components by using $A_x = A \cos \theta$, which gives

Equation:

$$\begin{aligned} A_x &= A \cos \theta_A = (53.0 \text{ m})(\cos 20.0^\circ) \\ &= (53.0 \text{ m})(0.940) = 49.8 \text{ m} \end{aligned}$$

and

Equation:

$$\begin{aligned} B_x &= B \cos \theta_B = (34.0 \text{ m})(\cos 63.0^\circ) \\ &= (34.0 \text{ m})(0.454) = 15.4 \text{ m.} \end{aligned}$$

Similarly, the y-components are found using $A_y = A \sin \theta_A$:

Equation:

$$\begin{aligned} A_y &= A \sin \theta_A = (53.0 \text{ m})(\sin 20.0^\circ) \\ &= (53.0 \text{ m})(0.342) = 18.1 \text{ m} \end{aligned}$$

and

Equation:

$$\begin{aligned} B_y &= B \sin \theta_B = (34.0 \text{ m})(\sin 63.0^\circ) \\ &= (34.0 \text{ m})(0.891) = 30.3 \text{ m.} \end{aligned}$$

The x- and y-components of the resultant are thus

Equation:

$$R_x = A_x + B_x = 49.8 \text{ m} + 15.4 \text{ m} = 65.2 \text{ m}$$

and

Equation:

$$R_y = A_y + B_y = 18.1 \text{ m} + 30.3 \text{ m} = 48.4 \text{ m.}$$

Now we can find the magnitude of the resultant by using the Pythagorean theorem:

Equation:

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(65.2)^2 + (48.4)^2} \text{ m}$$

so that

Equation:

$$R = 81.2 \text{ m.}$$

Finally, we find the direction of the resultant:

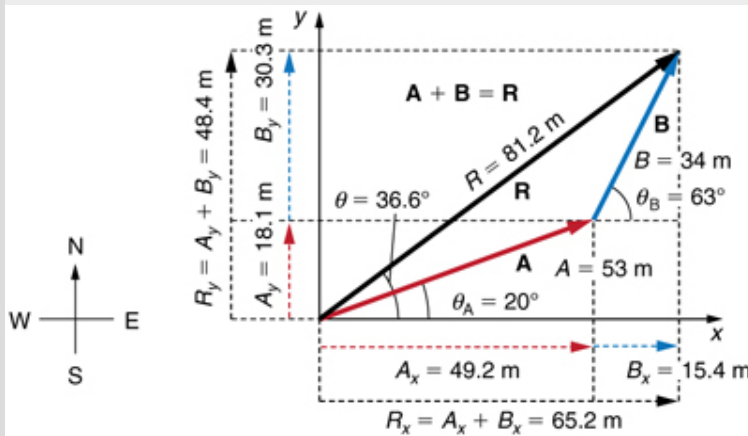
Equation:

$$\theta = \tan^{-1}(R_y/R_x) = +\tan^{-1}(48.4/65.2).$$

Thus,

Equation:

$$\theta = \tan^{-1}(0.742) = 36.6^\circ.$$



Using analytical methods, we see that the magnitude of **R** is 81.2 m and its direction is 36.6° north of east.

Discussion

This example illustrates the addition of vectors using perpendicular components. Vector subtraction using perpendicular components is very similar—it is just the addition of a negative vector.

Subtraction of vectors is accomplished by the addition of a negative vector.

That is, $\mathbf{A} - \mathbf{B} \equiv \mathbf{A} + (-\mathbf{B})$. Thus, *the method for the subtraction of vectors using perpendicular components is identical to that for addition.*

The components of $-\mathbf{B}$ are the negatives of the components of **B**. The x- and y-components of the resultant $\mathbf{A} - \mathbf{B} = \mathbf{R}$ are thus

Equation:

$$R_x = A_x + (-B_x)$$

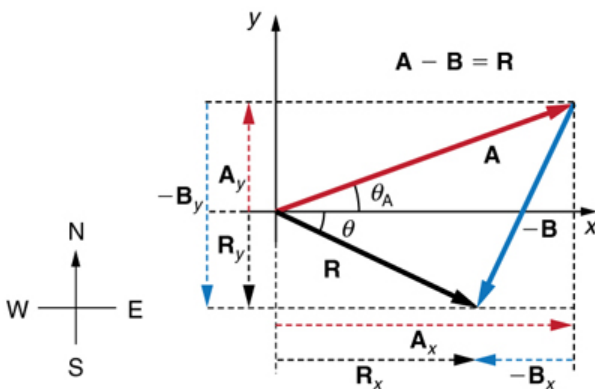
and

Equation:

$$R_y = A_y + (-B_y)$$

and the rest of the method outlined above is identical to that for addition.
(See [\[link\]](#).)

Analyzing vectors using perpendicular components is very useful in many areas of physics, because perpendicular quantities are often independent of one another. The next module, [Projectile Motion](#), is one of many in which using perpendicular components helps make the picture clear and simplifies the physics.



The subtraction of the two vectors shown in [\[link\]](#). The components of $-\mathbf{B}$ are the negatives of the components of **B**. The method of subtraction is the same as that for addition.

Note:**PhET Explorations: Vector Addition**

Learn how to add vectors. Drag vectors onto a graph, change their length and angle, and sum them together. The magnitude, angle, and components of each vector can be displayed in several formats.

[Vector](#)
[Additio](#)
[n](#)

Summary

- The analytical method of vector addition and subtraction involves using the Pythagorean theorem and trigonometric identities to determine the magnitude and direction of a resultant vector.
- The steps to add vectors **A** and **B** using the analytical method are as follows:

Step 1: Determine the coordinate system for the vectors. Then, determine the horizontal and vertical components of each vector using the equations

Equation:

$$A_x = A \cos \theta$$

$$B_x = B \cos \theta$$

and

Equation:

$$A_y = A \sin \theta$$

$$B_y = B \sin \theta.$$

Step 2: Add the horizontal and vertical components of each vector to determine the components R_x and R_y of the resultant vector, \mathbf{R} :

Equation:

$$R_x = A_x + B_x$$

and

Equation:

$$R_y = A_y + B_y.$$

Step 3: Use the Pythagorean theorem to determine the magnitude, R , of the resultant vector \mathbf{R} :

Equation:

$$R = \sqrt{R_x^2 + R_y^2}.$$

Step 4: Use a trigonometric identity to determine the direction, θ , of \mathbf{R} :

Equation:

$$\theta = \tan^{-1}(R_y/R_x).$$

Conceptual Questions

Exercise:

Problem:

Suppose you add two vectors **A** and **B**. What relative direction between them produces the resultant with the greatest magnitude? What is the maximum magnitude? What relative direction between them produces the resultant with the smallest magnitude? What is the minimum magnitude?

Exercise:**Problem:**

Give an example of a nonzero vector that has a component of zero.

Exercise:**Problem:**

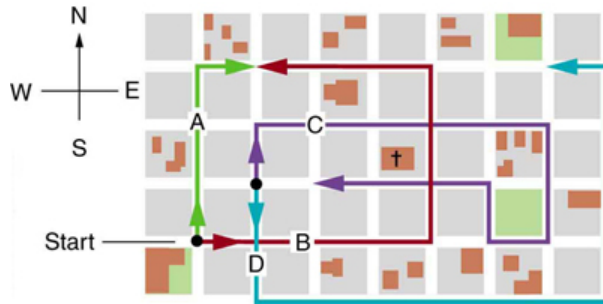
Explain why a vector cannot have a component greater than its own magnitude.

Exercise:**Problem:**

If the vectors **A** and **B** are perpendicular, what is the component of **A** along the direction of **B**? What is the component of **B** along the direction of **A**?

Problems & Exercises**Exercise:****Problem:**

Find the following for path C in [\[link\]](#): (a) the total distance traveled and (b) the magnitude and direction of the displacement from start to finish. In this part of the problem, explicitly show how you follow the steps of the analytical method of vector addition.



The various lines represent paths taken by different people walking in a city. All blocks are 120 m on a side.

Solution:

(a) 1.56 km

(b) 120 m east

Exercise:

Problem:

Find the following for path D in [\[link\]](#): (a) the total distance traveled and (b) the magnitude and direction of the displacement from start to finish. In this part of the problem, explicitly show how you follow the steps of the analytical method of vector addition.

Exercise:

Problem:

Find the north and east components of the displacement from San Francisco to Sacramento shown in [\[link\]](#).



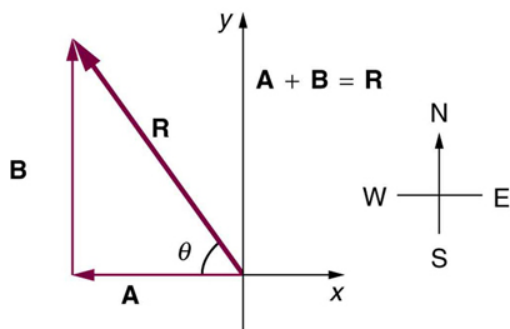
Solution:

North-component 87.0 km, east-component 87.0 km

Exercise:

Problem:

Solve the following problem using analytical techniques: Suppose you walk 18.0 m straight west and then 25.0 m straight north. How far are you from your starting point, and what is the compass direction of a line connecting your starting point to your final position? (If you represent the two legs of the walk as vector displacements **A** and **B**, as in [\[link\]](#), then this problem asks you to find their sum $\mathbf{R} = \mathbf{A} + \mathbf{B}$.)



The two displacements **A** and **B** add to give a total displacement **R** having

magnitude R and
direction θ .

Note that you can also solve this graphically. Discuss why the analytical technique for solving this problem is potentially more accurate than the graphical technique.

Exercise:

Problem:

Repeat [\[link\]](#) using analytical techniques, but reverse the order of the two legs of the walk and show that you get the same final result. (This problem shows that adding them in reverse order gives the same result—that is, $\mathbf{B} + \mathbf{A} = \mathbf{A} + \mathbf{B}$.) Discuss how taking another path to reach the same point might help to overcome an obstacle blocking your other path.

Solution:

30.8 m, 35.8 west of north

Exercise:

Problem:

You drive 7.50 km in a straight line in a direction 15° east of north. (a) Find the distances you would have to drive straight east and then straight north to arrive at the same point. (This determination is equivalent to finding the components of the displacement along the east and north directions.) (b) Show that you still arrive at the same point if the east and north legs are reversed in order.

Exercise:

Problem:

Do [\[link\]](#) again using analytical techniques and change the second leg of the walk to 25.0 m straight south. (This is equivalent to subtracting **B** from **A** —that is, finding $\mathbf{R} = \mathbf{A} - \mathbf{B}$) (b) Repeat again, but now you first walk 25.0 m north and then 18.0 m east. (This is equivalent to subtract **A** from **B** —that is, to find $\mathbf{A} = \mathbf{B} + \mathbf{C}$. Is that consistent with your result?)

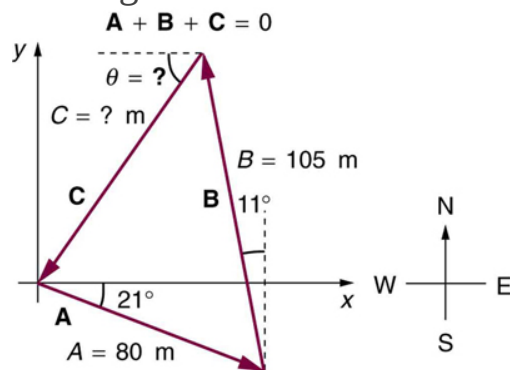
Solution:

(a) 30.8 m, 54.2° south of west

(b) 30.8 m, 54.2° north of east

Exercise:**Problem:**

A new landowner has a triangular piece of flat land she wishes to fence. Starting at the west corner, she measures the first side to be 80.0 m long and the next to be 105 m. These sides are represented as displacement vectors **A** from **B** in [\[link\]](#). She then correctly calculates the length and orientation of the third side **C**. What is her result?

**Exercise:**

Problem:

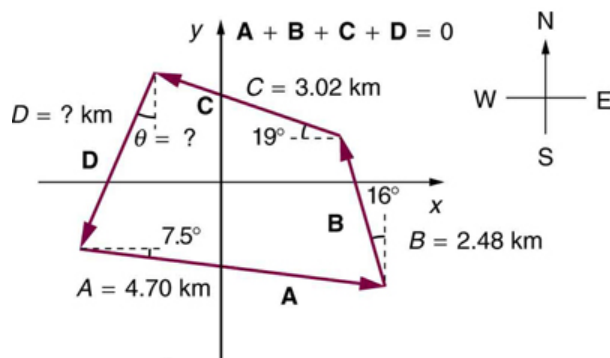
You fly 32.0 km in a straight line in still air in the direction 35.0° south of west. (a) Find the distances you would have to fly straight south and then straight west to arrive at the same point. (This determination is equivalent to finding the components of the displacement along the south and west directions.) (b) Find the distances you would have to fly first in a direction 45.0° south of west and then in a direction 45.0° west of north. These are the components of the displacement along a different set of axes—one rotated 45° .

Solution:

18.4 km south, then 26.2 km west (b) 31.5 km at 45.0° south of west, then 5.56 km at 45.0° west of north

Exercise:**Problem:**

A farmer wants to fence off his four-sided plot of flat land. He measures the first three sides, shown as **A**, **B**, and **C** in [\[link\]](#), and then correctly calculates the length and orientation of the fourth side **D**. What is his result?

**Exercise:**

Problem:

In an attempt to escape his island, Gilligan builds a raft and sets to sea. The wind shifts a great deal during the day, and he is blown along the following straight lines: 2.50 km 45.0° north of west; then 4.70 km 60.0° south of east; then 1.30 km 25.0° south of west; then 5.10 km straight east; then 1.70 km 5.00° east of north; then 7.20 km 55.0° south of west; and finally 2.80 km 10.0° north of east. What is his final position relative to the island?

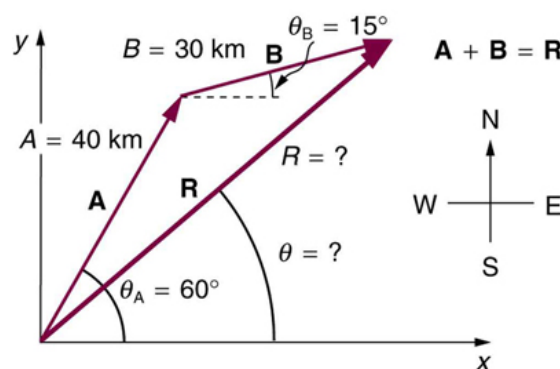
Solution:

7.34 km, 63.5° south of east

Exercise:

Problem:

Suppose a pilot flies 40.0 km in a direction 60° north of east and then flies 30.0 km in a direction 15° north of east as shown in [\[link\]](#). Find her total distance R from the starting point and the direction θ of the straight-line path to the final position. Discuss qualitatively how this flight would be altered by a wind from the north and how the effect of the wind would depend on both wind speed and the speed of the plane relative to the air mass.



Glossary

analytical method

the method of determining the magnitude and direction of a resultant vector using the Pythagorean theorem and trigonometric identities

Projectile Motion

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Identify and explain the properties of a projectile, such as acceleration due to gravity, range, maximum height, and trajectory.
- Determine the location and velocity of a projectile at different points in its trajectory.
- Apply the principle of independence of motion to solve projectile motion problems.

The information presented in this section supports the following AP® learning objectives:

- **3.A.1.1** The student is able to express the motion of an object using narrative, mathematical, and graphical representations. (S.P. 1.5, 2.1, 2.2)
- **3.A.1.3** The student is able to analyze experimental data describing the motion of an object and is able to express the results of the analysis using narrative, mathematical, and graphical representations. (S.P. 5.1)

Projectile motion is the **motion** of an object thrown or projected into the air, subject to only the acceleration of gravity. The object is called a **projectile**, and its path is called its **trajectory**. The motion of falling objects, as covered in [Problem-Solving Basics for One-Dimensional Kinematics](#), is a simple one-dimensional type of projectile motion in which there is no horizontal movement. In this section, we consider two-dimensional projectile motion, such as that of a football or other object for which **air resistance is negligible**.

The most important fact to remember here is that *motions along perpendicular axes are independent* and thus can be analyzed separately. This fact was discussed in [Kinematics in Two Dimensions: An Introduction](#), where vertical and horizontal motions were seen to be independent. The key to analyzing two-dimensional projectile motion is to break it into two motions, one along the horizontal axis and the other along the vertical. (This choice of axes is the most sensible, because acceleration due to gravity is vertical—thus, there will be no acceleration along the horizontal axis when air resistance is negligible.) As is customary, we call the horizontal axis the x -axis and the vertical axis the y -axis. [\[link\]](#) illustrates the notation for displacement, where \mathbf{s} is defined to be the total displacement and \mathbf{x} and \mathbf{y} are its components along the horizontal and vertical axes, respectively. The magnitudes of these vectors are s , x , and y . (Note that in the last section we used the notation \mathbf{A} to represent a vector with components \mathbf{A}_x and \mathbf{A}_y . If we continued this format, we would call displacement \mathbf{s} with components \mathbf{s}_x and \mathbf{s}_y . However, to simplify the notation, we will simply represent the component vectors as \mathbf{x} and \mathbf{y} .)

Of course, to describe motion we must deal with velocity and acceleration, as well as with displacement. We must find their components along the x - and y -axes, too. We will assume all forces except gravity (such as air resistance and friction, for example) are negligible. The components of acceleration are then very simple:

$a_y = -g = -9.80 \text{ m/s}^2$. (Note that this definition assumes that the upwards direction is defined as the positive direction. If you arrange the coordinate system instead such that the downwards direction is positive, then acceleration due to gravity takes a positive value.) Because gravity is vertical, $a_x = 0$. Both accelerations are constant, so the kinematic equations can be used.

Note:

Review of Kinematic Equations (constant a)

Equation:

$$x = x_0 + \bar{v}t$$

Equation:

$$\bar{v} = \frac{v_0 + v}{2}$$

Equation:

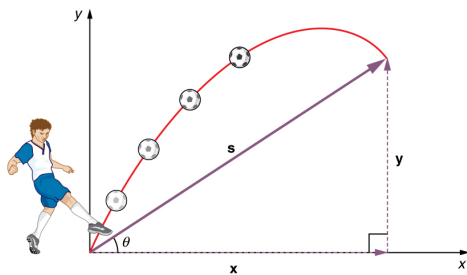
$$v = v_0 + at$$

Equation:

$$x = x_0 + v_0t + \frac{1}{2}at^2$$

Equation:

$$v^2 = v_0^2 + 2a(x - x_0).$$



The total displacement s of a soccer ball at a point along its path. The vector s has components x and y along the horizontal and vertical axes. Its magnitude is s , and it makes an angle θ with the horizontal.

Given these assumptions, the following steps are then used to analyze projectile motion:

Step 1. Resolve or break the motion into horizontal and vertical components along the x - and y -axes. These axes are perpendicular, so $A_x = A \cos \theta$ and $A_y = A \sin \theta$ are used. The magnitude of the components of displacement s along these axes are x and y . The magnitudes of the components of the velocity \mathbf{v} are $v_x = v \cos \theta$ and $v_y = v \sin \theta$, where v is the magnitude of the velocity and θ is its direction, as shown in [\[link\]](#). Initial values are denoted with a subscript 0, as usual.

Step 2. Treat the motion as two independent one-dimensional motions, one horizontal and the other vertical. The kinematic equations for horizontal and vertical motion take the following forms:

Equation:

$$\text{Horizontal Motion}(a_x = 0)$$

Equation:

$$x = x_0 + v_x t$$

Equation:

$$v_x = v_{0x} = v_x = \text{velocity is a constant.}$$

Equation:

$$\text{Vertical Motion (assuming positive is up } a_y = -g = -9.80\text{m/s}^2)$$

Equation:

$$y = y_0 + \frac{1}{2}(v_{0y} + v_y)t$$

Equation:

$$v_y = v_{0y} - gt$$

Equation:

$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2$$

Equation:

$$v_y^2 = v_{0y}^2 - 2g(y - y_0).$$

Step 3. Solve for the unknowns in the two separate motions—one horizontal and one vertical. Note that the only common variable between the motions is time t . The problem solving procedures here are the same as for one-dimensional **kinematics** and are illustrated in the solved examples below.

Step 4. Recombine the two motions to find the total displacement \mathbf{s} and velocity \mathbf{v} . Because the x - and y -motions are perpendicular, we determine these vectors by using the techniques outlined in the [Vector Addition and Subtraction: Analytical Methods](#) and employing $A = \sqrt{A_x^2 + A_y^2}$ and $\theta = \tan^{-1}(A_y/A_x)$ in the following form, where θ is the direction of the displacement \mathbf{s} and θ_v is the direction of the velocity \mathbf{v} :

Total displacement and velocity

Equation:

$$s = \sqrt{x^2 + y^2}$$

Equation:

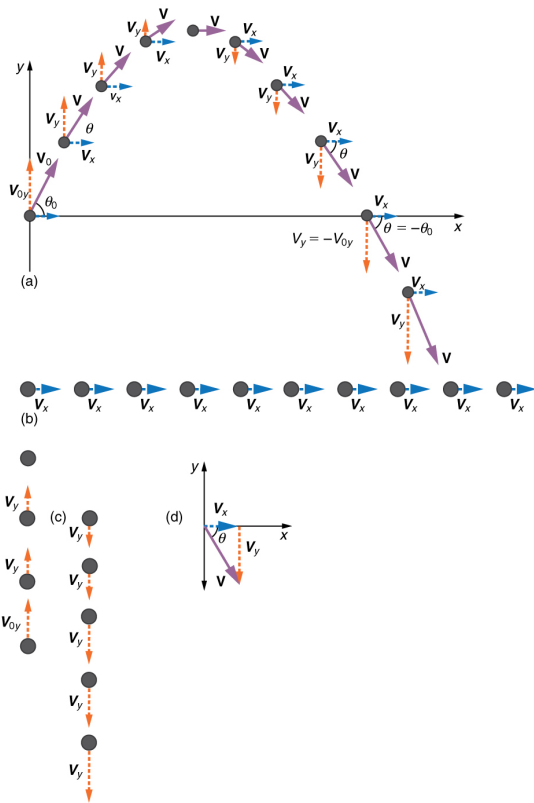
$$\theta = \tan^{-1}(y/x)$$

Equation:

$$v = \sqrt{v_x^2 + v_y^2}$$

Equation:

$$\theta_v = \tan^{-1}(v_y/v_x).$$



(a) We analyze two-dimensional projectile motion by breaking it into two independent one-dimensional motions along the vertical and horizontal axes. (b) The horizontal motion is simple, because $a_x = 0$ and v_x is thus constant. (c) The velocity in the vertical direction begins to decrease as the object rises; at its highest point, the vertical velocity is zero. As the object falls towards the Earth again, the vertical velocity increases again in magnitude but points in the opposite direction to the initial vertical velocity. (d) The x - and y -motions are recombined to give the total velocity at any given point on the trajectory.

Example:

A Fireworks Projectile Explodes High and Away

During a fireworks display, a shell is shot into the air with an initial speed of 70.0 m/s at an angle of 75.0° above the horizontal, as illustrated in [\[link\]](#). The fuse is timed to ignite the shell just as it reaches its highest point above the ground. (a) Calculate the height at which the shell explodes. (b) How much time passed between the launch of the shell and the explosion? (c) What is the horizontal displacement of the shell when it explodes?

Strategy

Because air resistance is negligible for the unexploded shell, the analysis method outlined above can be used. The motion can be broken into horizontal and vertical motions in which $a_x = 0$ and $a_y = -g$. We can then define x_0

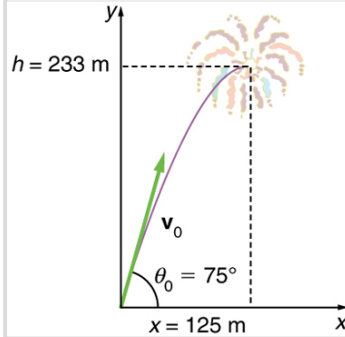
and y_0 to be zero and solve for the desired quantities.

Solution for (a)

By “height” we mean the altitude or vertical position y above the starting point. The highest point in any trajectory, called the apex, is reached when $v_y = 0$. Since we know the initial and final velocities as well as the initial position, we use the following equation to find y :

Equation:

$$v_y^2 = v_{0y}^2 - 2g(y - y_0).$$



The trajectory of a fireworks shell. The fuse is set to explode the shell at the highest point in its trajectory, which is found to be at a height of 233 m and 125 m away horizontally.

Because y_0 and v_y are both zero, the equation simplifies to

Equation:

$$0 = v_{0y}^2 - 2gy.$$

Solving for y gives

Equation:

$$y = \frac{v_{0y}^2}{2g}.$$

Now we must find v_{0y} , the component of the initial velocity in the y -direction. It is given by $v_{0y} = v_0 \sin \theta$, where v_{0y} is the initial velocity of 70.0 m/s, and $\theta_0 = 75.0^\circ$ is the initial angle. Thus,

Equation:

$$v_{0y} = v_0 \sin \theta_0 = (70.0 \text{ m/s})(\sin 75^\circ) = 67.6 \text{ m/s}.$$

and y is

Equation:

$$y = \frac{(67.6 \text{ m/s})^2}{2(9.80 \text{ m/s}^2)},$$

so that

Equation:

$$y = 233\text{m}.$$

Discussion for (a)

Note that because up is positive, the initial velocity is positive, as is the maximum height, but the acceleration due to gravity is negative. Note also that the maximum height depends only on the vertical component of the initial velocity, so that any projectile with a 67.6 m/s initial vertical component of velocity will reach a maximum height of 233 m (neglecting air resistance). The numbers in this example are reasonable for large fireworks displays, the shells of which do reach such heights before exploding. In practice, air resistance is not completely negligible, and so the initial velocity would have to be somewhat larger than that given to reach the same height.

Solution for (b)

As in many physics problems, there is more than one way to solve for the time to the highest point. In this case, the easiest method is to use $y = y_0 + \frac{1}{2}(v_{0y} + v_y)t$. Because y_0 is zero, this equation reduces to simply

Equation:

$$y = \frac{1}{2}(v_{0y} + v_y)t.$$

Note that the final vertical velocity, v_y , at the highest point is zero. Thus,

Equation:

$$\begin{aligned} t &= \frac{2y}{(v_{0y} + v_y)} = \frac{2(233 \text{ m})}{(67.6 \text{ m/s})} \\ &= 6.90 \text{ s}. \end{aligned}$$

Discussion for (b)

This time is also reasonable for large fireworks. When you are able to see the launch of fireworks, you will notice several seconds pass before the shell explodes. (Another way of finding the time is by using $y = y_0 + v_{0y}t - \frac{1}{2}gt^2$, and solving the quadratic equation for t .)

Solution for (c)

Because air resistance is negligible, $a_x = 0$ and the horizontal velocity is constant, as discussed above. The horizontal displacement is horizontal velocity multiplied by time as given by $x = x_0 + v_x t$, where x_0 is equal to zero:

Equation:

$$x = v_x t,$$

where v_x is the x-component of the velocity, which is given by $v_x = v_0 \cos \theta_0$. Now,

Equation:

$$v_x = v_0 \cos \theta_0 = (70.0 \text{ m/s})(\cos 75.0^\circ) = 18.1 \text{ m/s}.$$

The time t for both motions is the same, and so x is

Equation:

$$x = (18.1 \text{ m/s})(6.90 \text{ s}) = 125 \text{ m}.$$

Discussion for (c)

The horizontal motion is a constant velocity in the absence of air resistance. The horizontal displacement found here could be useful in keeping the fireworks fragments from falling on spectators. Once the shell explodes, air resistance has a major effect, and many fragments will land directly below.

In solving part (a) of the preceding example, the expression we found for y is valid for any projectile motion where air resistance is negligible. Call the maximum height $y = h$; then,

Equation:

$$h = \frac{v_{0y}^2}{2g}.$$

This equation defines the *maximum height of a projectile* and depends only on the vertical component of the initial velocity.

Note:

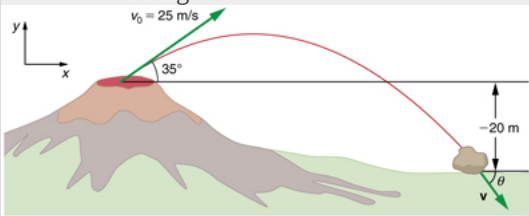
Defining a Coordinate System

It is important to set up a coordinate system when analyzing projectile motion. One part of defining the coordinate system is to define an origin for the x and y positions. Often, it is convenient to choose the initial position of the object as the origin such that $x_0 = 0$ and $y_0 = 0$. It is also important to define the positive and negative directions in the x and y directions. Typically, we define the positive vertical direction as upwards, and the positive horizontal direction is usually the direction of the object's motion. When this is the case, the vertical acceleration, g , takes a negative value (since it is directed downwards towards the Earth). However, it is occasionally useful to define the coordinates differently. For example, if you are analyzing the motion of a ball thrown downwards from the top of a cliff, it may make sense to define the positive direction downwards since the motion of the ball is solely in the downwards direction. If this is the case, g takes a positive value.

Example:

Calculating Projectile Motion: Hot Rock Projectile

Kilauea in Hawaii is the world's most continuously active volcano. Very active volcanoes characteristically eject red-hot rocks and lava rather than smoke and ash. Suppose a large rock is ejected from the volcano with a speed of 25.0 m/s and at an angle 35.0° above the horizontal, as shown in [\[link\]](#). The rock strikes the side of the volcano at an altitude 20.0 m lower than its starting point. (a) Calculate the time it takes the rock to follow this path. (b) What are the magnitude and direction of the rock's velocity at impact?



The trajectory of a rock ejected from the Kilauea volcano.

Strategy

Again, resolving this two-dimensional motion into two independent one-dimensional motions will allow us to solve for the desired quantities. The time a projectile is in the air is governed by its vertical motion alone. We will solve for t first. While the rock is rising and falling vertically, the horizontal motion continues at a constant velocity. This example asks for the final velocity. Thus, the vertical and horizontal results will be recombined to obtain v and θ_v at the final time t determined in the first part of the example.

Solution for (a)

While the rock is in the air, it rises and then falls to a final position 20.0 m lower than its starting altitude. We can find the time for this by using

Equation:

$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2.$$

If we take the initial position y_0 to be zero, then the final position is $y = -20.0$ m. Now the initial vertical velocity is the vertical component of the initial velocity, found from $v_{0y} = v_0 \sin \theta_0 = (25.0 \text{ m/s})(\sin 35.0^\circ) = 14.3 \text{ m/s}$. Substituting known values yields

Equation:

$$-20.0 \text{ m} = (14.3 \text{ m/s})t - (4.90 \text{ m/s}^2)t^2.$$

Rearranging terms gives a quadratic equation in t :

Equation:

$$(4.90 \text{ m/s}^2)t^2 - (14.3 \text{ m/s})t - (20.0 \text{ m}) = 0.$$

This expression is a quadratic equation of the form $at^2 + bt + c = 0$, where the constants are $a = 4.90$, $b = -14.3$, and $c = -20.0$. Its solutions are given by the quadratic formula:

Equation:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This equation yields two solutions: $t = 3.96$ and $t = -1.03$. (It is left as an exercise for the reader to verify these solutions.) The time is $t = 3.96$ s or -1.03 s. The negative value of time implies an event before the start of motion, and so we discard it. Thus,

Equation:

$$t = 3.96 \text{ s}.$$

Discussion for (a)

The time for projectile motion is completely determined by the vertical motion. So any projectile that has an initial vertical velocity of 14.3 m/s and lands 20.0 m below its starting altitude will spend 3.96 s in the air.

Solution for (b)

From the information now in hand, we can find the final horizontal and vertical velocities v_x and v_y and combine them to find the total velocity v and the angle θ_0 it makes with the horizontal. Of course, v_x is constant so we can solve for it at any horizontal location. In this case, we chose the starting point since we know both the initial velocity and initial angle. Therefore:

Equation:

$$v_x = v_0 \cos \theta_0 = (25.0 \text{ m/s})(\cos 35^\circ) = 20.5 \text{ m/s}.$$

The final vertical velocity is given by the following equation:

Equation:

$$v_y = v_{0y} - gt,$$

where v_{0y} was found in part (a) to be 14.3 m/s . Thus,

Equation:

$$v_y = 14.3 \text{ m/s} - (9.80 \text{ m/s}^2)(3.96 \text{ s})$$

so that

Equation:

$$v_y = -24.5 \text{ m/s}.$$

To find the magnitude of the final velocity v we combine its perpendicular components, using the following equation:

Equation:

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(20.5 \text{ m/s})^2 + (-24.5 \text{ m/s})^2},$$

which gives

Equation:

$$v = 31.9 \text{ m/s}.$$

The direction θ_v is found from the equation:

Equation:

$$\theta_v = \tan^{-1}(v_y/v_x)$$

so that

Equation:

$$\theta_v = \tan^{-1}(-24.5/20.5) = \tan^{-1}(-1.19).$$

Thus,

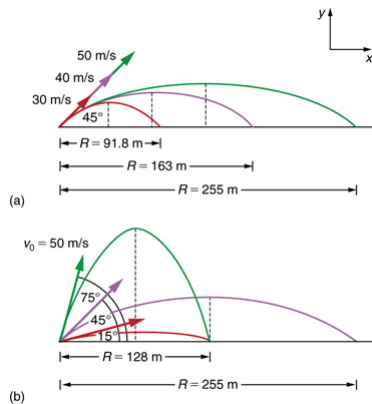
Equation:

$$\theta_v = -50.1^\circ.$$

Discussion for (b)

The negative angle means that the velocity is 50.1° below the horizontal. This result is consistent with the fact that the final vertical velocity is negative and hence downward—as you would expect because the final altitude is 20.0 m lower than the initial altitude. (See [\[link\]](#).)

One of the most important things illustrated by projectile motion is that vertical and horizontal motions are independent of each other. Galileo was the first person to fully comprehend this characteristic. He used it to predict the range of a projectile. On level ground, we define **range** to be the horizontal distance R traveled by a projectile. Galileo and many others were interested in the range of projectiles primarily for military purposes—such as aiming cannons. However, investigating the range of projectiles can shed light on other interesting phenomena, such as the orbits of satellites around the Earth. Let us consider projectile range further.



Trajectories of projectiles on level ground. (a) The greater the initial speed v_0 , the greater the range for a given initial angle. (b) The effect of initial angle θ_0 on the range of a projectile with a given initial speed. Note that

the range is the same for 15° and 75° , although the maximum heights of those paths are different.

How does the initial velocity of a projectile affect its range? Obviously, the greater the initial speed v_0 , the greater the range, as shown in [\[link\]](#)(a). The initial angle θ_0 also has a dramatic effect on the range, as illustrated in [\[link\]](#)(b). For a fixed initial speed, such as might be produced by a cannon, the maximum range is obtained with $\theta_0 = 45^\circ$. This is true only for conditions neglecting air resistance. If air resistance is considered, the maximum angle is approximately 38° . Interestingly, for every initial angle except 45° , there are two angles that give the same range—the sum of those angles is 90° . The range also depends on the value of the acceleration of gravity g . The lunar astronaut Alan Shepherd was able to drive a golf ball a great distance on the Moon because gravity is weaker there. The range R of a projectile on *level ground* for which air resistance is negligible is given by

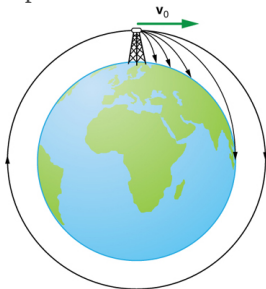
Equation:

$$R = \frac{v_0^2 \sin 2\theta_0}{g},$$

where v_0 is the initial speed and θ_0 is the initial angle relative to the horizontal. The proof of this equation is left as an end-of-chapter problem (hints are given), but it does fit the major features of projectile range as described.

When we speak of the range of a projectile on level ground, we assume that R is very small compared with the circumference of the Earth. If, however, the range is large, the Earth curves away below the projectile and acceleration of gravity changes direction along the path. The range is larger than predicted by the range equation given above because the projectile has farther to fall than it would on level ground. (See [\[link\]](#).) If the initial speed is great enough, the projectile goes into orbit. This possibility was recognized centuries before it could be accomplished. When an object is in orbit, the Earth curves away from underneath the object at the same rate as it falls. The object thus falls continuously but never hits the surface. These and other aspects of orbital motion, such as the rotation of the Earth, will be covered analytically and in greater depth later in this text.

Once again we see that thinking about one topic, such as the range of a projectile, can lead us to others, such as the Earth orbits. In [Addition of Velocities](#), we will examine the addition of velocities, which is another important aspect of two-dimensional kinematics and will also yield insights beyond the immediate topic.



Projectile to satellite. In each case shown here, a projectile is launched from a very high tower to avoid air resistance. With increasing initial speed, the

range increases and becomes longer than it would be on level ground because the Earth curves away underneath its path. With a large enough initial speed, orbit is achieved.

Note:

PhET Explorations: Projectile Motion

Blast a Buick out of a cannon! Learn about projectile motion by firing various objects. Set the angle, initial speed, and mass. Add air resistance. Make a game out of this simulation by trying to hit a target.

[Projectile Motion](#)

Test Prep for AP Courses

Exercise:

Problem:

In an experiment, a student launches a ball with an initial horizontal velocity of 5.00 meters/sec at an elevation 2.00 meters above ground. Draw and clearly label with appropriate values and units a graph of the ball's horizontal velocity vs. time and the ball's vertical velocity vs. time. The graph should cover the motion from the instant after the ball is launched until the instant before it hits the ground. Assume the downward direction is negative for this problem.

Summary

- Projectile motion is the motion of an object through the air that is subject only to the acceleration of gravity.
- To solve projectile motion problems, perform the following steps:

Determine a coordinate system. Then, resolve the position and/or velocity of the object in the horizontal and vertical components.	s are given by the quantities	x and y , and the components of the velocity	\mathbf{v} are given by	$v_x = v \cos \theta$ and $v_y = v \sin \theta$,	v is the magnitude of the velocity and	θ is its direction.
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The components of position

Analyze the motion of the projectile in the horizontal direction using the following equations:

Equation:

Horizontal motion ($a_x = 0$)

Equation:

$$x = x_0 + v_x t$$

Equation:

$v_x = v_{0x} = \mathbf{v}_x = \text{velocity is a constant.}$

Analyze the motion of the projectile in the vertical direction using the following equations:

Equation:

Vertical motion (Assuming positive direction is up; $a_y = -g = -9.80 \text{ m/s}^2$) $y = y_0 + \frac{1}{2}(v_{0y}$

Equation:

the projectile in the vertical direction using the following equations:

Equation:

$$s = \sqrt{x^2 + y^2}$$

Equation:

$$\theta = \tan^{-1}(y/x)$$

Equation:

$$v = \sqrt{v_x^2 + v_y^2}$$

Equation:

$$\theta_v = \tan^{-1}(v_y/v_x).$$

Recombine the horizontal and vertical components of location and/or velocity using the following equations:

- The maximum height h of a projectile launched with initial vertical velocity v_{0y} is given by **Equation:**

$$h = \frac{v_{0y}^2}{2g}.$$

- The maximum horizontal distance traveled by a projectile is called the **range**. The range R of a projectile on level ground launched at an angle θ_0 above the horizontal with initial speed v_0 is given by **Equation:**

$$R = \frac{v_0^2 \sin 2\theta_0}{g}.$$

Conceptual Questions

Exercise:

Problem:

Answer the following questions for projectile motion on level ground assuming negligible air resistance (the initial angle being neither 0° nor 90°): (a) Is the velocity ever zero? (b) When is the velocity a minimum? A maximum? (c) Can the velocity ever be the same as the initial velocity at a time other than at $t = 0$? (d) Can the speed ever be the same as the initial speed at a time other than at $t = 0$?

Exercise:

Problem:

Answer the following questions for projectile motion on level ground assuming negligible air resistance (the initial angle being neither 0° nor 90°): (a) Is the acceleration ever zero? (b) Is the acceleration ever in the same direction as a component of velocity? (c) Is the acceleration ever opposite in direction to a component of velocity?

Exercise:**Problem:**

For a fixed initial speed, the range of a projectile is determined by the angle at which it is fired. For all but the maximum, there are two angles that give the same range. Considering factors that might affect the ability of an archer to hit a target, such as wind, explain why the smaller angle (closer to the horizontal) is preferable. When would it be necessary for the archer to use the larger angle? Why does the punter in a football game use the higher trajectory?

Exercise:**Problem:**

During a lecture demonstration, a professor places two coins on the edge of a table. She then flicks one of the coins horizontally off the table, simultaneously nudging the other over the edge. Describe the subsequent motion of the two coins, in particular discussing whether they hit the floor at the same time.

Problems & Exercises**Exercise:****Problem:**

A projectile is launched at ground level with an initial speed of 50.0 m/s at an angle of 30.0° above the horizontal. It strikes a target above the ground 3.00 seconds later. What are the x and y distances from where the projectile was launched to where it lands?

Solution:

$$\begin{aligned}x &= 1.30 \text{ m} \times 10^2 \\y &= 30.9 \text{ m}.\end{aligned}$$

Exercise:**Problem:**

A ball is kicked with an initial velocity of 16 m/s in the horizontal direction and 12 m/s in the vertical direction. (a) At what speed does the ball hit the ground? (b) For how long does the ball remain in the air? (c) What maximum height is attained by the ball?

Exercise:**Problem:**

A ball is thrown horizontally from the top of a 60.0-m building and lands 100.0 m from the base of the building. Ignore air resistance. (a) How long is the ball in the air? (b) What must have been the initial horizontal component of the velocity? (c) What is the vertical component of the velocity just before the ball hits the ground? (d) What is the velocity (including both the horizontal and vertical components) of the ball just before it hits the ground?

Solution:

- (a) 3.50 s
- (b) 28.6 m/s (c) 34.3 m/s
- (d) 44.7 m/s, 50.2° below horizontal

Exercise:

Problem:

(a) A daredevil is attempting to jump his motorcycle over a line of buses parked end to end by driving up a 32° ramp at a speed of 40.0 m/s (144 km/h). How many buses can he clear if the top of the takeoff ramp is at the same height as the bus tops and the buses are 20.0 m long? (b) Discuss what your answer implies about the margin of error in this act—that is, consider how much greater the range is than the horizontal distance he must travel to miss the end of the last bus. (Neglect air resistance.)

Exercise:

Problem:

An archer shoots an arrow at a 75.0 m distant target; the bull's-eye of the target is at same height as the release height of the arrow. (a) At what angle must the arrow be released to hit the bull's-eye if its initial speed is 35.0 m/s? In this part of the problem, explicitly show how you follow the steps involved in solving projectile motion problems. (b) There is a large tree halfway between the archer and the target with an overhanging horizontal branch 3.50 m above the release height of the arrow. Will the arrow go over or under the branch?

Solution:

- (a) 18.4°
- (b) The arrow will go over the branch.

Exercise:

Problem:

A rugby player passes the ball 7.00 m across the field, where it is caught at the same height as it left his hand. (a) At what angle was the ball thrown if its initial speed was 12.0 m/s, assuming that the smaller of the two possible angles was used? (b) What other angle gives the same range, and why would it not be used? (c) How long did this pass take?

Exercise:

Problem: Verify the ranges for the projectiles in [\[link\]](#)(a) for $\theta = 45^\circ$ and the given initial velocities.

Solution:

$$R = \frac{v_0^2}{\sin 2\theta_0 g}$$

$$\text{For } \theta = 45^\circ, \quad R = \frac{v_0^2}{g}$$

$$R = 91.8 \text{ m for } v_0 = 30 \text{ m/s}; \quad R = 163 \text{ m for } v_0 = 40 \text{ m/s}; \quad R = 255 \text{ m for } v_0 = 50 \text{ m/s}.$$

Exercise:

Problem:

Verify the ranges shown for the projectiles in [\[link\]](#)(b) for an initial velocity of 50 m/s at the given initial angles.

Exercise:

Problem:

The cannon on a battleship can fire a shell a maximum distance of 32.0 km. (a) Calculate the initial velocity of the shell. (b) What maximum height does it reach? (At its highest, the shell is above 60% of the atmosphere—but air resistance is not really negligible as assumed to make this problem easier.) (c) The ocean is not flat, because the Earth is curved. Assume that the radius of the Earth is 6.37×10^3 km. How many meters lower will its surface be 32.0 km from the ship along a horizontal line parallel to the surface at the ship? Does your answer imply that error introduced by the assumption of a flat Earth in projectile motion is significant here?

Solution:

(a) 560 m/s

(b) 8.00×10^3 m

(c) 80.0 m. This error is not significant because it is only 1% of the answer in part (b).

Exercise:**Problem:**

An arrow is shot from a height of 1.5 m toward a cliff of height H . It is shot with a velocity of 30 m/s at an angle of 60° above the horizontal. It lands on the top edge of the cliff 4.0 s later. (a) What is the height of the cliff? (b) What is the maximum height reached by the arrow along its trajectory? (c) What is the arrow's impact speed just before hitting the cliff?

Exercise:**Problem:**

In the standing broad jump, one squats and then pushes off with the legs to see how far one can jump. Suppose the extension of the legs from the crouch position is 0.600 m and the acceleration achieved from this position is 1.25 times the acceleration due to gravity, g . How far can they jump? State your assumptions. (Increased range can be achieved by swinging the arms in the direction of the jump.)

Solution:

1.50 m, assuming launch angle of 45°

Exercise:**Problem:**

The world long jump record is 8.95 m (Mike Powell, USA, 1991). Treated as a projectile, what is the maximum range obtainable by a person if he has a take-off speed of 9.5 m/s? State your assumptions.

Exercise:**Problem:**

Serving at a speed of 170 km/h, a tennis player hits the ball at a height of 2.5 m and an angle θ below the horizontal. The baseline from which the ball is served is 11.9 m from the net, which is 0.91 m high. What is the angle θ such that the ball just crosses the net? Will the ball land in the service box, which has an outermost service line is 6.40 m from the net?

Solution:

$\theta = 6.1^\circ$

yes, the ball lands at 5.3 m from the net

Exercise:**Problem:**

A football quarterback is moving straight backward at a speed of 2.00 m/s when he throws a pass to a player 18.0 m straight downfield. (a) If the ball is thrown at an angle of 25° relative to the ground and is caught at the same height as it is released, what is its initial speed relative to the ground? (b) How long does it take to get to the receiver? (c) What is its maximum height above its point of release?

Exercise:**Problem:**

Gun sights are adjusted to aim high to compensate for the effect of gravity, effectively making the gun accurate only for a specific range. (a) If a gun is sighted to hit targets that are at the same height as the gun and 100.0 m away, how low will the bullet hit if aimed directly at a target 150.0 m away? The muzzle velocity of the bullet is 275 m/s. (b) Discuss qualitatively how a larger muzzle velocity would affect this problem and what would be the effect of air resistance.

Solution:

(a) -0.486 m

(b) The larger the muzzle velocity, the smaller the deviation in the vertical direction, because the time of flight would be smaller. Air resistance would have the effect of decreasing the time of flight, therefore increasing the vertical deviation.

Exercise:**Problem:**

An eagle is flying horizontally at a speed of 3.00 m/s when the fish in her talons wiggles loose and falls into the lake 5.00 m below. Calculate the velocity of the fish relative to the water when it hits the water.

Exercise:**Problem:**

An owl is carrying a mouse to the chicks in its nest. Its position at that time is 4.00 m west and 12.0 m above the center of the 30.0 cm diameter nest. The owl is flying east at 3.50 m/s at an angle 30.0° below the horizontal when it accidentally drops the mouse. Is the owl lucky enough to have the mouse hit the nest? To answer this question, calculate the horizontal position of the mouse when it has fallen 12.0 m.

Solution:

4.23 m. No, the owl is not lucky; he misses the nest.

Exercise:**Problem:**

Suppose a soccer player kicks the ball from a distance 30 m toward the goal. Find the initial speed of the ball if it just passes over the goal, 2.4 m above the ground, given the initial direction to be 40° above the horizontal.

Exercise:**Problem:**

Can a goalkeeper at her/ his goal kick a soccer ball into the opponent's goal without the ball touching the ground? The distance will be about 95 m. A goalkeeper can give the ball a speed of 30 m/s.

Solution:

No, the maximum range (neglecting air resistance) is about 92 m.

Exercise:

Problem:

The free throw line in basketball is 4.57 m (15 ft) from the basket, which is 3.05 m (10 ft) above the floor. A player standing on the free throw line throws the ball with an initial speed of 8.15 m/s, releasing it at a height of 2.44 m (8 ft) above the floor. At what angle above the horizontal must the ball be thrown to exactly hit the basket? Note that most players will use a large initial angle rather than a flat shot because it allows for a larger margin of error. Explicitly show how you follow the steps involved in solving projectile motion problems.

Exercise:

Problem:

In 2007, Michael Carter (U.S.) set a world record in the shot put with a throw of 24.77 m. What was the initial speed of the shot if he released it at a height of 2.10 m and threw it at an angle of 38.0° above the horizontal? (Although the maximum distance for a projectile on level ground is achieved at 45° when air resistance is neglected, the actual angle to achieve maximum range is smaller; thus, 38° will give a longer range than 45° in the shot put.)

Solution:

15.0 m/s

Exercise:

Problem:

A basketball player is running at 5.00 m/s directly toward the basket when he jumps into the air to dunk the ball. He maintains his horizontal velocity. (a) What vertical velocity does he need to rise 0.750 m above the floor? (b) How far from the basket (measured in the horizontal direction) must he start his jump to reach his maximum height at the same time as he reaches the basket?

Exercise:

Problem:

A football player punts the ball at a 45.0° angle. Without an effect from the wind, the ball would travel 60.0 m horizontally. (a) What is the initial speed of the ball? (b) When the ball is near its maximum height it experiences a brief gust of wind that reduces its horizontal velocity by 1.50 m/s. What distance does the ball travel horizontally?

Solution:

(a) 24.2 m/s

(b) The ball travels a total of 57.4 m with the brief gust of wind.

Exercise:

Problem:

Prove that the trajectory of a projectile is parabolic, having the form $y = ax + bx^2$. To obtain this expression, solve the equation $x = v_{0x}t$ for t and substitute it into the expression for $y = v_{0y}t - (1/2)gt^2$ (These equations describe the x and y positions of a projectile that starts at the origin.) You should obtain an equation of the form $y = ax + bx^2$ where a and b are constants.

Exercise:

Problem:

Derive $R = \frac{v_0^2 \sin 2\theta_0}{g}$ for the range of a projectile on level ground by finding the time t at which y becomes zero and substituting this value of t into the expression for $x - x_0$, noting that $R = x - x_0$

Solution:

$$y - y_0 = 0 = v_{0y}t - \frac{1}{2}gt^2 = (v_0 \sin \theta)t - \frac{1}{2}gt^2,$$

$$\text{so that } t = \frac{2(v_0 \sin \theta)}{g}$$

$x - x_0 = v_{0x}t = (v_0 \cos \theta)t = R$, and substituting for t gives:

$$R = v_0 \cos \theta \left(\frac{2v_0 \sin \theta}{g} \right) = \frac{2v_0^2 \sin \theta \cos \theta}{g}$$

since $2 \sin \theta \cos \theta = \sin 2\theta$, the range is:

$$R = \frac{v_0^2 \sin 2\theta}{g}.$$

Exercise:**Problem:**

Unreasonable Results (a) Find the maximum range of a super cannon that has a muzzle velocity of 4.0 km/s. (b) What is unreasonable about the range you found? (c) Is the premise unreasonable or is the available equation inapplicable? Explain your answer. (d) If such a muzzle velocity could be obtained, discuss the effects of air resistance, thinning air with altitude, and the curvature of the Earth on the range of the super cannon.

Exercise:**Problem:**

Construct Your Own Problem Consider a ball tossed over a fence. Construct a problem in which you calculate the ball's needed initial velocity to just clear the fence. Among the things to determine are; the height of the fence, the distance to the fence from the point of release of the ball, and the height at which the ball is released. You should also consider whether it is possible to choose the initial speed for the ball and just calculate the angle at which it is thrown. Also examine the possibility of multiple solutions given the distances and heights you have chosen.

Glossary

air resistance

a frictional force that slows the motion of objects as they travel through the air; when solving basic physics problems, air resistance is assumed to be zero

kinematics

the study of motion without regard to mass or force

motion

displacement of an object as a function of time

projectile

an object that travels through the air and experiences only acceleration due to gravity

projectile motion

the motion of an object that is subject only to the acceleration of gravity

range

the maximum horizontal distance that a projectile travels

trajectory

the path of a projectile through the air

Addition of Velocities

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Apply principles of vector addition to determine relative velocity.
- Explain the significance of the observer in the measurement of velocity.

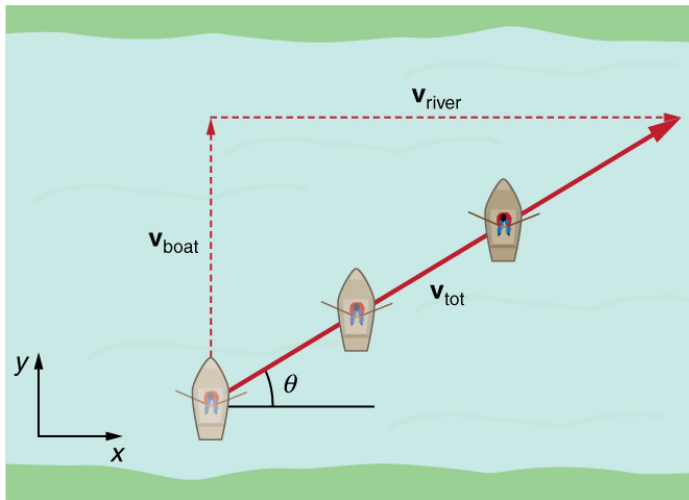
The information presented in this section supports the following AP® learning objectives and science practices:

- **3.A.1.1** The student is able to express the motion of an object using narrative, mathematical, and graphical representations. (S.P. 1.5, 2.1, 2.2)
- **3.A.1.3** The student is able to analyze experimental data describing the motion of an object and is able to express the results of the analysis using narrative, mathematical, and graphical representations. (S.P. 5.1)

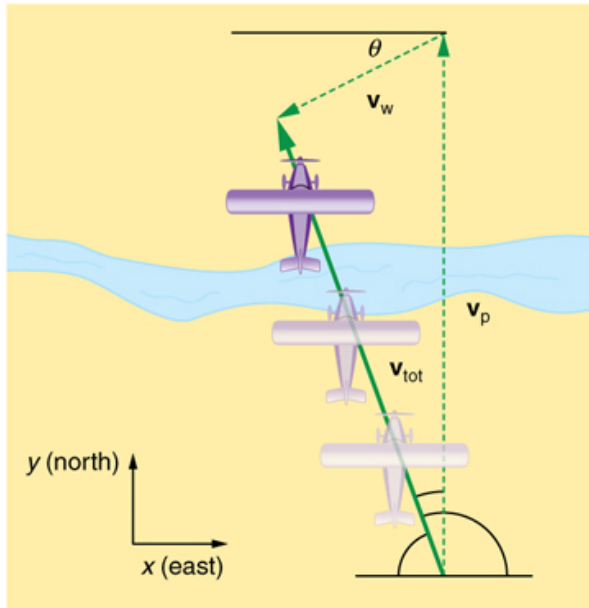
Relative Velocity

If a person rows a boat across a rapidly flowing river and tries to head directly for the other shore, the boat instead moves *diagonally* relative to the shore, as in [\[link\]](#). The boat does not move in the direction in which it is pointed. The reason, of course, is that the river carries the boat downstream. Similarly, if a small airplane flies overhead in a strong crosswind, you can

sometimes see that the plane is not moving in the direction in which it is pointed, as illustrated in [\[link\]](#). The plane is moving straight ahead relative to the air, but the movement of the air mass relative to the ground carries it sideways.



A boat trying to head straight across a river will actually move diagonally relative to the shore as shown. Its total velocity (solid arrow) relative to the shore is the sum of its velocity relative to the river plus the velocity of the river relative to the shore.



An airplane heading straight north is instead carried to the west and slowed down by wind.

The plane does not move relative to the ground in the direction it points; rather, it moves in the direction of its total velocity (solid arrow).

In each of these situations, an object has a **velocity** relative to a medium (such as a river) and that medium has a velocity relative to an observer on solid ground. The velocity of the object *relative to the observer* is the sum of these velocity vectors, as indicated in [\[link\]](#) and [\[link\]](#). These situations are only two of many in which it is useful to add velocities. In this module, we first re-examine how to add velocities and then consider certain aspects of what relative velocity means.

How do we add velocities? Velocity is a vector (it has both magnitude and direction); the rules of **vector addition** discussed in [Vector Addition and Subtraction: Graphical Methods](#) and [Vector Addition and Subtraction: Analytical Methods](#) apply to the addition of velocities, just as they do for

any other vectors. In one-dimensional motion, the addition of velocities is simple—they add like ordinary numbers. For example, if a field hockey player is moving at 5 m/s straight toward the goal and drives the ball in the same direction with a velocity of 30 m/s relative to her body, then the velocity of the ball is 35 m/s relative to the stationary, profusely sweating goalkeeper standing in front of the goal.

In two-dimensional motion, either graphical or analytical techniques can be used to add velocities. We will concentrate on analytical techniques. The following equations give the relationships between the magnitude and direction of velocity (v and θ) and its components (v_x and v_y) along the x - and y -axes of an appropriately chosen coordinate system:

Equation:

$$v_x = v \cos \theta$$

Equation:

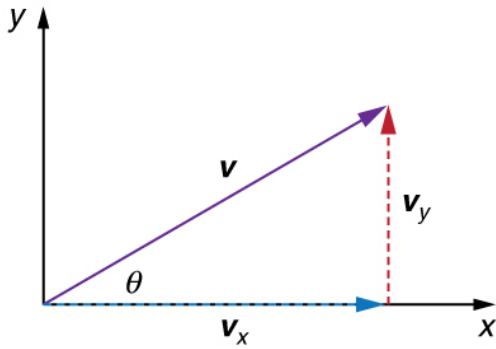
$$v_y = v \sin \theta$$

Equation:

$$v = \sqrt{v_x^2 + v_y^2}$$

Equation:

$$\theta = \tan^{-1}(v_y/v_x).$$



The velocity, v , of an object traveling at an angle θ to the horizontal axis is the sum of component vectors \mathbf{v}_x and \mathbf{v}_y .

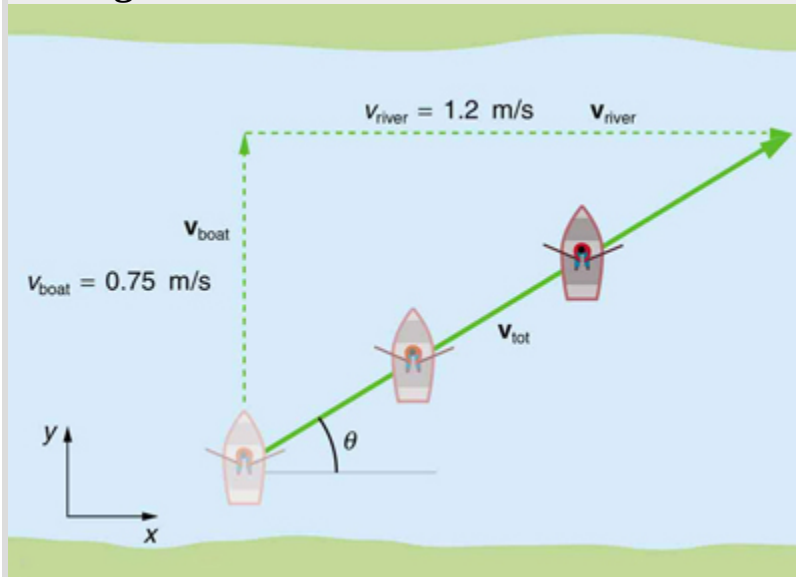
These equations are valid for any vectors and are adapted specifically for velocity. The first two equations are used to find the components of a velocity when its magnitude and direction are known. The last two are used to find the magnitude and direction of velocity when its components are known.

Note:**Take-Home Experiment: Relative Velocity of a Boat**

Fill a bathtub half-full of water. Take a toy boat or some other object that floats in water. Unplug the drain so water starts to drain. Try pushing the boat from one side of the tub to the other and perpendicular to the flow of water. Which way do you need to push the boat so that it ends up immediately opposite? Compare the directions of the flow of water, heading of the boat, and actual velocity of the boat.

Example:

Adding Velocities: A Boat on a River



A boat attempts to travel straight across a river at a speed 0.75 m/s. The current in the river, however, flows at a speed of 1.20 m/s to the right. What is the total displacement of the boat relative to the shore?

Refer to [\[link\]](#), which shows a boat trying to go straight across the river. Let us calculate the magnitude and direction of the boat's velocity relative to an observer on the shore, \mathbf{v}_{tot} . The velocity of the boat, \mathbf{v}_{boat} , is 0.75 m/s in the y -direction relative to the river and the velocity of the river, $\mathbf{v}_{\text{river}}$, is 1.20 m/s to the right.

Strategy

We start by choosing a coordinate system with its x -axis parallel to the velocity of the river, as shown in [\[link\]](#). Because the boat is directed straight toward the other shore, its velocity relative to the water is parallel to the y -axis and perpendicular to the velocity of the river. Thus, we can add the two velocities by using the equations $v_{\text{tot}} = \sqrt{v_x^2 + v_y^2}$ and $\theta = \tan^{-1}(v_y/v_x)$ directly.

Solution

The magnitude of the total velocity is

Equation:

$$v_{\text{tot}} = \sqrt{v_x^2 + v_y^2},$$

where

Equation:

$$v_x = v_{\text{river}} = 1.20 \text{ m/s}$$

and

Equation:

$$v_y = v_{\text{boat}} = 0.750 \text{ m/s}.$$

Thus,

Equation:

$$v_{\text{tot}} = \sqrt{(1.20 \text{ m/s})^2 + (0.750 \text{ m/s})^2}$$

yielding

Equation:

$$v_{\text{tot}} = 1.42 \text{ m/s}.$$

The direction of the total velocity θ is given by:

Equation:

$$\theta = \tan^{-1}(v_y/v_x) = \tan^{-1}(0.750/1.20).$$

This equation gives

Equation:

$$\theta = 32.0^\circ.$$

Discussion

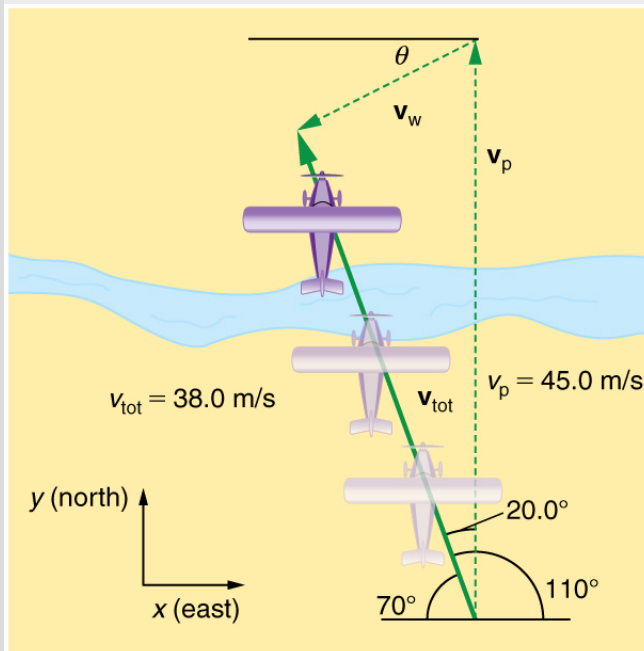
Both the magnitude v and the direction θ of the total velocity are consistent with [\[link\]](#). Note that because the velocity of the river is large compared

with the velocity of the boat, it is swept rapidly downstream. This result is evidenced by the small angle (only 32.0°) the total velocity has relative to the riverbank.

Example:

Calculating Velocity: Wind Velocity Causes an Airplane to Drift

Calculate the wind velocity for the situation shown in [\[link\]](#). The plane is known to be moving at 45.0 m/s due north relative to the air mass, while its velocity relative to the ground (its total velocity) is 38.0 m/s in a direction 20.0° west of north.



An airplane is known to be heading north at 45.0 m/s , though its velocity relative to the ground is 38.0 m/s at an angle west of north. What is the speed and direction of the wind?

Strategy

In this problem, somewhat different from the previous example, we know the total velocity \mathbf{v}_{tot} and that it is the sum of two other velocities, \mathbf{v}_w (the wind) and \mathbf{v}_p (the plane relative to the air mass). The quantity \mathbf{v}_p is known, and we are asked to find \mathbf{v}_w . None of the velocities are perpendicular, but it is possible to find their components along a common set of perpendicular axes. If we can find the components of \mathbf{v}_w , then we can combine them to solve for its magnitude and direction. As shown in [\[link\]](#), we choose a coordinate system with its x-axis due east and its y-axis due north (parallel to \mathbf{v}_p). (You may wish to look back at the discussion of the addition of vectors using perpendicular components in [Vector Addition and Subtraction: Analytical Methods](#).)

Solution

Because \mathbf{v}_{tot} is the vector sum of the \mathbf{v}_w and \mathbf{v}_p , its x- and y-components are the sums of the x- and y-components of the wind and plane velocities. Note that the plane only has vertical component of velocity so $v_{px} = 0$ and $v_{py} = v_p$. That is,

Equation:

$$v_{\text{tot}x} = v_{wx}$$

and

Equation:

$$v_{\text{tot}y} = v_{wy} + v_p.$$

We can use the first of these two equations to find v_{wx} :

Equation:

$$v_{wx} = v_{\text{tot}x} = v_{\text{tot}} \cos 110^\circ.$$

Because $v_{\text{tot}} = 38.0 \text{ m/s}$ and $\cos 110^\circ = -0.342$ we have

Equation:

$$v_{wx} = (38.0 \text{ m/s})(-0.342) = -13.0 \text{ m/s}.$$

The minus sign indicates motion west which is consistent with the diagram.

Now, to find v_{wy} we note that

Equation:

$$v_{\text{tot}y} = v_{wy} + v_p$$

Here $v_{\text{tot}y} = v_{\text{tot}} \sin 110^\circ$; thus,

Equation:

$$v_{wy} = (38.0 \text{ m/s})(0.940) - 45.0 \text{ m/s} = -9.29 \text{ m/s}.$$

This minus sign indicates motion south which is consistent with the diagram.

Now that the perpendicular components of the wind velocity v_{wx} and v_{wy} are known, we can find the magnitude and direction of \mathbf{v}_w . First, the magnitude is

Equation:

$$\begin{aligned} v_w &= \sqrt{v_{wx}^2 + v_{wy}^2} \\ &= \sqrt{(-13.0 \text{ m/s})^2 + (-9.29 \text{ m/s})^2} \end{aligned}$$

so that

Equation:

$$v_w = 16.0 \text{ m/s}.$$

The direction is:

Equation:

$$\theta = \tan^{-1}(v_{wy}/v_{wx}) = \tan^{-1}(-9.29/-13.0)$$

giving

Equation:

$$\theta = 35.6^\circ.$$

Discussion

The wind's speed and direction are consistent with the significant effect the wind has on the total velocity of the plane, as seen in [\[link\]](#). Because the

plane is fighting a strong combination of crosswind and head-wind, it ends up with a total velocity significantly less than its velocity relative to the air mass as well as heading in a different direction.

Note that in both of the last two examples, we were able to make the mathematics easier by choosing a coordinate system with one axis parallel to one of the velocities. We will repeatedly find that choosing an appropriate coordinate system makes problem solving easier. For example, in projectile motion we always use a coordinate system with one axis parallel to gravity.

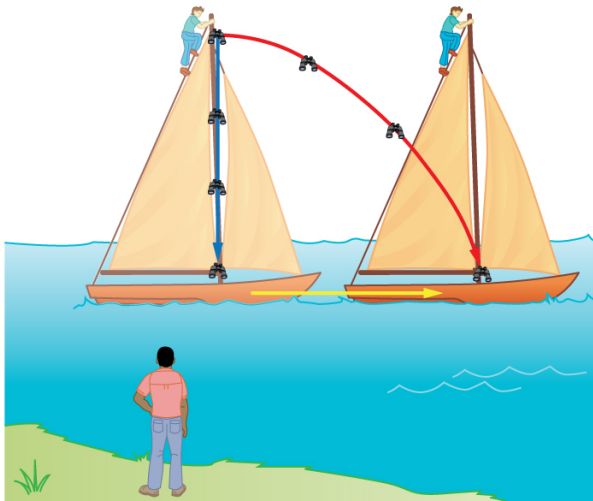
Relative Velocities and Classical Relativity

When adding velocities, we have been careful to specify that the *velocity is relative to some reference frame*. These velocities are called **relative velocities**. For example, the velocity of an airplane relative to an air mass is different from its velocity relative to the ground. Both are quite different from the velocity of an airplane relative to its passengers (which should be close to zero). Relative velocities are one aspect of **relativity**, which is defined to be the study of how different observers moving relative to each other measure the same phenomenon.

Nearly everyone has heard of relativity and immediately associates it with Albert Einstein (1879–1955), the greatest physicist of the 20th century. Einstein revolutionized our view of nature with his *modern* theory of relativity, which we shall study in later chapters. The relative velocities in this section are actually aspects of classical relativity, first discussed correctly by Galileo and Isaac Newton. **Classical relativity** is limited to situations where speeds are less than about 1% of the speed of light—that is, less than 3,000 km/s. Most things we encounter in daily life move slower than this speed.

Let us consider an example of what two different observers see in a situation analyzed long ago by Galileo. Suppose a sailor at the top of a mast on a moving ship drops his binoculars. Where will it hit the deck? Will it hit

at the base of the mast, or will it hit behind the mast because the ship is moving forward? The answer is that if air resistance is negligible, the binoculars will hit at the base of the mast at a point directly below its point of release. Now let us consider what two different observers see when the binoculars drop. One observer is on the ship and the other on shore. The binoculars have no horizontal velocity relative to the observer on the ship, and so he sees them fall straight down the mast. (See [\[link\]](#).) To the observer on shore, the binoculars and the ship have the *same* horizontal velocity, so both move the same distance forward while the binoculars are falling. This observer sees the curved path shown in [\[link\]](#). Although the paths look different to the different observers, each sees the same result—the binoculars hit at the base of the mast and not behind it. To get the correct description, it is crucial to correctly specify the velocities relative to the observer.



Classical relativity. The same motion as viewed by two different observers. An observer on the moving ship sees the binoculars dropped from the top of its mast fall straight down. An observer on shore sees the binoculars take the curved path,

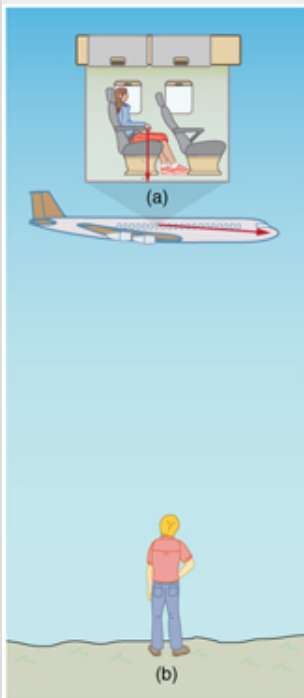
moving forward with the ship.

Both observers see the binoculars strike the deck at the base of the mast. The initial horizontal velocity is different relative to the two observers. (The ship is shown moving rather fast to emphasize the effect.)

Example:

Calculating Relative Velocity: An Airline Passenger Drops a Coin

An airline passenger drops a coin while the plane is moving at 260 m/s. What is the velocity of the coin when it strikes the floor 1.50 m below its point of release: (a) Measured relative to the plane? (b) Measured relative to the Earth?



The motion of

a coin
dropped
inside an
airplane as
viewed by
two different
observers. (a)
An observer
in the plane
sees the coin
fall straight
down. (b) An
observer on
the ground
sees the coin
move almost
horizontally.

Strategy

Both problems can be solved with the techniques for falling objects and projectiles. In part (a), the initial velocity of the coin is zero relative to the plane, so the motion is that of a falling object (one-dimensional). In part (b), the initial velocity is 260 m/s horizontal relative to the Earth and gravity is vertical, so this motion is a projectile motion. In both parts, it is best to use a coordinate system with vertical and horizontal axes.

Solution for (a)

Using the given information, we note that the initial velocity and position are zero, and the final position is 1.50 m. The final velocity can be found using the equation:

Equation:

$$v_y^2 = v_{0y}^2 - 2g(y - y_0).$$

Substituting known values into the equation, we get

Equation:

$$v_y^2 = 0^2 - 2(9.80 \text{ m/s}^2)(-1.50 \text{ m} - 0 \text{ m}) = 29.4 \text{ m}^2/\text{s}^2$$

yielding

Equation:

$$v_y = -5.42 \text{ m/s}.$$

We know that the square root of 29.4 has two roots: 5.42 and -5.42. We choose the negative root because we know that the velocity is directed downwards, and we have defined the positive direction to be upwards. There is no initial horizontal velocity relative to the plane and no horizontal acceleration, and so the motion is straight down relative to the plane.

Solution for (b)

Because the initial vertical velocity is zero relative to the ground and vertical motion is independent of horizontal motion, the final vertical velocity for the coin relative to the ground is $v_y = -5.42 \text{ m/s}$, the same as found in part (a). In contrast to part (a), there now is a horizontal component of the velocity. However, since there is no horizontal acceleration, the initial and final horizontal velocities are the same and $v_x = 260 \text{ m/s}$. The x - and y -components of velocity can be combined to find the magnitude of the final velocity:

Equation:

$$v = \sqrt{v_x^2 + v_y^2}.$$

Thus,

Equation:

$$v = \sqrt{(260 \text{ m/s})^2 + (-5.42 \text{ m/s})^2}$$

yielding

Equation:

$$v = 260.06 \text{ m/s}.$$

The direction is given by:

Equation:

$$\theta = \tan^{-1}(v_y/v_x) = \tan^{-1}(-5.42/260)$$

so that

Equation:

$$\theta = \tan^{-1}(-0.0208) = -1.19^\circ.$$

Discussion

In part (a), the final velocity relative to the plane is the same as it would be if the coin were dropped from rest on the Earth and fell 1.50 m. This result fits our experience; objects in a plane fall the same way when the plane is flying horizontally as when it is at rest on the ground. This result is also true in moving cars. In part (b), an observer on the ground sees a much different motion for the coin. The plane is moving so fast horizontally to begin with that its final velocity is barely greater than the initial velocity. Once again, we see that in two dimensions, vectors do not add like ordinary numbers—the final velocity v in part (b) is *not* $(260 - 5.42)$ m/s; rather, it is 260.06 m/s. The velocity's magnitude had to be calculated to five digits to see any difference from that of the airplane. The motions as seen by different observers (one in the plane and one on the ground) in this example are analogous to those discussed for the binoculars dropped from the mast of a moving ship, except that the velocity of the plane is much larger, so that the two observers see *very* different paths. (See [\[link\]](#).) In addition, both observers see the coin fall 1.50 m vertically, but the one on the ground also sees it move forward 144 m (this calculation is left for the reader). Thus, one observer sees a vertical path, the other a nearly horizontal path.

Note:

Making Connections: Relativity and Einstein

Because Einstein was able to clearly define how measurements are made (some involve light) and because the speed of light is the same for all

observers, the outcomes are spectacularly unexpected. Time varies with observer, energy is stored as increased mass, and more surprises await.

Note:

PhET Explorations: Motion in 2D

Try the new "Ladybug Motion 2D" simulation for the latest updated version. Learn about position, velocity, and acceleration vectors. Move the ball with the mouse or let the simulation move the ball in four types of motion (2 types of linear, simple harmonic, circle).

[Motion in 2D](#)

Summary

- Velocities in two dimensions are added using the same analytical vector techniques, which are rewritten as

Equation:

$$v_x = v \cos \theta$$

Equation:

$$v_y = v \sin \theta$$

Equation:

$$v = \sqrt{v_x^2 + v_y^2}$$

Equation:

$$\theta = \tan^{-1}(v_y/v_x).$$

- Relative velocity is the velocity of an object as observed from a particular reference frame, and it varies dramatically with reference frame.
- **Relativity** is the study of how different observers measure the same phenomenon, particularly when the observers move relative to one another. **Classical relativity** is limited to situations where speed is less than about 1% of the speed of light (3000 km/s).

Conceptual Questions

Exercise:

Problem:

What frame or frames of reference do you instinctively use when driving a car? When flying in a commercial jet airplane?

Exercise:

Problem:

A basketball player dribbling down the court usually keeps his eyes fixed on the players around him. He is moving fast. Why doesn't he need to keep his eyes on the ball?

Exercise:

Problem:

If someone is riding in the back of a pickup truck and throws a softball straight backward, is it possible for the ball to fall straight down as viewed by a person standing at the side of the road? Under what condition would this occur? How would the motion of the ball appear to the person who threw it?

Exercise:**Problem:**

The hat of a jogger running at constant velocity falls off the back of his head. Draw a sketch showing the path of the hat in the jogger's frame of reference. Draw its path as viewed by a stationary observer.

Exercise:**Problem:**

A clod of dirt falls from the bed of a moving truck. It strikes the ground directly below the end of the truck. What is the direction of its velocity relative to the truck just before it hits? Is this the same as the direction of its velocity relative to ground just before it hits? Explain your answers.

Problems & Exercises**Exercise:****Problem:**

Bryan Allen pedaled a human-powered aircraft across the English Channel from the cliffs of Dover to Cap Gris-Nez on June 12, 1979. (a) He flew for 169 min at an average velocity of 3.53 m/s in a direction 45° south of east. What was his total displacement? (b) Allen encountered a headwind averaging 2.00 m/s almost precisely in the opposite direction of his motion relative to the Earth. What was his average velocity relative to the air? (c) What was his total displacement relative to the air mass?

Solution:

(a) 35.8 km, 45° south of east

(b) 5.53 m/s, 45° south of east

(c) 56.1 km, 45° south of east

Exercise:

Problem:

A seagull flies at a velocity of 9.00 m/s straight into the wind. (a) If it takes the bird 20.0 min to travel 6.00 km relative to the Earth, what is the velocity of the wind? (b) If the bird turns around and flies with the wind, how long will he take to return 6.00 km? (c) Discuss how the wind affects the total round-trip time compared to what it would be with no wind.

Exercise:

Problem:

Near the end of a marathon race, the first two runners are separated by a distance of 45.0 m. The front runner has a velocity of 3.50 m/s, and the second a velocity of 4.20 m/s. (a) What is the velocity of the second runner relative to the first? (b) If the front runner is 250 m from the finish line, who will win the race, assuming they run at constant velocity? (c) What distance ahead will the winner be when she crosses the finish line?

Solution:

(a) 0.70 m/s faster

(b) Second runner wins

(c) 4.17 m

Exercise:

Problem:

Verify that the coin dropped by the airline passenger in the [\[link\]](#) travels 144 m horizontally while falling 1.50 m in the frame of reference of the Earth.

Exercise:

Problem:

A football quarterback is moving straight backward at a speed of 2.00 m/s when he throws a pass to a player 18.0 m straight downfield. The ball is thrown at an angle of 25.0° relative to the ground and is caught at the same height as it is released. What is the initial velocity of the ball *relative to the quarterback* ?

Solution:

17.0 m/s, 22.1°

Exercise:**Problem:**

A ship sets sail from Rotterdam, The Netherlands, heading due north at 7.00 m/s relative to the water. The local ocean current is 1.50 m/s in a direction 40.0° north of east. What is the velocity of the ship relative to the Earth?

Exercise:**Problem:**

(a) A jet airplane flying from Darwin, Australia, has an air speed of 260 m/s in a direction 5.0° south of west. It is in the jet stream, which is blowing at 35.0 m/s in a direction 15° south of east. What is the velocity of the airplane relative to the Earth? (b) Discuss whether your answers are consistent with your expectations for the effect of the wind on the plane's path.

Solution:

(a) 230 m/s, 8.0° south of west

(b) The wind should make the plane travel slower and more to the south, which is what was calculated.

Exercise:

Problem:

(a) In what direction would the ship in [\[link\]](#) have to travel in order to have a velocity straight north relative to the Earth, assuming its speed relative to the water remains 7.00 m/s ? (b) What would its speed be relative to the Earth?

Exercise:**Problem:**

(a) Another airplane is flying in a jet stream that is blowing at 45.0 m/s in a direction 20° south of east (as in [\[link\]](#)). Its direction of motion relative to the Earth is 45.0° south of west, while its direction of travel relative to the air is 5.00° south of west. What is the airplane's speed relative to the air mass? (b) What is the airplane's speed relative to the Earth?

Solution:

(a) 63.5 m/s

(b) 29.6 m/s

Exercise:**Problem:**

A sandal is dropped from the top of a 15.0-m -high mast on a ship moving at 1.75 m/s due south. Calculate the velocity of the sandal when it hits the deck of the ship: (a) relative to the ship and (b) relative to a stationary observer on shore. (c) Discuss how the answers give a consistent result for the position at which the sandal hits the deck.

Exercise:

Problem:

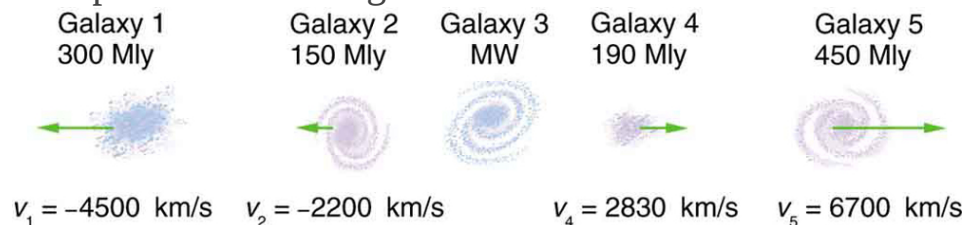
The velocity of the wind relative to the water is crucial to sailboats. Suppose a sailboat is in an ocean current that has a velocity of 2.20 m/s in a direction 30.0° east of north relative to the Earth. It encounters a wind that has a velocity of 4.50 m/s in a direction of 50.0° south of west relative to the Earth. What is the velocity of the wind relative to the water?

Solution:

6.68 m/s, 53.3° south of west

Exercise:**Problem:**

The great astronomer Edwin Hubble discovered that all distant galaxies are receding from our Milky Way Galaxy with velocities proportional to their distances. [\[link\]](#) illustrates this for five galaxies lying along a straight line, with the Milky Way Galaxy at the center. Using the data from the figure, calculate the velocities: (a) relative to galaxy 2 and (b) relative to galaxy 5. The results mean that observers on all galaxies will see themselves at the center of the expanding universe, and they would likely be aware of relative velocities, concluding that it is not possible to locate the center of expansion with the given information.



Five galaxies on a straight line, showing their distances and velocities relative to the Milky Way (MW) Galaxy. The distances are in millions of light years (Mly), where a light year is the distance light

travels in one year. The velocities are nearly proportional to the distances. The sizes of the galaxies are greatly exaggerated; an average galaxy is about 0.1 Mly across.

Exercise:

Problem:

- (a) Use the distance and velocity data in [\[link\]](#) to find the rate of expansion as a function of distance.
- (b) If you extrapolate back in time, how long ago would all of the galaxies have been at approximately the same position? The two parts of this problem give you some idea of how the Hubble constant for universal expansion and the time back to the Big Bang are determined, respectively.

Solution:

(a) $H_{\text{average}} = 14.9 \frac{\text{km/s}}{\text{Mly}}$

(b) 20.2 billion years

Exercise:

Problem:

An athlete crosses a 25-m-wide river by swimming perpendicular to the water current at a speed of 0.5 m/s relative to the water. He reaches the opposite side at a distance 40 m downstream from his starting point. How fast is the water in the river flowing with respect to the ground? What is the speed of the swimmer with respect to a friend at rest on the ground?

Exercise:

Problem:

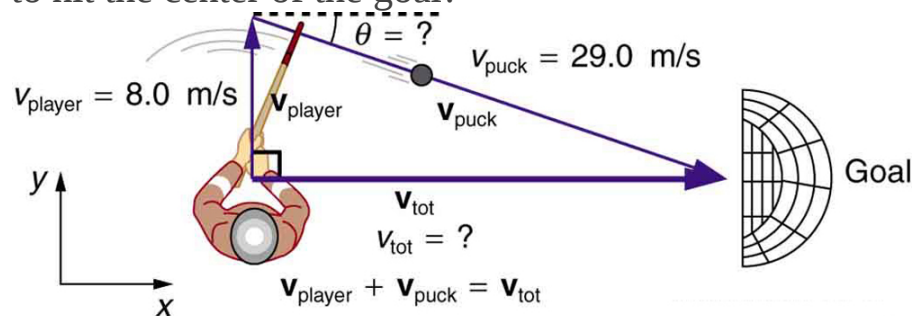
A ship sailing in the Gulf Stream is heading 25.0° west of north at a speed of 4.00 m/s relative to the water. Its velocity relative to the Earth is 4.80 m/s 5.00° west of north. What is the velocity of the Gulf Stream? (The velocity obtained is typical for the Gulf Stream a few hundred kilometers off the east coast of the United States.)

Solution:

1.72 m/s , 42.3° north of east

Exercise:**Problem:**

An ice hockey player is moving at 8.00 m/s when he hits the puck toward the goal. The speed of the puck relative to the player is 29.0 m/s . The line between the center of the goal and the player makes a 90.0° angle relative to his path as shown in [\[link\]](#). What angle must the puck's velocity make relative to the player (in his frame of reference) to hit the center of the goal?



An ice hockey player moving across the rink must shoot backward to give the puck a velocity toward the goal.

Exercise:

Problem:

Unreasonable Results Suppose you wish to shoot supplies straight up to astronauts in an orbit 36,000 km above the surface of the Earth. (a) At what velocity must the supplies be launched? (b) What is unreasonable about this velocity? (c) Is there a problem with the relative velocity between the supplies and the astronauts when the supplies reach their maximum height? (d) Is the premise unreasonable or is the available equation inapplicable? Explain your answer.

Exercise:**Problem:**

Unreasonable Results A commercial airplane has an air speed of 280 m/s due east and flies with a strong tailwind. It travels 3000 km in a direction 5° south of east in 1.50 h. (a) What was the velocity of the plane relative to the ground? (b) Calculate the magnitude and direction of the tailwind's velocity. (c) What is unreasonable about both of these velocities? (d) Which premise is unreasonable?

Exercise:**Problem:**

Construct Your Own Problem Consider an airplane headed for a runway in a cross wind. Construct a problem in which you calculate the angle the airplane must fly relative to the air mass in order to have a velocity parallel to the runway. Among the things to consider are the direction of the runway, the wind speed and direction (its velocity) and the speed of the plane relative to the air mass. Also calculate the speed of the airplane relative to the ground. Discuss any last minute maneuvers the pilot might have to perform in order for the plane to land with its wheels pointing straight down the runway.

Glossary

classical relativity

the study of relative velocities in situations where speeds are less than about 1% of the speed of light—that is, less than 3000 km/s

relative velocity

the velocity of an object as observed from a particular reference frame

relativity

the study of how different observers moving relative to each other measure the same phenomenon

velocity

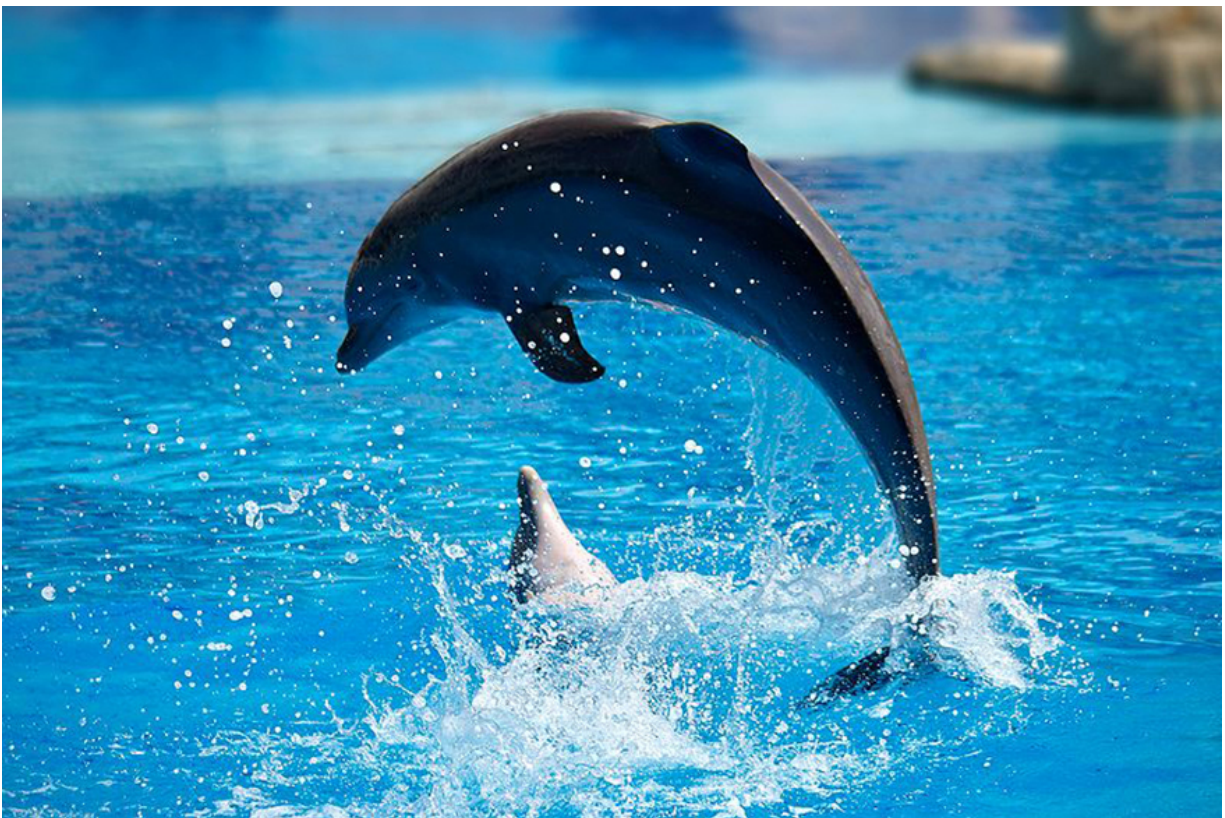
speed in a given direction

vector addition

the rules that apply to adding vectors together

Connection for AP® Courses
class="introduction"

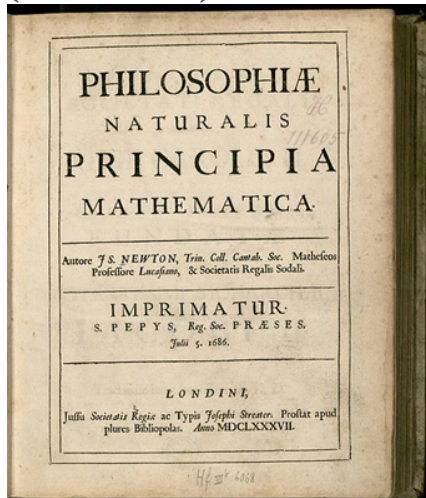
Newton's laws of motion describe the motion of the dolphin's path.
(credit: Jin Jang)



Motion draws our attention. Motion itself can be beautiful, causing us to marvel at the forces needed to achieve spectacular motion, such as that of a

jumping dolphin, a leaping pole vaulter, a bird in flight, or an orbiting satellite. The study of motion is kinematics, but kinematics only *describes* the way objects move—their velocity and their acceleration. **Dynamics** considers the forces that affect the motion of moving objects and systems. Newton's laws of motion are the foundation of dynamics. These laws provide an example of the breadth and simplicity of principles under which nature functions. They are also universal laws in that they apply to situations on Earth as well as in space.

Isaac Newton's (1642–1727) laws of motion were just one part of the monumental work that has made him legendary. The development of Newton's laws marks the transition from the Renaissance into the modern era. This transition was characterized by a revolutionary change in the way people thought about the physical universe. For many centuries natural philosophers had debated the nature of the universe based largely on certain rules of logic, with great weight given to the thoughts of earlier classical philosophers such as Aristotle (384–322 BC). Among the many great thinkers who contributed to this change were Newton and Galileo Galilei (1564–1642).



Isaac Newton's
monumental work,
*Philosophiæ
Naturalis Principia
Mathematica*, was
published in 1687. It

proposed scientific
laws that are still
used today to
describe the motion
of objects. (credit:
Service commun de
la documentation de
l'Université de
Strasbourg)

Galileo was instrumental in establishing *observation* as the absolute determinant of truth, rather than “logical” argument. Galileo’s use of the telescope was his most notable achievement in demonstrating the importance of observation. He discovered moons orbiting Jupiter and made other observations that were inconsistent with certain ancient ideas and religious dogma. For this reason, and because of the manner in which he dealt with those in authority, Galileo was tried by the Inquisition and punished. He spent the final years of his life under a form of house arrest. Because others before Galileo had also made discoveries by *observing* the nature of the universe and because repeated observations verified those of Galileo, his work could not be suppressed or denied. After his death, his work was verified by others, and his ideas were eventually accepted by the church and scientific communities.

Galileo also contributed to the formulation of what is now called Newton’s first law of motion. Newton made use of the work of his predecessors, which enabled him to develop laws of motion, discover the law of gravity, invent calculus, and make great contributions to the theories of light and color. It is amazing that many of these developments were made by Newton working alone, without the benefit of the usual interactions that take place among scientists today.

Newton’s laws are introduced along with Big Idea 3, that interactions can be described by forces. These laws provide a theoretical basis for studying motion depending on interactions between the objects. In particular,

Newton's laws are applicable to all forces in inertial frames of references (Enduring Understanding 3.A). We will find that all forces are vectors; that is, forces always have both a magnitude and a direction (Essential Knowledge 3.A.2). Furthermore, we will learn that all forces are a result of interactions between two or more objects (Essential Knowledge 3.A.3). These interactions between any two objects are described by Newton's third law, stating that the forces exerted on these objects are equal in magnitude and opposite in direction to each other (Essential Knowledge 3.A.4).

We will discover that there is an empirical cause-effect relationship between the net force exerted on an object of mass m and its acceleration, with this relationship described by Newton's second law (Enduring Understanding 3.B). This supports Big Idea 1, that inertial mass is a property of an object or a system. The mass of an object or a system is one of the factors affecting changes in motion when an object or a system interacts with other objects or systems (Essential Knowledge 1.C.1). Another is the net force on an object, which is the vector sum of all the forces exerted on the object (Essential Knowledge 3.B.1). To analyze this, we use free-body diagrams to visualize the forces exerted on a given object in order to find the net force and analyze the object's motion (Essential Knowledge 3.B.2).

Thinking of these objects as systems is a concept introduced in this chapter, where a system is a collection of elements that could be considered as a single object without any internal structure (Essential Knowledge 5.A.1). This will support Big Idea 5, that changes that occur to the system due to interactions are governed by conservation laws. These conservation laws will be the focus of later chapters in this book. They explain whether quantities are conserved in the given system or change due to transfer to or from another system due to interactions between the systems (Enduring Understanding 5.A).

Furthermore, when a situation involves more than one object, it is important to define the system and analyze the motion of a whole system, not its elements, based on analysis of external forces on the system. This supports Big Idea 4, that interactions between systems cause changes in those systems. All kinematics variables in this case describe the motion of the center of mass of the system (Essential Knowledge 4.A.1, Essential

Knowledge 4.A.2). The internal forces between the elements of the system do not affect the velocity of the center of mass (Essential Knowledge 4.A.3). The velocity of the center of mass will change only if there is a net external force exerted on the system (Enduring Understanding 4.A).

We will learn that some of these interactions can be explained by the existence of fields extending through space, supporting Big Idea 2. For example, any object that has mass creates a gravitational field in space (Enduring Understanding 2.B). Any material object (one that has mass) placed in the gravitational field will experience gravitational force (Essential Knowledge 2.B.1).

Forces may be categorized as contact or long-distance (Enduring Understanding 3.C). In this chapter we will work with both. An example of a long-distance force is gravitation (Essential Knowledge 3.C.1). Contact forces, such as tension, friction, normal force, and the force of a spring, result from interatomic electric forces at the microscopic level (Essential Knowledge 3.C.4).

It was not until the advent of modern physics early in the twentieth century that it was discovered that Newton's laws of motion produce a good approximation to motion only when the objects are moving at speeds much, much less than the speed of light and when those objects are larger than the size of most molecules (about 10^{-9} m in diameter). These constraints define the realm of classical mechanics, as discussed in [Introduction to the Nature of Science and Physics](#). At the beginning of the twentieth century, Albert Einstein (1879–1955) developed the theory of relativity and, along with many other scientists, quantum theory. Quantum theory does not have the constraints present in classical physics. All of the situations we consider in this chapter, and all those preceding the introduction of relativity in **Special Relativity**, are in the realm of classical physics.

The development of special relativity and empirical observations at atomic scales led to the idea that there are four basic forces that account for all known phenomena. These forces are called fundamental (Enduring Understanding 3.G). The properties of gravitational (Essential Knowledge 3.G.1) and electromagnetic (Essential Knowledge 3.G.2) forces are explained in more detail.

Big Idea 1 Objects and systems have properties such as mass and charge. Systems may have internal structure.

Essential Knowledge 1.C.1 Inertial mass is the property of an object or a system that determines how its motion changes when it interacts with other objects or systems.

Big Idea 2 Fields existing in space can be used to explain interactions.

Enduring Understanding 2.A A field associates a value of some physical quantity with every point in space. Field models are useful for describing interactions that occur at a distance (long-range forces) as well as a variety of other physical phenomena.

Essential Knowledge 2.A.1 A vector field gives, as a function of position (and perhaps time), the value of a physical quantity that is described by a vector.

Essential Knowledge 2.A.2 A scalar field gives the value of a physical quantity.

Enduring Understanding 2.B A gravitational field is caused by an object with mass.

Essential Knowledge 2.B.1 A gravitational field g at the location of an object with mass m causes a gravitational force of magnitude mg to be exerted on the object in the direction of the field.

Big Idea 3 The interactions of an object with other objects can be described by forces.

Enduring Understanding 3.A All forces share certain common characteristics when considered by observers in inertial reference frames.

Essential Knowledge 3.A.2 Forces are described by vectors.

Essential Knowledge 3.A.3 A force exerted on an object is always due to the interaction of that object with another object.

Essential Knowledge 3.A.4 If one object exerts a force on a second object, the second object always exerts a force of equal magnitude on the first object in the opposite direction.

Enduring Understanding 3.B Classically, the acceleration of an object interacting with other objects can be predicted by using $a = \sum F/m$.

Essential Knowledge 3.B.1 If an object of interest interacts with several other objects, the net force is the vector sum of the individual forces.

Essential Knowledge 3.B.2 Free-body diagrams are useful tools for visualizing the forces being exerted on a single object and writing the equations that represent a physical situation.

Enduring Understanding 3.C At the macroscopic level, forces can be categorized as either long-range (action-at-a-distance) forces or contact forces.

Essential Knowledge 3.C.1 Gravitational force describes the interaction of one object that has mass with another object that has mass.

Essential Knowledge 3.C.4 Contact forces result from the interaction of one object touching another object, and they arise from interatomic electric forces. These forces include tension, friction, normal, spring (Physics 1), and buoyant (Physics 2).

Enduring Understanding 3.G Certain types of forces are considered fundamental.

Essential Knowledge 3.G.1 Gravitational forces are exerted at all scales and dominate at the largest distance and mass scales.

Essential Knowledge 3.G.2 Electromagnetic forces are exerted at all scales and can dominate at the human scale.

Big Idea 4 Interactions between systems can result in changes in those systems.

Enduring Understanding 4.A The acceleration of the center of mass of a system is related to the net force exerted on the system, where

$$a = \sum F/m.$$

Essential Knowledge 4.A.1 The linear motion of a system can be described by the displacement, velocity, and acceleration of its center of mass.

Essential Knowledge 4.A.2 The acceleration is equal to the rate of change of velocity with time, and velocity is equal to the rate of change of position with time.

Essential Knowledge 4.A.3 Forces that systems exert on each other are due to interactions between objects in the systems. If the interacting objects are parts of the same system, there will be no change in the center-of-mass velocity of that system.

Big Idea 5 Changes that occur as a result of interactions are constrained by conservation laws.

Enduring Understanding 5.A Certain quantities are conserved, in the sense that the changes of those quantities in a given system are always equal to the transfer of that quantity to or from the system by all possible interactions with other systems.

Essential Knowledge 5.A.1 A system is an object or a collection of objects. The objects are treated as having no internal structure.

Development of Force Concept

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Understand the definition of force.

The information presented in this section supports the following AP® learning objectives and science practices:

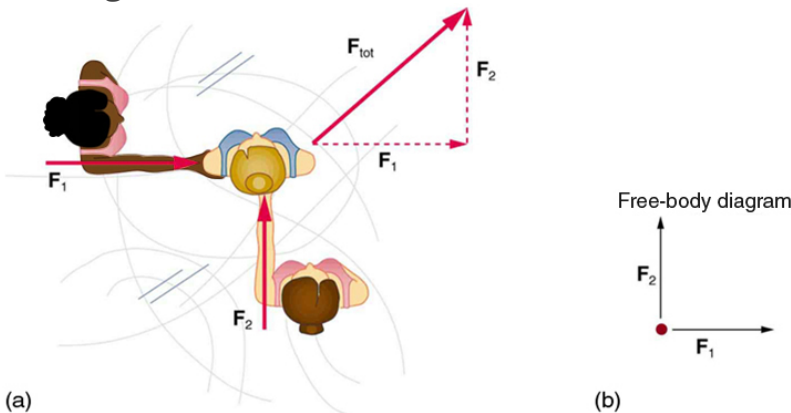
- **3.A.2.1** The student is able to represent forces in diagrams or mathematically using appropriately labeled vectors with magnitude, direction, and units during the analysis of a situation. **(S.P. 1.1)**
- **3.A.3.2** The student is able to challenge a claim that an object can exert a force on itself. **(S.P. 6.1)**
- **3.A.3.3** The student is able to describe a force as an interaction between two objects and identify both objects for any force. **(S.P. 1.4)**
- **3.B.2.1** The student is able to create and use free-body diagrams to analyze physical situations to solve problems with motion qualitatively and quantitatively. **(S.P. 1.1, 1.4, 2.2)**

Dynamics is the study of the forces that cause objects and systems to move. To understand this, we need a working definition of force. Our intuitive definition of **force**—that is, a push or a pull—is a good place to start. We know that a push or pull has both magnitude and direction (therefore, it is a vector quantity) and can vary considerably in each regard. For example, a

cannon exerts a strong force on a cannonball that is launched into the air. In contrast, Earth exerts only a tiny downward pull on a flea. Our everyday experiences also give us a good idea of how multiple forces add. If two people push in different directions on a third person, as illustrated in [\[link\]](#), we might expect the total force to be in the direction shown. Since force is a vector, it adds just like other vectors, as illustrated in [\[link\]](#)(a) for two ice skaters. Forces, like other vectors, are represented by arrows and can be added using the familiar head-to-tail method or by trigonometric methods. These ideas were developed in [Two-Dimensional Kinematics](#).

By definition, force is always the result of an interaction of two or more objects. No object possesses force on its own. For example, a cannon does not possess force, but it can exert force on a cannonball. Earth does not possess force on its own, but exerts force on a football or on any other massive object. The skaters in [Figure 4.3](#) exert force on one another as they interact.

No object can exert force on itself. When you clap your hands, one hand exerts force on the other. When a train accelerates, it exerts force on the track and vice versa. A bowling ball is accelerated by the hand throwing it; once the hand is no longer in contact with the bowling ball, it is no longer accelerating the bowling ball or exerting force on it. The ball continues moving forward due to inertia.

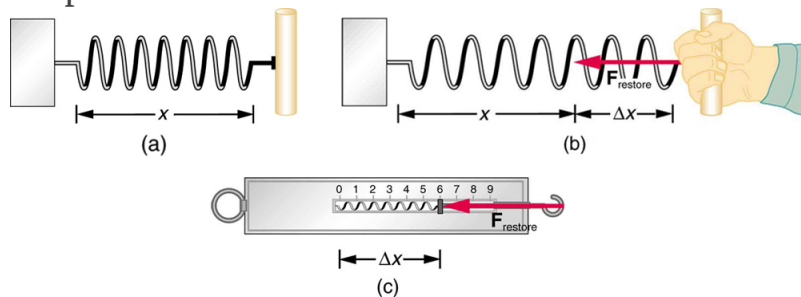


Part (a) shows an overhead view of two ice skaters pushing on a third. Forces are vectors and add like other vectors, so the total force on the third skater is in the

direction shown. In part (b), we see a free-body diagram representing the forces acting on the third skater.

[\[link\]](#) (b) is our first example of a **free-body diagram**, which is a technique used to illustrate all the **external forces** acting on a body. The body is represented by a single isolated point (or free body), and only those forces acting *on* the body from the outside (external forces) are shown. (These forces are the only ones shown, because only external forces acting on the body affect its motion. We can ignore any internal forces within the body.) Free-body diagrams are very useful in analyzing forces acting on a system and are employed extensively in the study and application of Newton's laws of motion.

A more quantitative definition of force can be based on some standard force, just as distance is measured in units relative to a standard distance. One possibility is to stretch a spring a certain fixed distance, as illustrated in [\[link\]](#), and use the force it exerts to pull itself back to its relaxed shape—called a *restoring force*—as a standard. The magnitude of all other forces can be stated as multiples of this standard unit of force. Many other possibilities exist for standard forces. (One that we will encounter in [Magnetism](#) is the magnetic force between two wires carrying electric current.) Some alternative definitions of force will be given later in this chapter.



The force exerted by a stretched spring can be used as a standard unit of force. (a) This spring has a length x when undistorted. (b) When stretched a distance Δx , the spring

exerts a restoring force, $\mathbf{F}_{\text{restore}}$, which is reproducible. (c) A spring scale is one device that uses a spring to measure force. The force $\mathbf{F}_{\text{restore}}$ is exerted on whatever is attached to the hook. Here $\mathbf{F}_{\text{restore}}$ has a magnitude of 6 units in the force standard being employed.

Note:

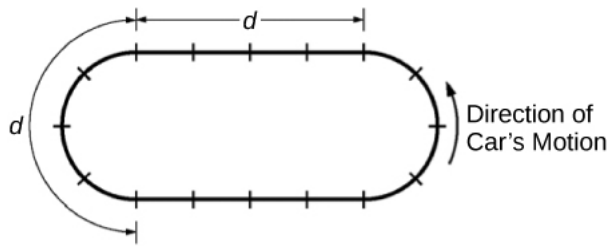
Take-Home Experiment: Force Standards

To investigate force standards and cause and effect, get two identical rubber bands. Hang one rubber band vertically on a hook. Find a small household item that could be attached to the rubber band using a paper clip, and use this item as a weight to investigate the stretch of the rubber band. Measure the amount of stretch produced in the rubber band with one, two, and four of these (identical) items suspended from the rubber band. What is the relationship between the number of items and the amount of stretch? How large a stretch would you expect for the same number of items suspended from two rubber bands? What happens to the amount of stretch of the rubber band (with the weights attached) if the weights are also pushed to the side with a pencil?

Test Prep for AP Courses

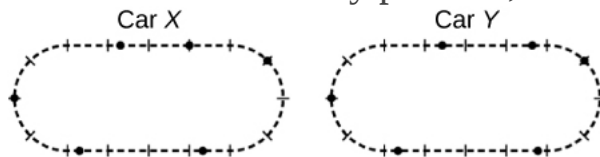
Exercise:

Problem:



The figure above represents a racetrack with semicircular sections connected by straight sections. Each section has length d , and markers along the track are spaced $d/4$ apart. Two people drive cars counterclockwise around the track, as shown. Car X goes around the curves at constant speed v_c , increases speed at constant acceleration for half of each straight section to reach a maximum speed of $2v_c$, then brakes at constant acceleration for the other half of each straight section to return to speed v_c . Car Y also goes around the curves at constant speed v_c , increases its speed at constant acceleration for one-fourth of each straight section to reach the same maximum speed $2v_c$, stays at that speed for half of each straight section, then brakes at constant acceleration for the remaining fourth of each straight section to return to speed v_c .

(a) On the figures below, draw an arrow showing the direction of the net force on each of the cars at the positions noted by the dots. If the net force is zero at any position, label the dot with 0.



The position of the six dots on the Car Y track on the right are as follows:

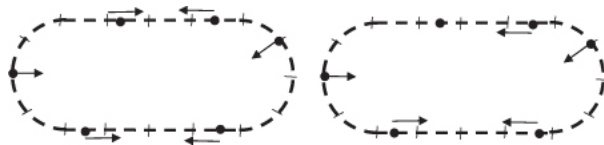
- The first dot on the left center of the track is at the same position as it is on the Car X track.
- The second dot is just slight to the right of the Car X dot (less than a dash) past three perpendicular hash marks moving to the right.
- The third dot is about one and two-thirds perpendicular hash marks to the right of the center top perpendicular hash mark.

- The fourth dot is in the same position as the Car X figure (one perpendicular hash mark above the center right perpendicular hash mark).
- The fifth dot is about one and two-third perpendicular hash marks to the right of the center bottom perpendicular hash mark.
- The sixth dot is in the same position as the Car Y dot (one and two third perpendicular hash marks to the left of the center bottom hash mark).

(b)

- Indicate which car, if either, completes one trip around the track in less time, and justify your answer qualitatively without using equations.
- Justify your answer about which car, if either, completes one trip around the track in less time quantitatively with appropriate equations.

Solution:



Car X is shown on the left, and
Car Y is shown on the right.

i.

Car X takes longer to accelerate and does not spend any time traveling at top speed. Car Y accelerates over a shorter time and spends time going at top speed. So Car Y must cover the straightaways in a shorter time. Curves take the same time, so Car Y must overall take a shorter time.

ii.

The only difference in the calculations for the time of one segment of linear acceleration is the difference in distances. That shows that Car X takes longer to accelerate. The equation $\frac{d}{4v_c} = t_c$ corresponds to Car Y traveling for a time at top speed.

Substituting $a = \frac{v_c}{t_1}$ into the displacement equation in part (b) ii gives $D = \frac{3}{2}v_c t_1$. This shows that a car takes less time to reach its maximum speed when it accelerates over a shorter distance. Therefore, Car Y reaches its maximum speed more quickly, and spends more time at its maximum speed than Car X does, as argued in part (b) i.

Exercise:

Problem:

Which of the following is an example of a body exerting a force on itself?

- a. a person standing up from a seated position
- b. a car accelerating while driving
- c. both of the above
- d. none of the above

Exercise:

Problem:

A hawk accelerates as it glides in the air. Does the force causing the acceleration come from the hawk itself? Explain.

Solution:

A body cannot exert a force on itself. The hawk may accelerate as a result of several forces. The hawk may accelerate toward Earth as a result of the force due to gravity. The hawk may accelerate as a result of the additional force exerted on it by wind. The hawk may accelerate as a result of orienting its body to create less air resistance, thus increasing the net force forward.

Exercise:

Problem:

What causes the force that moves a boat forward when someone rows it?

- a. The force is caused by the rower's arms.
- b. The force is caused by an interaction between the oars and gravity.
- c. The force is caused by an interaction between the oars and the water the boat is traveling in.
- d. The force is caused by friction.

Section Summary

- **Dynamics** is the study of how forces affect the motion of objects.
- **Force** is a push or pull that can be defined in terms of various standards, and it is a vector having both magnitude and direction.
- **External forces** are any outside forces that act on a body. A **free-body diagram** is a drawing of all external forces acting on a body.

Conceptual Questions

Exercise:

Problem:

Propose a force standard different from the example of a stretched spring discussed in the text. Your standard must be capable of producing the same force repeatedly.

Exercise:

Problem:

What properties do forces have that allow us to classify them as vectors?

Glossary

dynamics

the study of how forces affect the motion of objects and systems

external force

a force acting on an object or system that originates outside of the object or system

free-body diagram

a sketch showing all of the external forces acting on an object or system; the system is represented by a dot, and the forces are represented by vectors extending outward from the dot

force

a push or pull on an object with a specific magnitude and direction; can be represented by vectors; can be expressed as a multiple of a standard force

Newton's First Law of Motion: Inertia

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define mass and inertia.
- Understand Newton's first law of motion.

Experience suggests that an object at rest will remain at rest if left alone, and that an object in motion tends to slow down and stop unless some effort is made to keep it moving. What **Newton's first law of motion** states, however, is the following:

Note:

Newton's First Law of Motion

There exists an inertial frame of reference such that a body at rest remains at rest, or, if in motion, remains in motion at a constant velocity unless acted on by a net external force.

Note the repeated use of the verb “remains.” We can think of this law as preserving the status quo of motion.

The first law of motion postulates the existence of at least one frame of reference which we call an inertial reference frame, relative to which the motion of an object not subject to forces is a straight line at a constant

speed. An inertial reference frame is any reference frame that is not itself accelerating. A car traveling at constant velocity is an inertial reference frame. A car slowing down for a stoplight, or speeding up after the light turns green, will be accelerating and is not an inertial reference frame. Finally, when the car goes around a turn, which is due to an acceleration changing the direction of the velocity vector, it is not an inertial reference frame. Note that Newton's laws of motion are only valid for inertial reference frames.

Rather than contradicting our experience, **Newton's first law of motion** states that there must be a *cause* (which is a net external force) *for there to be any change in velocity (either a change in magnitude or direction)* in an inertial reference frame. We will define *net external force* in the next section. An object sliding across a table or floor slows down due to the net force of friction acting on the object. If friction disappeared, would the object still slow down?

The idea of cause and effect is crucial in accurately describing what happens in various situations. For example, consider what happens to an object sliding along a rough horizontal surface. The object quickly grinds to a halt. If we spray the surface with talcum powder to make the surface smoother, the object slides farther. If we make the surface even smoother by rubbing lubricating oil on it, the object slides farther yet. Extrapolating to a frictionless surface, we can imagine the object sliding in a straight line indefinitely. Friction is thus the *cause* of the slowing (consistent with Newton's first law). The object would not slow down at all if friction were completely eliminated. Consider an air hockey table. When the air is turned off, the puck slides only a short distance before friction slows it to a stop. However, when the air is turned on, it creates a nearly frictionless surface, and the puck glides long distances without slowing down. Additionally, if we know enough about the friction, we can accurately predict how quickly the object will slow down. Friction is an external force.

Newton's first law is completely general and can be applied to anything from an object sliding on a table to a satellite in orbit to blood pumped from the heart. Experiments have thoroughly verified that any change in velocity (speed or direction) must be caused by an external force. The idea of

generally applicable or universal laws is important not only here—it is a basic feature of all laws of physics. Identifying these laws is like recognizing patterns in nature from which further patterns can be discovered. The genius of Galileo, who first developed the idea for the first law, and Newton, who clarified it, was to ask the fundamental question, “What is the cause?” Thinking in terms of cause and effect is a worldview fundamentally different from the typical ancient Greek approach when questions such as “Why does a tiger have stripes?” would have been answered in Aristotelian fashion, “That is the nature of the beast.” True perhaps, but not a useful insight.

Mass

The property of a body to remain at rest or to remain in motion with constant velocity is called **inertia**. Newton’s first law is often called the **law of inertia**. As we know from experience, some objects have more inertia than others. It is obviously more difficult to change the motion of a large boulder than that of a basketball, for example. The inertia of an object is measured by its **mass**.

An object with a small mass will exhibit less inertia and be more affected by other objects. An object with a large mass will exhibit greater inertia and be less affected by other objects. This inertial mass of an object is a measure of how difficult it is to alter the uniform motion of the object by an external force.

Roughly speaking, mass is a measure of the amount of “stuff” (or matter) in something. The quantity or amount of matter in an object is determined by the numbers of atoms and molecules of various types it contains. Unlike weight, mass does not vary with location. The mass of an object is the same on Earth, in orbit, or on the surface of the Moon. In practice, it is very difficult to count and identify all of the atoms and molecules in an object, so masses are not often determined in this manner. Operationally, the masses of objects are determined by comparison with the standard kilogram.

Exercise:

Check Your Understanding

Problem:

Which has more mass: a kilogram of cotton balls or a kilogram of gold?

Solution:**Answer**

They are equal. A kilogram of one substance is equal in mass to a kilogram of another substance. The quantities that might differ between them are volume and density.

Section Summary

- **Newton's first law of motion** states that in an inertial frame of reference a body at rest remains at rest, or, if in motion, remains in motion at a constant velocity unless acted on by a net external force. This is also known as the **law of inertia**.
- **Inertia** is the tendency of an object to remain at rest or remain in motion. Inertia is related to an object's mass.
- **Mass** is the quantity of matter in a substance.

Conceptual Questions**Exercise:**

Problem: How are inertia and mass related?

Exercise:**Problem:**

What is the relationship between weight and mass? Which is an intrinsic, unchanging property of a body?

Glossary

inertia

the tendency of an object to remain at rest or remain in motion

law of inertia

see Newton's first law of motion

mass

the quantity of matter in a substance; measured in kilograms

Newton's first law of motion

in an inertial frame of reference, a body at rest remains at rest, or, if in motion, remains in motion at a constant velocity unless acted on by a net external force; also known as the law of inertia

Newton's Second Law of Motion: Concept of a System

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define net force, external force, and system.
- Understand Newton's second law of motion.
- Apply Newton's second law to determine the weight of an object.

Newton's second law of motion is closely related to Newton's first law of motion. It mathematically states the cause and effect relationship between force and changes in motion. Newton's second law of motion is more quantitative and is used extensively to calculate what happens in situations involving a force. Before we can write down Newton's second law as a simple equation giving the exact relationship of force, mass, and acceleration, we need to sharpen some ideas that have already been mentioned.

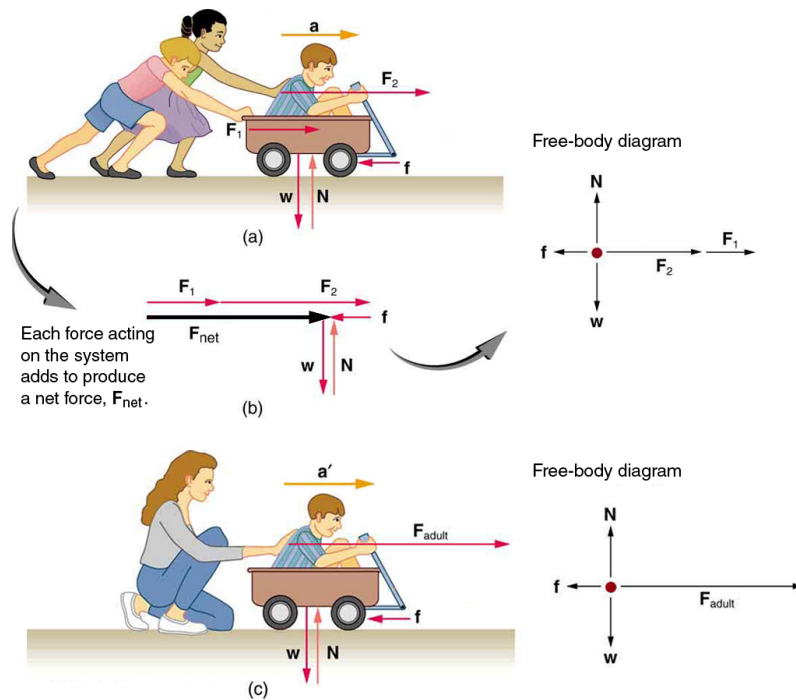
First, what do we mean by a change in motion? The answer is that a change in motion is equivalent to a change in velocity. A change in velocity means, by definition, that there is an **acceleration**. Newton's first law says that a net external force causes a change in motion; thus, we see that a *net external force causes acceleration*.

Another question immediately arises. What do we mean by an external force? An intuitive notion of external is correct—an **external force** acts from outside the **system** of interest. For example, in [\[link\]](#)(a) the system of

interest is the wagon plus the child in it. The two forces exerted by the other children are external forces. An internal force acts between elements of the system. Again looking at [\[link\]](#)(a), the force the child in the wagon exerts to hang onto the wagon is an internal force between elements of the system of interest. Only external forces affect the motion of a system, according to Newton's first law. (The internal forces actually cancel, as we shall see in the next section.) *You must define the boundaries of the system before you can determine which forces are external.* Sometimes the system is obvious, whereas other times identifying the boundaries of a system is more subtle. The concept of a system is fundamental to many areas of physics, as is the correct application of Newton's laws. This concept will be revisited many times on our journey through physics.

When we describe the acceleration of a system, we are modeling the system as a single point which contains all of the mass of that system. The point we choose for this is the point about which the system's mass is evenly distributed. For example, in a rigid object, this center of mass is the point where the object will stay balanced even if only supported at this point. For a sphere or disk made of homogenous material, this point is of course at the center. Similarly, for a rod made of homogenous material, the center of mass will be at the midpoint.

For the rider in the wagon in [Figure 4.5](#), the center of mass is probably between the rider's hips. Due to internal forces, the rider's hand or hair may accelerate slightly differently, but it is the acceleration of the system's center of mass that interests us. This is true whether the system is a vehicle carrying passengers, a bowl of grapes, or a planet. When we draw a free-body diagram of a system, we represent the system's center of mass with a single point and use vectors to indicate the forces exerted on that center of mass. (See [Figure 4.5](#).)



Different forces exerted on the same mass produce different accelerations. (a) Two children push a wagon with a child in it. Arrows representing all external forces are shown. The system of interest is the wagon and its rider. The weight w of the system and the support of the ground N are also shown for completeness and are assumed to cancel. The vector f represents the friction acting on the wagon, and it acts to the left, opposing the motion of the wagon. (b) All of the external forces acting on the system add together to produce a net force, F_{net} . The free-body diagram shows all of the forces acting on the system of interest. The dot represents the center of mass of the system. Each force vector extends from this dot. Because there are two forces acting to the right, we draw the vectors collinearly. (c) A larger net external force produces a larger

acceleration ($\mathbf{a}' > \mathbf{a}$) when an adult pushes the child.

Now, it seems reasonable that acceleration should be directly proportional to and in the same direction as the net (total) external force acting on a system. This assumption has been verified experimentally and is illustrated in [\[link\]](#). In part (a), a smaller force causes a smaller acceleration than the larger force illustrated in part (c). For completeness, the vertical forces are also shown; they are assumed to cancel since there is no acceleration in the vertical direction. The vertical forces are the weight \mathbf{w} and the support of the ground \mathbf{N} , and the horizontal force \mathbf{f} represents the force of friction. These will be discussed in more detail in later sections. For now, we will define **friction** as a force that opposes the motion past each other of objects that are touching. [\[link\]](#)(b) shows how vectors representing the external forces add together to produce a net force, \mathbf{F}_{net} .

To obtain an equation for Newton's second law, we first write the relationship of acceleration and net external force as the proportionality **Equation:**

$$\mathbf{a} \propto \mathbf{F}_{\text{net}},$$

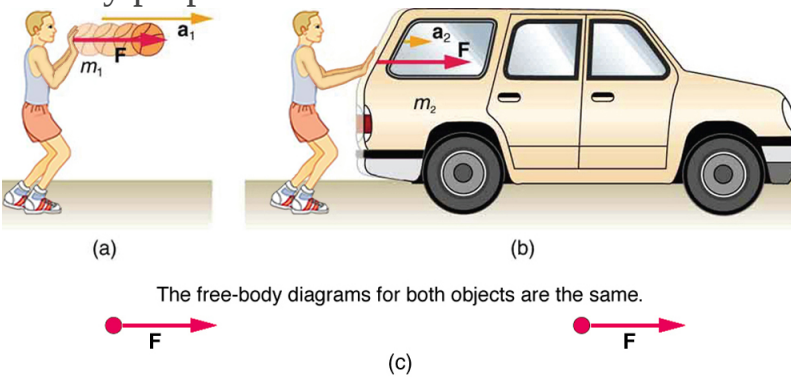
where the symbol \propto means “proportional to,” and \mathbf{F}_{net} is the **net external force**. (The net external force is the vector sum of all external forces and can be determined graphically, using the head-to-tail method, or analytically, using components. The techniques are the same as for the addition of other vectors, and are covered in [Two-Dimensional Kinematics](#).) This proportionality states what we have said in words—*acceleration is directly proportional to the net external force*. Once the system of interest is chosen, it is important to identify the external forces and ignore the internal ones. It is a tremendous simplification not to have to consider the numerous internal forces acting between objects within the system, such as muscular forces within the child's body, let alone the myriad of forces between atoms in the objects, but by doing so, we can easily solve some very complex problems with only minimal error due to our simplification

Now, it also seems reasonable that acceleration should be inversely proportional to the mass of the system. In other words, the larger the mass (the inertia), the smaller the acceleration produced by a given force. And indeed, as illustrated in [\[link\]](#), the same net external force applied to a car produces a much smaller acceleration than when applied to a basketball. The proportionality is written as

Equation:

$$\mathbf{a} \propto \frac{1}{m}$$

where m is the mass of the system. Experiments have shown that acceleration is exactly inversely proportional to mass, just as it is exactly linearly proportional to the net external force.



The same force exerted on systems of different masses produces different accelerations. (a) A basketball player pushes on a basketball to make a pass. (The effect of gravity on the ball is ignored.) (b) The same player exerts an identical force on a stalled SUV and produces a far smaller acceleration (even if friction is negligible). (c) The free-body diagrams are identical, permitting direct comparison of the two situations. A series of patterns for the free-body diagram will emerge as you do more problems.

Both of these proportionalities have been experimentally verified repeatedly and consistently, for a broad range of systems and scales. Thus, it has been **experimentally** found that the acceleration of an object depends *only* on the net external force and the mass of the object. Combining the two proportionalities just given yields Newton's second law of motion.

Note:

Applying the Science Practices: Testing the Relationship Between Mass, Acceleration, and Force

Plan three simple experiments using objects you have at home to test relationships between mass, acceleration, and force.

(a) Design an experiment to test the relationship between mass and acceleration. What will be the independent variable in your experiment? What will be the dependent variable? What controls will you put in place to ensure force is constant?

(b) Design a similar experiment to test the relationship between mass and force. What will be the independent variable in your experiment? What will be the dependent variable? What controls will you put in place to ensure acceleration is constant?

(c) Design a similar experiment to test the relationship between force and acceleration. What will be the independent variable in your experiment? What will be the dependent variable? Will you have any trouble ensuring that the mass is constant?

What did you learn?

Note:

Newton's Second Law of Motion

The acceleration of a system is directly proportional to and in the same direction as the net external force acting on the system, and inversely proportional to its mass.

In equation form, Newton's second law of motion is

Equation:

$$\mathbf{a} = \frac{\mathbf{F}_{\text{net}}}{m}.$$

This is often written in the more familiar form

Equation:

$$\mathbf{F}_{\text{net}} = m\mathbf{a}.$$

When only the magnitude of force and acceleration are considered, this equation is simply

Equation:

$$F_{\text{net}} = ma.$$

Although these last two equations are really the same, the first gives more insight into what Newton's second law means. The law is a *cause and effect relationship* among three quantities that is not simply based on their definitions. The validity of the second law is completely based on experimental verification.

Note:

Applying the Science Practices: Systems and Free-Body Diagrams

First, consider a person on a sled sliding downhill. What is the system in this situation? Try to draw a free-body diagram describing this system, labeling all the forces and their directions. Which of the forces are internal? Which are external?

Next, consider a person on a sled being pushed along level ground by a friend. What is the system in this situation? Try to draw a free-body diagram describing this system, labelling all the forces and their directions. Which of the forces are internal? Which are external?

Units of Force

$\mathbf{F}_{\text{net}} = m\mathbf{a}$ is used to define the units of force in terms of the three basic units for mass, length, and time. The SI unit of force is called the **newton** (abbreviated N) and is the force needed to accelerate a 1-kg system at the rate of 1m/s^2 . That is, since $\mathbf{F}_{\text{net}} = m\mathbf{a}$,

Equation:

$$1 \text{ N} = 1 \text{ kg} \cdot \text{m/s}^2.$$

While almost the entire world uses the newton for the unit of force, in the United States the most familiar unit of force is the pound (lb), where $1 \text{ N} = 0.225 \text{ lb}$.

Weight and the Gravitational Force

When an object is dropped, it accelerates toward the center of Earth. Newton's second law states that a net force on an object is responsible for its acceleration. If air resistance is negligible, the net force on a falling object is the gravitational force, commonly called its **weight \mathbf{w}** . Weight can be denoted as a vector \mathbf{w} because it has a direction; *down* is, by definition, the direction of gravity, and hence weight is a downward force. The magnitude of weight is denoted as w . Galileo was instrumental in showing that, in the absence of air resistance, all objects fall with the same acceleration g . Using Galileo's result and Newton's second law, we can derive an equation for weight.

Consider an object with mass m falling downward toward Earth. It experiences only the downward force of gravity, which has magnitude w . Newton's second law states that the magnitude of the net external force on an object is $F_{\text{net}} = ma$.

Since the object experiences only the downward force of gravity, $F_{\text{net}} = w$. We know that the acceleration of an object due to gravity is g , or $a = g$. Substituting these into Newton's second law gives

Note:**Weight**

This is the equation for *weight*—the gravitational force on a mass m :

Equation:

$$w = mg.$$

Since $g = 9.80 \text{ m/s}^2$ on Earth, the weight of a 1.0 kg object on Earth is 9.8 N, as we see:

Equation:

$$w = mg = (1.0 \text{ kg})(9.80 \text{ m/s}^2) = 9.8 \text{ N}.$$

Recall that g can take a positive or negative value, depending on the positive direction in the coordinate system. Be sure to take this into consideration when solving problems with weight.

When the net external force on an object is its weight, we say that it is in **free-fall**. That is, the only force acting on the object is the force of gravity. In the real world, when objects fall downward toward Earth, they are never truly in free-fall because there is always some upward force from the air acting on the object.

The acceleration due to gravity g varies slightly over the surface of Earth, so that the weight of an object depends on location and is not an intrinsic property of the object. Weight varies dramatically if one leaves Earth's surface. On the Moon, for example, the acceleration due to gravity is only 1.67 m/s^2 . A 1.0-kg mass thus has a weight of 9.8 N on Earth and only about 1.7 N on the Moon.

The broadest definition of weight in this sense is that *the weight of an object is the gravitational force on it from the nearest large body*, such as Earth, the Moon, the Sun, and so on. This is the most common and useful definition of weight in physics. It differs dramatically, however, from the definition of weight used by NASA and the popular media in relation to space travel and exploration. When they speak of “weightlessness” and

“microgravity,” they are really referring to the phenomenon we call “free-fall” in physics. We shall use the above definition of weight, and we will make careful distinctions between free-fall and actual weightlessness.

It is important to be aware that weight and mass are very different physical quantities, although they are closely related. Mass is the quantity of matter (how much “stuff”) and does not vary in classical physics, whereas weight is the gravitational force and does vary depending on gravity. It is tempting to equate the two, since most of our examples take place on Earth, where the weight of an object only varies a little with the location of the object. Furthermore, the terms *mass* and *weight* are used interchangeably in everyday language; for example, our medical records often show our “weight” in kilograms, but never in the correct units of newtons.

Note:**Common Misconceptions: Mass vs. Weight**

Mass and weight are often used interchangeably in everyday language. However, in science, these terms are distinctly different from one another. Mass is a measure of how much matter is in an object. The typical measure of mass is the kilogram (or the “slug” in English units). Weight, on the other hand, is a measure of the force of gravity acting on an object. Weight is equal to the mass of an object (m) multiplied by the acceleration due to gravity (g). Like any other force, weight is measured in terms of newtons (or pounds in English units).

Assuming the mass of an object is kept intact, it will remain the same, regardless of its location. However, because weight depends on the acceleration due to gravity, the weight of an object *can change* when the object enters into a region with stronger or weaker gravity. For example, the acceleration due to gravity on the Moon is 1.67 m/s^2 (which is much less than the acceleration due to gravity on Earth, 9.80 m/s^2). If you measured your weight on Earth and then measured your weight on the Moon, you would find that you “weigh” much less, even though you do not look any skinnier. This is because the force of gravity is weaker on the Moon. In fact, when people say that they are “losing weight,” they really

mean that they are losing “mass” (which in turn causes them to weigh less).

Note:

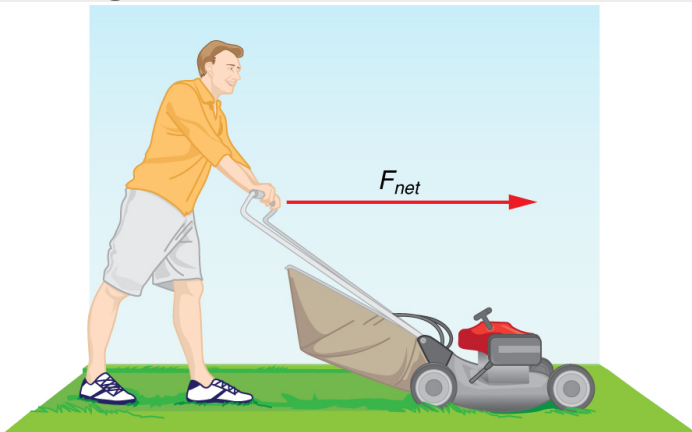
Take-Home Experiment: Mass and Weight

What do bathroom scales measure? When you stand on a bathroom scale, what happens to the scale? It depresses slightly. The scale contains springs that compress in proportion to your weight—similar to rubber bands expanding when pulled. The springs provide a measure of your weight (for an object which is not accelerating). This is a force in newtons (or pounds). In most countries, the measurement is divided by 9.80 to give a reading in mass units of kilograms. The scale measures weight but is calibrated to provide information about mass. While standing on a bathroom scale, push down on a table next to you. What happens to the reading? Why? Would your scale measure the same “mass” on Earth as on the Moon?

Example:

What Acceleration Can a Person Produce when Pushing a Lawn Mower?

Suppose that the net external force (push minus friction) exerted on a lawn mower is 51 N (about 11 lb) parallel to the ground. The mass of the mower is 24 kg. What is its acceleration?



The net force on a lawn mower is 51

N to the right. At what rate does the lawn mower accelerate to the right?

Strategy

Since \mathbf{F}_{net} and m are given, the acceleration can be calculated directly from Newton's second law as stated in $\mathbf{F}_{\text{net}} = m\mathbf{a}$.

Solution

The magnitude of the acceleration a is $a = \frac{F_{\text{net}}}{m}$. Entering known values gives

Equation:

$$a = \frac{51 \text{ N}}{24 \text{ kg}}$$

Substituting the units $\text{kg} \cdot \text{m}/\text{s}^2$ for N yields

Equation:

$$a = \frac{51 \text{ kg} \cdot \text{m}/\text{s}^2}{24 \text{ kg}} = 2.1 \text{ m}/\text{s}^2.$$

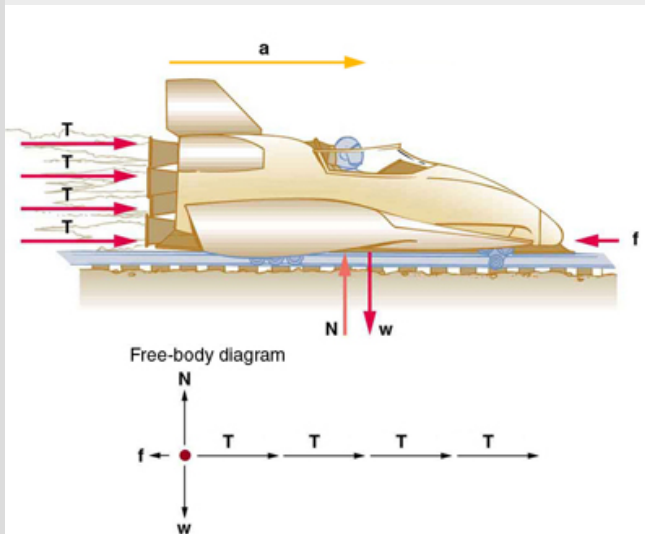
Discussion

The direction of the acceleration is the same direction as that of the net force, which is parallel to the ground. There is no information given in this example about the individual external forces acting on the system, but we can say something about their relative magnitudes. For example, the force exerted by the person pushing the mower must be greater than the friction opposing the motion (since we know the mower moves forward), and the vertical forces must cancel if there is to be no acceleration in the vertical direction (the mower is moving only horizontally). The acceleration found is small enough to be reasonable for a person pushing a mower. Such an effort would not last too long because the person's top speed would soon be reached.

Example:

What Rocket Thrust Accelerates This Sled?

Prior to manned space flights, rocket sleds were used to test aircraft, missile equipment, and physiological effects on human subjects at high speeds. They consisted of a platform that was mounted on one or two rails and propelled by several rockets. Calculate the magnitude of force exerted by each rocket, called its thrust \mathbf{T} , for the four-rocket propulsion system shown in [\[link\]](#). The sled's initial acceleration is 49 m/s^2 , the mass of the system is 2100 kg , and the force of friction opposing the motion is known to be 650 N .



A sled experiences a rocket thrust that accelerates it to the right. Each rocket creates an identical thrust \mathbf{T} . As in other situations where there is only horizontal acceleration, the vertical forces cancel. The ground exerts an upward force \mathbf{N} on the system that is equal in magnitude and opposite in direction to its weight, \mathbf{w} . The system here is the sled, its rockets, and rider, so none of the forces *between* these objects are considered. The arrow representing friction (\mathbf{f}) is drawn larger than scale.

Strategy

Although there are forces acting vertically and horizontally, we assume the vertical forces cancel since there is no vertical acceleration. This leaves us with only horizontal forces and a simpler one-dimensional problem.

Directions are indicated with plus or minus signs, with right taken as the positive direction. See the free-body diagram in the figure.

Solution

Since acceleration, mass, and the force of friction are given, we start with Newton's second law and look for ways to find the thrust of the engines.

Since we have defined the direction of the force and acceleration as acting "to the right," we need to consider only the magnitudes of these quantities in the calculations. Hence we begin with

Equation:

$$F_{\text{net}} = ma,$$

where F_{net} is the net force along the horizontal direction. We can see from [\[link\]](#) that the engine thrusts add, while friction opposes the thrust. In equation form, the net external force is

Equation:

$$F_{\text{net}} = 4T - f.$$

Substituting this into Newton's second law gives

Equation:

$$F_{\text{net}} = ma = 4T - f.$$

Using a little algebra, we solve for the total thrust $4T$:

Equation:

$$4T = ma + f.$$

Substituting known values yields

Equation:

$$4T = ma + f = (2100 \text{ kg})(49 \text{ m/s}^2) + 650 \text{ N}.$$

So the total thrust is

Equation:

$$4T = 1.0 \times 10^5 \text{ N},$$

and the individual thrusts are

Equation:

$$T = \frac{1.0 \times 10^5 \text{ N}}{4} = 2.6 \times 10^4 \text{ N}.$$

Discussion

The numbers are quite large, so the result might surprise you. Experiments such as this were performed in the early 1960s to test the limits of human endurance and the setup designed to protect human subjects in jet fighter emergency ejections. Speeds of 1000 km/h were obtained, with accelerations of 45 g 's. (Recall that g , the acceleration due to gravity, is 9.80 m/s^2 . When we say that an acceleration is 45 g 's, it is $45 \times 9.80 \text{ m/s}^2$, which is approximately 440 m/s^2 .) While living subjects are not used any more, land speeds of 10,000 km/h have been obtained with rocket sleds. In this example, as in the preceding one, the system of interest is obvious. We will see in later examples that choosing the system of interest is crucial—and the choice is not always obvious.

Newton's second law of motion is more than a definition; it is a relationship among acceleration, force, and mass. It can help us make predictions. Each of those physical quantities can be defined independently, so the second law tells us something basic and universal about nature. The next section introduces the third and final law of motion.

Note:

Applying the Science Practices: Sums of Forces

Recall that forces are vector quantities, and therefore the net force acting on a system should be the vector sum of the forces.

(a) Design an experiment to test this hypothesis. What sort of a system would be appropriate and convenient to have multiple forces applied to it?

What features of the system should be held constant? What could be varied? Can forces be arranged in multiple directions so that, while the hypothesis is still tested, the resulting calculations are not too inconvenient?

(b) Another group of students has done such an experiment, using a motion capture system looking down at an air hockey table to measure the motion of the 0.10-kg puck. The table was aligned with the cardinal directions, and a compressed air hose was placed in the center of each side, capable of varying levels of force output and fixed so that it was aimed at the center of the table.

Forces	Measured acceleration (magnitudes)
3 N north, 4 N west	$48 \pm 4 \text{ m/s}^2$
5 N south, 12 N east	$132 \pm 6 \text{ m/s}^2$
6 N north, 12 N east, 4 N west	$99 \pm 3 \text{ m/s}^2$

Given the data in the table, is the hypothesis confirmed? What were the directions of the accelerations?

Section Summary

- Acceleration, **a**, is defined as a change in velocity, meaning a change in its magnitude or direction, or both.
- An external force is one acting on a system from outside the system, as opposed to internal forces, which act between components within the

system.

- Newton's second law of motion states that the acceleration of a system is directly proportional to and in the same direction as the net external force acting on the system, and inversely proportional to its mass.
- In equation form, Newton's second law of motion is $\mathbf{a} = \frac{\mathbf{F}_{\text{net}}}{m}$.
- This is often written in the more familiar form: $\mathbf{F}_{\text{net}} = m\mathbf{a}$.
- The weight \mathbf{w} of an object is defined as the force of gravity acting on an object of mass m . The object experiences an acceleration due to gravity \mathbf{g} :

Equation:

$$\mathbf{w} = m\mathbf{g}.$$

- If the only force acting on an object is due to gravity, the object is in free fall.
- Friction is a force that opposes the motion past each other of objects that are touching.

Conceptual Questions

Exercise:

Problem:

Which statement is correct? (a) Net force causes motion. (b) Net force causes change in motion. Explain your answer and give an example.

Exercise:

Problem:

Why can we neglect forces such as those holding a body together when we apply Newton's second law of motion?

Exercise:

Problem:

Explain how the choice of the “system of interest” affects which forces must be considered when applying Newton’s second law of motion.

Exercise:

Problem:

Describe a situation in which the net external force on a system is not zero, yet its speed remains constant.

Exercise:

Problem:

A system can have a nonzero velocity while the net external force on it is zero. Describe such a situation.

Exercise:

Problem:

A rock is thrown straight up. What is the net external force acting on the rock when it is at the top of its trajectory?

Exercise:

Problem:

(a) Give an example of different net external forces acting on the same system to produce different accelerations. (b) Give an example of the same net external force acting on systems of different masses, producing different accelerations. (c) What law accurately describes both effects? State it in words and as an equation.

Exercise:

Problem:

If the acceleration of a system is zero, are no external forces acting on it? What about internal forces? Explain your answers.

Exercise:

Problem:

If a constant, nonzero force is applied to an object, what can you say about the velocity and acceleration of the object?

Exercise:**Problem:**

The gravitational force on the basketball in [\[link\]](#) is ignored. When gravity *is* taken into account, what is the direction of the net external force on the basketball—above horizontal, below horizontal, or still horizontal?

Problem Exercises

You may assume data taken from illustrations is accurate to three digits.

Exercise:**Problem:**

A 63.0-kg sprinter starts a race with an acceleration of 4.20 m/s^2 . What is the net external force on him?

Solution:

265 N

Exercise:**Problem:**

If the sprinter from the previous problem accelerates at that rate for 20 m, and then maintains that velocity for the remainder of the 100-m dash, what will be his time for the race?

Exercise:

Problem:

A cleaner pushes a 4.50-kg laundry cart in such a way that the net external force on it is 60.0 N. Calculate the magnitude of its acceleration.

Solution:

$$13.3 \text{ m/s}^2$$

Exercise:**Problem:**

Since astronauts in orbit are apparently weightless, a clever method of measuring their masses is needed to monitor their mass gains or losses to adjust diets. One way to do this is to exert a known force on an astronaut and measure the acceleration produced. Suppose a net external force of 50.0 N is exerted and the astronaut's acceleration is measured to be 0.893 m/s^2 . (a) Calculate her mass. (b) By exerting a force on the astronaut, the vehicle in which they orbit experiences an equal and opposite force. Discuss how this would affect the measurement of the astronaut's acceleration. Propose a method in which recoil of the vehicle is avoided.

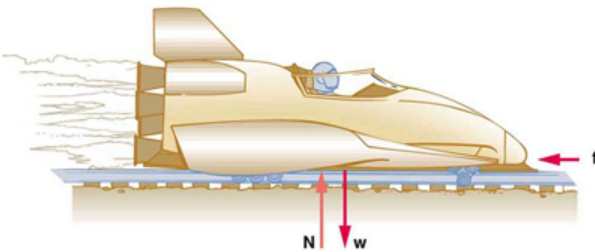
Exercise:**Problem:**

In [\[link\]](#), the net external force on the 24-kg mower is stated to be 51 N. If the force of friction opposing the motion is 24 N, what force F (in newtons) is the person exerting on the mower? Suppose the mower is moving at 1.5 m/s when the force F is removed. How far will the mower go before stopping?

Exercise:

Problem:

The same rocket sled drawn in [\[link\]](#) is decelerated at a rate of 196 m/s^2 . What force is necessary to produce this deceleration? Assume that the rockets are off. The mass of the system is 2100 kg.

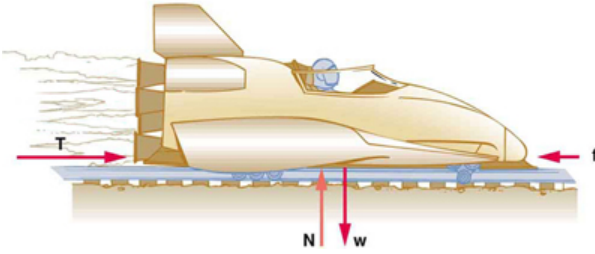
**Exercise:****Problem:**

(a) If the rocket sled shown in [\[link\]](#) starts with only one rocket burning, what is the magnitude of its acceleration? Assume that the mass of the system is 2100 kg, the thrust T is $2.4 \times 10^4 \text{ N}$, and the force of friction opposing the motion is known to be 650 N. (b) Why is the acceleration not one-fourth of what it is with all rockets burning?

Solution:

(a) 12 m/s^2 .

(b) The acceleration is not one-fourth of what it was with all rockets burning because the frictional force is still as large as it was with all rockets burning.



Exercise:

Problem:

What is the deceleration of the rocket sled if it comes to rest in 1.1 s from a speed of 1000 km/h? (Such deceleration caused one test subject to black out and have temporary blindness.)

Exercise:

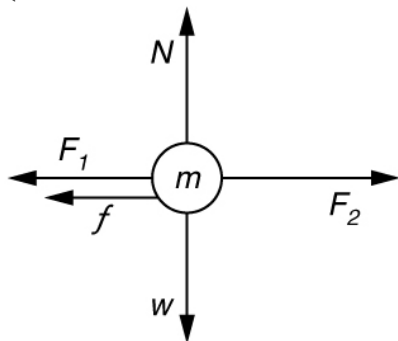
Problem:

Suppose two children push horizontally, but in exactly opposite directions, on a third child in a wagon. The first child exerts a force of 75.0 N, the second a force of 90.0 N, friction is 12.0 N, and the mass of the third child plus wagon is 23.0 kg. (a) What is the system of interest if the acceleration of the child in the wagon is to be calculated? (b) Draw a free-body diagram, including all forces acting on the system. (c) Calculate the acceleration. (d) What would the acceleration be if friction were 15.0 N?

Solution:

(a) The system is the child in the wagon plus the wagon.

(b)



(c) $a = 0.130 \text{ m/s}^2$ in the direction of the second child's push.

(d) $a = 0.00 \text{ m/s}^2$

Exercise:

Problem:

A powerful motorcycle can produce an acceleration of 3.50 m/s^2 while traveling at 90.0 km/h . At that speed the forces resisting motion, including friction and air resistance, total 400 N . (Air resistance is analogous to air friction. It always opposes the motion of an object.) What is the magnitude of the force the motorcycle exerts backward on the ground to produce its acceleration if the mass of the motorcycle with rider is 245 kg ?

Exercise:

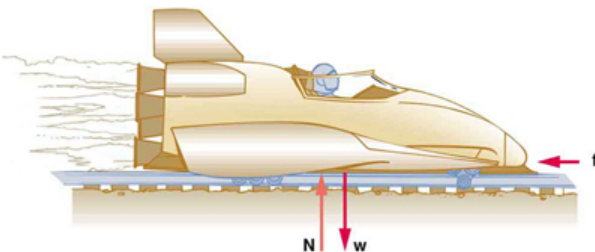
Problem:

The rocket sled shown in [\[link\]](#) accelerates at a rate of 49.0 m/s^2 . Its passenger has a mass of 75.0 kg . (a) Calculate the horizontal component of the force the seat exerts against his body. Compare this with his weight by using a ratio. (b) Calculate the direction and magnitude of the total force the seat exerts against his body.

Solution:

(a) $3.68 \times 10^3 \text{ N}$. This force is 5.00 times greater than his weight.

(b) 3750 N ; 11.3° above horizontal



Exercise:**Problem:**

Repeat the previous problem for the situation in which the rocket sled decelerates at a rate of 201 m/s^2 . In this problem, the forces are exerted by the seat and restraining belts.

Exercise:**Problem:**

The weight of an astronaut plus his space suit on the Moon is only 250 N. How much do they weigh on Earth? What is the mass on the Moon? On Earth?

Solution:

$1.5 \times 10^3 \text{ N}$, 150 kg, 150 kg

Exercise:**Problem:**

Suppose the mass of a fully loaded module in which astronauts take off from the Moon is 10,000 kg. The thrust of its engines is 30,000 N. (a) Calculate its the magnitude of acceleration in a vertical takeoff from the Moon. (b) Could it lift off from Earth? If not, why not? If it could, calculate the magnitude of its acceleration.

Glossary

acceleration

the rate at which an object's velocity changes over a period of time

free-fall

a situation in which the only force acting on an object is the force due to gravity

friction

a force past each other of objects that are touching; examples include rough surfaces and air resistance

net external force

the vector sum of all external forces acting on an object or system; causes a mass to accelerate

Newton's second law of motion

the net external force \mathbf{F}_{net} on an object with mass m is proportional to and in the same direction as the acceleration of the object, \mathbf{a} , and inversely proportional to the mass; defined mathematically as

$$\mathbf{a} = \frac{\mathbf{F}_{\text{net}}}{m}$$

system

defined by the boundaries of an object or collection of objects being observed; all forces originating from outside of the system are considered external forces

weight

the force \mathbf{w} due to gravity acting on an object of mass m ; defined mathematically as: $\mathbf{w} = m\mathbf{g}$, where \mathbf{g} is the magnitude and direction of the acceleration due to gravity

Newton's Third Law of Motion: Symmetry in Forces

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Understand Newton's third law of motion.
- Apply Newton's third law to define systems and solve problems of motion.

The information presented in this section supports the following AP® learning objectives and science practices:

- **3.A.2.1** The student is able to represent forces in diagrams or mathematically using appropriately labeled vectors with magnitude, direction, and units during the analysis of a situation. **(S.P. 1.1)**
- **3.A.3.1** The student is able to analyze a scenario and make claims (develop arguments, justify assertions) about the forces exerted on an object by other objects for different types of forces or components of forces. **(S.P. 6.4, 7.2)**
- **3.A.3.3** The student is able to describe a force as an interaction between two objects and identify both objects for any force. **(S.P. 1.4)**
- **3.A.4.1** The student is able to construct explanations of physical situations involving the interaction of bodies using Newton's third law and the representation of action-reaction pairs of forces. **(S.P. 1.4, 6.2)**
- **3.A.4.2** The student is able to use Newton's third law to make claims and predictions about the action-reaction pairs of forces when two objects interact. **(S.P. 6.4, 7.2)**
- **3.A.4.3** The student is able to analyze situations involving interactions among several objects by using free-body

diagrams that include the application of Newton's third law to identify forces. **(S.P. 1.4)**

- **3.B.2.1** The student is able to create and use free-body diagrams to analyze physical situations to solve problems with motion qualitatively and quantitatively. **(S.P. 1.1, 1.4, 2.2)**
- **4.A.2.1** The student is able to make predictions about the motion of a system based on the fact that acceleration is equal to the change in velocity per unit time, and velocity is equal to the change in position per unit time. **(S.P. 6.4)**
- **4.A.2.2** The student is able to evaluate using given data whether all the forces on a system or whether all the parts of a system have been identified. **(S.P. 5.3)**
- **4.A.3.1** The student is able to apply Newton's second law to systems to calculate the change in the center-of-mass velocity when an external force is exerted on the system. **(S.P. 2.2)**

There is a passage in the musical *Man of la Mancha* that relates to Newton's third law of motion. Sancho, in describing a fight with his wife to Don Quixote, says, "Of course I hit her back, Your Grace, but she's a lot harder than me and you know what they say, 'Whether the stone hits the pitcher or the pitcher hits the stone, it's going to be bad for the pitcher.'" This is exactly what happens whenever one body exerts a force on another—the first also experiences a force (equal in magnitude and opposite in direction). Numerous common experiences, such as stubbing a toe or throwing a ball, confirm this. It is precisely stated in **Newton's third law of motion**.

Note:

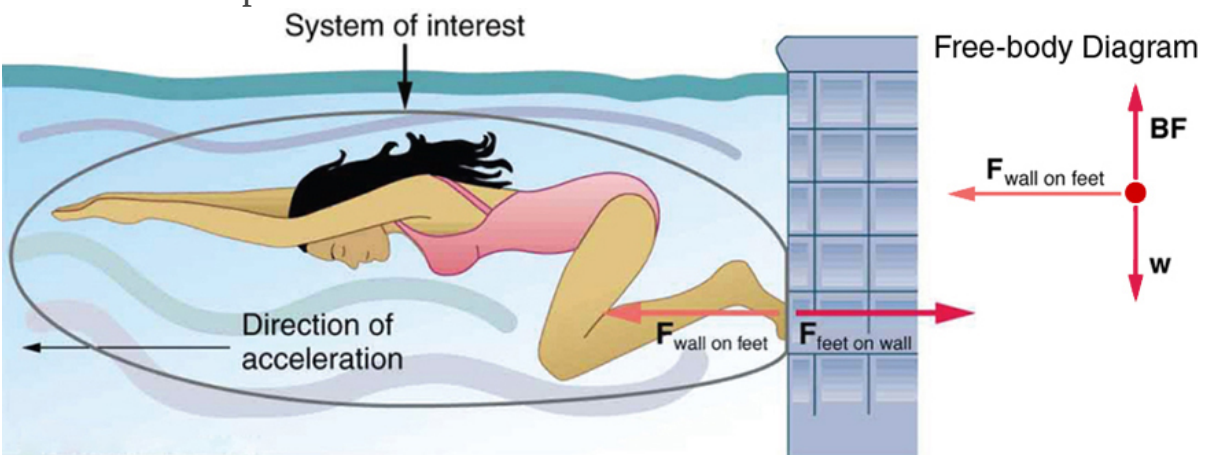
Newton's Third Law of Motion

Whenever one body exerts a force on a second body, the first body experiences a force that is equal in magnitude and opposite in direction to

the force that it exerts.

This law represents a certain *symmetry in nature*: Forces always occur in pairs, and one body cannot exert a force on another without experiencing a force itself. We sometimes refer to this law loosely as “action-reaction,” where the force exerted is the action and the force experienced as a consequence is the reaction. Newton’s third law has practical uses in analyzing the origin of forces and understanding which forces are external to a system.

We can readily see Newton’s third law at work by taking a look at how people move about. Consider a swimmer pushing off from the side of a pool, as illustrated in [\[link\]](#). She pushes against the pool wall with her feet and accelerates in the direction *opposite* to that of her push. The wall has exerted an equal and opposite force back on the swimmer. You might think that two equal and opposite forces would cancel, but they do not *because they act on different systems*. In this case, there are two systems that we could investigate: the swimmer or the wall. If we select the swimmer to be the system of interest, as in the figure, then $\mathbf{F}_{\text{wall on feet}}$ is an external force on this system and affects its motion. The swimmer moves in the direction of $\mathbf{F}_{\text{wall on feet}}$. In contrast, the force $\mathbf{F}_{\text{feet on wall}}$ acts on the wall and not on our system of interest. Thus $\mathbf{F}_{\text{feet on wall}}$ does not directly affect the motion of the system and does not cancel $\mathbf{F}_{\text{wall on feet}}$. Note that the swimmer pushes in the direction opposite to that in which she wishes to move. The reaction to her push is thus in the desired direction.



When the swimmer exerts a force $\mathbf{F}_{\text{feet on wall}}$ on the wall, she accelerates in the direction opposite to that of her push. This means the net external force on her is in the direction opposite to $\mathbf{F}_{\text{feet on wall}}$. This opposition occurs because, in accordance with Newton's third law of motion, the wall exerts a force $\mathbf{F}_{\text{wall on feet}}$ on her, equal in magnitude but in the direction opposite to the one she exerts on it. The line around the swimmer indicates the system of interest. Note that $\mathbf{F}_{\text{feet on wall}}$ does not act on this system (the swimmer) and, thus, does not cancel $\mathbf{F}_{\text{wall on feet}}$. Thus the free-body diagram shows only $\mathbf{F}_{\text{wall on feet}}$, \mathbf{w} , the gravitational force, and \mathbf{BF} , the buoyant force of the water supporting the swimmer's weight. The vertical forces \mathbf{w} and \mathbf{BF} cancel since there is no vertical motion.

Similarly, when a person stands on Earth, the Earth exerts a force on the person, pulling the person toward the Earth. As stated by Newton's third law of motion, the person also exerts a force that is equal in magnitude, but opposite in direction, pulling the Earth up toward the person. Since the mass of the Earth is so great, however, and $F = ma$, the acceleration of the Earth toward the person is not noticeable.

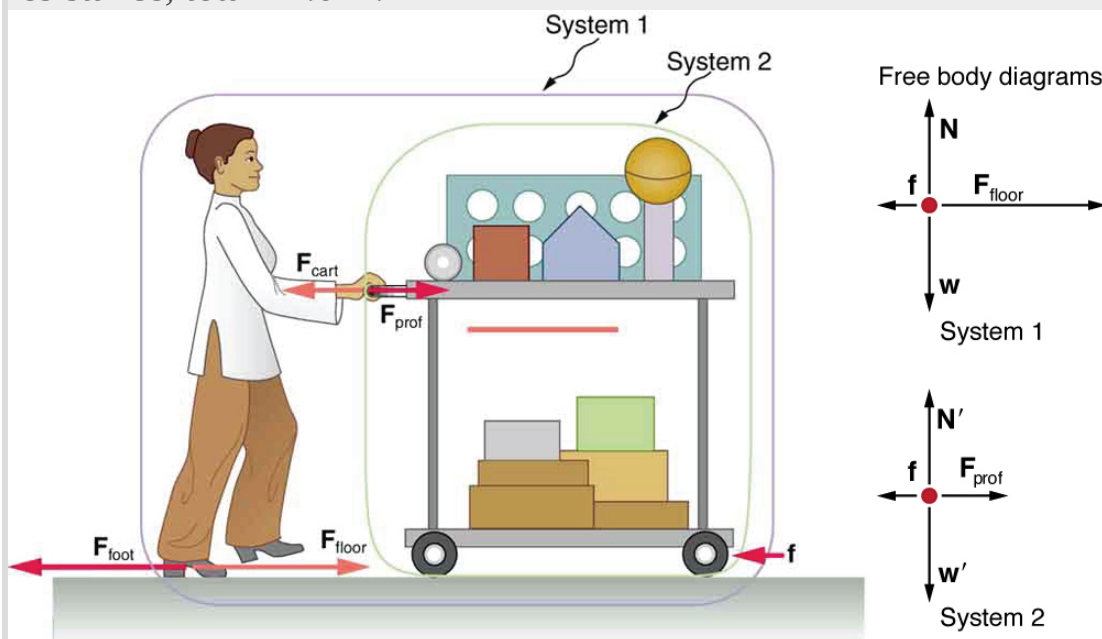
Other examples of Newton's third law are easy to find. As a professor paces in front of a whiteboard, she exerts a force backward on the floor. The floor exerts a reaction force forward on the professor that causes her to accelerate forward. Similarly, a car accelerates because the ground pushes forward on the drive wheels in reaction to the drive wheels pushing backward on the ground. You can see evidence of the wheels pushing backward when tires spin on a gravel road and throw rocks backward. In another example, rockets move forward by expelling gas backward at high velocity. This means the rocket exerts a large backward force on the gas in the rocket combustion chamber, and the gas therefore exerts a large reaction force forward on the rocket. This reaction force is called **thrust**. It is a common misconception that rockets propel themselves by pushing on the ground or on the air behind them. They actually work better in a vacuum, where they can more readily expel the exhaust gases. Helicopters similarly create lift by pushing air down, thereby experiencing an upward reaction force. Birds

and airplanes also fly by exerting force on air in a direction opposite to that of whatever force they need. For example, the wings of a bird force air downward and backward in order to get lift and move forward. An octopus propels itself in the water by ejecting water through a funnel from its body, similar to a jet ski. In a situation similar to Sancho's, professional cage fighters experience reaction forces when they punch, sometimes breaking their hand by hitting an opponent's body.

Example:

Getting Up To Speed: Choosing the Correct System

A physics professor pushes a cart of demonstration equipment to a lecture hall, as seen in [\[link\]](#). Her mass is 65.0 kg, the cart's is 12.0 kg, and the equipment's is 7.0 kg. Calculate the acceleration produced when the professor exerts a backward force of 150 N on the floor. All forces opposing the motion, such as friction on the cart's wheels and air resistance, total 24.0 N.



A professor pushes a cart of demonstration equipment. The lengths of the arrows are proportional to the magnitudes of the forces (except for f , since it is too small to draw to scale). Different questions are asked in each example; thus, the system

of interest must be defined differently for each. System 1 is appropriate for this example, since it asks for the acceleration of the entire group of objects. Only $\mathbf{F}_{\text{floor}}$ and \mathbf{f} are external forces acting on System 1 along the line of motion. All other forces either cancel or act on the outside world. System 2 is chosen for [\[link\]](#) so that \mathbf{F}_{prof} will be an external force and enter into Newton's second law. Note that the free-body diagrams, which allow us to apply Newton's second law, vary with the system chosen.

Strategy

Since they accelerate as a unit, we define the system to be the professor, cart, and equipment. This is System 1 in [\[link\]](#). The professor pushes backward with a force \mathbf{F}_{foot} of 150 N. According to Newton's third law, the floor exerts a forward reaction force $\mathbf{F}_{\text{floor}}$ of 150 N on System 1. Because all motion is horizontal, we can assume there is no net force in the vertical direction. The problem is therefore one-dimensional along the horizontal direction. As noted, \mathbf{f} opposes the motion and is thus in the opposite direction of $\mathbf{F}_{\text{floor}}$. Note that we do not include the forces \mathbf{F}_{prof} or \mathbf{F}_{cart} because these are internal forces, and we do not include \mathbf{F}_{foot} because it acts on the floor, not on the system. There are no other significant forces acting on System 1. If the net external force can be found from all this information, we can use Newton's second law to find the acceleration as requested. See the free-body diagram in the figure.

Solution

Newton's second law is given by

Equation:

$$a = \frac{F_{\text{net}}}{m}.$$

The net external force on System 1 is deduced from [\[link\]](#) and the discussion above to be

Equation:

$$F_{\text{net}} = F_{\text{floor}} - f = 150 \text{ N} - 24.0 \text{ N} = 126 \text{ N}.$$

The mass of System 1 is

Equation:

$$m = (65.0 + 12.0 + 7.0) \text{ kg} = 84 \text{ kg}.$$

These values of F_{net} and m produce an acceleration of

Equation:

$$a = \frac{F_{\text{net}}}{m},$$
$$a = \frac{126 \text{ N}}{84 \text{ kg}} = 1.5 \text{ m/s}^2.$$

Discussion

None of the forces between components of System 1, such as between the professor's hands and the cart, contribute to the net external force because they are internal to System 1. Another way to look at this is to note that forces between components of a system cancel because they are equal in magnitude and opposite in direction. For example, the force exerted by the professor on the cart results in an equal and opposite force back on her. In this case both forces act on the same system and, therefore, cancel. Thus internal forces (between components of a system) cancel. Choosing System 1 was crucial to solving this problem.

Example:

Force on the Cart—Choosing a New System

Calculate the force the professor exerts on the cart in [\[link\]](#) using data from the previous example if needed.

Strategy

If we now define the system of interest to be the cart plus equipment (System 2 in [\[link\]](#)), then the net external force on System 2 is the force the professor exerts on the cart minus friction. The force she exerts on the cart, \mathbf{F}_{prof} , is an external force acting on System 2. \mathbf{F}_{prof} was internal to System 1, but it is external to System 2 and will enter Newton's second law for System 2.

Solution

Newton's second law can be used to find \mathbf{F}_{prof} . Starting with

Equation:

$$a = \frac{F_{\text{net}}}{m}$$

and noting that the magnitude of the net external force on System 2 is

Equation:

$$F_{\text{net}} = F_{\text{prof}} - f,$$

we solve for F_{prof} , the desired quantity:

Equation:

$$F_{\text{prof}} = F_{\text{net}} + f.$$

The value of f is given, so we must calculate net F_{net} . That can be done since both the acceleration and mass of System 2 are known. Using Newton's second law we see that

Equation:

$$F_{\text{net}} = ma,$$

where the mass of System 2 is 19.0 kg ($m = 12.0 \text{ kg} + 7.0 \text{ kg}$) and its acceleration was found to be $a = 1.5 \text{ m/s}^2$ in the previous example. Thus,

Equation:

$$F_{\text{net}} = ma,$$

Equation:

$$F_{\text{net}} = (19.0 \text{ kg})(1.5 \text{ m/s}^2) = 29 \text{ N}.$$

Now we can find the desired force:

Equation:

$$F_{\text{prof}} = F_{\text{net}} + f,$$

Equation:

$$F_{\text{prof}} = 29 \text{ N} + 24.0 \text{ N} = 53 \text{ N}.$$

Discussion

It is interesting that this force is significantly less than the 150-N force the professor exerted backward on the floor. Not all of that 150-N force is transmitted to the cart; some of it accelerates the professor.

The choice of a system is an important analytical step both in solving problems and in thoroughly understanding the physics of the situation (which is not necessarily the same thing).

Note:

PhET Explorations: Gravity Force Lab

Visualize the gravitational force that two objects exert on each other.

Change properties of the objects in order to see how it changes the gravity force.

[Gravit](#)
[y Force](#)
[Lab](#)

Test Prep for AP Courses**Exercise:****Problem:**

What object or objects commonly exert forces on the following objects in motion? (a) a soccer ball being kicked, (b) a dolphin jumping, (c) a parachutist drifting to Earth.

Solution:

(a) A soccer player, gravity, air, and friction commonly exert forces on a soccer ball being kicked.

(b) Gravity and the surrounding water commonly exert forces on a dolphin jumping. (The dolphin moves its muscles to exert a force on the water. The water exerts an equal force on the dolphin, resulting in the dolphin's motion.)

(c) Gravity and air exert forces on a parachutist drifting to Earth.

Exercise:

Problem:

A ball with a mass of 0.25 kg hits a gym ceiling with a force of 78.0 N. What happens next?

- a. The ball accelerates downward with a force of 80.5 N.
- b. The ball accelerates downward with a force of 78.0 N.
- c. The ball accelerates downward with a force of 2.45 N.
- d. It depends on the height of the ceiling.

Exercise:

Problem: Which of the following is true?

- a. Earth exerts a force due to gravity on your body, and your body exerts a smaller force on the Earth, because your mass is smaller than the mass of the Earth.
- b. The Moon orbits the Earth because the Earth exerts a force on the Moon and the Moon exerts a force equal in magnitude and direction on the Earth.
- c. A rocket taking off exerts a force on the Earth equal to the force the Earth exerts on the rocket.
- d. An airplane cruising at a constant speed is not affected by gravity.

Solution:

(c)

Exercise:

Problem:

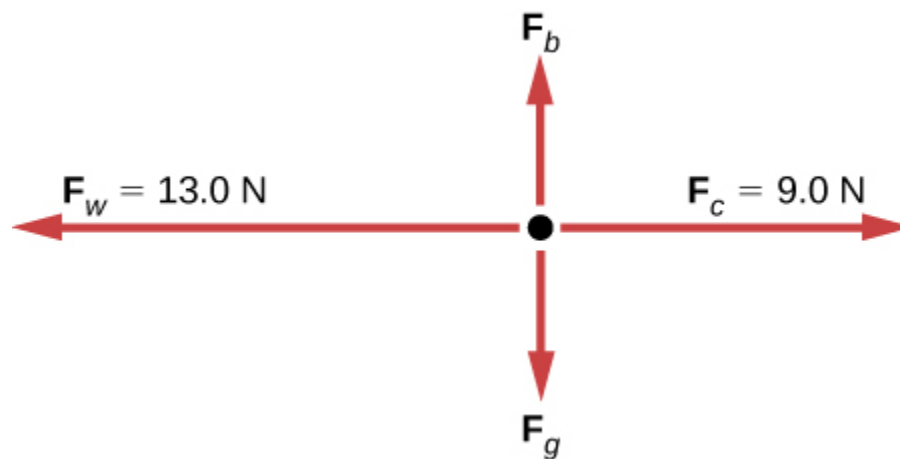
Stationary skater A pushes stationary skater B, who then accelerates at 5.0 m/s^2 . Skater A does not move. Since forces act in action-reaction pairs, explain why Skater A did not move?

Exercise:

Problem:

The current in a river exerts a force of 9.0 N on a balloon floating in the river. A wind exerts a force of 13.0 N on the balloon in the opposite direction. Draw a free-body diagram to show the forces acting on the balloon. Use your free-body diagram to predict the effect on the balloon.

Solution:



The diagram consists of a black dot in the center and two small red arrows pointing up (F_b) and down (F_g) and two long red arrows pointing right ($F_c = 9.0 \text{ N}$) and left ($F_w = 13.0 \text{ N}$).

In the diagram, F_g represents the force due to gravity on the balloon, and F_b represents the buoyant force. These two forces are equal in magnitude and opposite in direction. F_c represents the force of the current. F_w represents the force of the wind. The net force on the

balloon will be $F_w - F_c = 4.0 \text{ N}$ and the balloon will accelerate in the direction the wind is blowing.

Exercise:

Problem:

A force is applied to accelerate an object on a smooth icy surface. When the force stops, which of the following will be true? (Assume zero friction.)

- a. The object's acceleration becomes zero.
- b. The object's speed becomes zero.
- c. The object's acceleration continues to increase at a constant rate.
- d. The object accelerates, but in the opposite direction.

Exercise:

Problem:

A parachutist's fall to Earth is determined by two opposing forces. A gravitational force of 539 N acts on the parachutist. After 2 s, she opens her parachute and experiences an air resistance of 615 N. At what speed is the parachutist falling after 10 s?

Solution:

Since $m = F/a$, the parachutist has a mass of $539 \text{ N}/9.8 \text{ m/s}^2 = 55 \text{ kg}$.

For the first 2 s, the parachutist accelerates at 9.8 m/s^2 .

$$\begin{aligned} v &= at \\ &= 9.8 \frac{\text{m}}{\text{s}^2} \bullet 2\text{s} \\ &= 19.6 \frac{\text{m}}{\text{s}} \end{aligned}$$

Her speed after 2 s is 19.6 m/s.

From 2 s to 10 s, the net force on the parachutist is 539 N – 615 N, or 76 N upward.

$$\begin{aligned}a &= \frac{F}{m} \\&= \frac{-76 \text{ N}}{55 \text{ kg}} \\&= -1.4 \frac{\text{m}}{\text{s}^2}\end{aligned}$$

Since $v = v_0 + at$, $v = 17.6 \text{ m/s}^2 + (-1.4 \text{ m/s}^2)(8\text{s}) = 6.5 \text{ m/s}^2$.

At 10 s, the parachutist is falling to Earth at 8.4 m/s.

Exercise:

Problem:

A flight attendant pushes a cart down the aisle of a plane in flight. In determining the acceleration of the cart relative to the plane, which factor do you not need to consider?

- a. The friction of the cart's wheels.
- b. The force with which the flight attendant's feet push on the floor.
- c. The velocity of the plane.
- d. The mass of the items in the cart.

Exercise:

Problem:

A landscaper is easing a wheelbarrow full of soil down a hill. Define the system you would analyze and list all the forces that you would need to include to calculate the acceleration of the wheelbarrow.

Solution:

The system includes the gardener and the wheelbarrow with its contents. The following forces are important to include: the weight of the wheelbarrow, the weight of the gardener, the normal force for the wheelbarrow and the gardener, the force of the gardener pushing

against the ground and the equal force of the ground pushing back against the gardener, and any friction in the wheelbarrow's wheels.

Exercise:

Problem:

Two water-skiers, with masses of 48 kg and 61 kg, are preparing to be towed behind the same boat. When the boat accelerates, the rope the skiers hold onto accelerates with it and exerts a net force of 290 N on the skiers. At what rate will the skiers accelerate?

- a. 10.8 m/s^2
- b. 2.7 m/s^2
- c. 6.0 m/s^2 and 4.8 m/s^2
- d. 5.3 m/s^2

Exercise:

Problem:

A figure skater has a mass of 40 kg and her partner's mass is 50 kg. She pushes against the ice with a force of 120 N, causing her and her partner to move forward. Calculate the pair's acceleration. Assume that all forces opposing the motion, such as friction and air resistance, total 5.0 N.

Solution:

The system undergoing acceleration is the two figure skaters together.

$$\text{Net force} = 120 \text{ N} - 5.0 \text{ N} = 115 \text{ N}.$$

$$\text{Total mass} = 40 \text{ kg} + 50 \text{ kg} = 90 \text{ kg}.$$

Using Newton's second law, we have that

$$\begin{aligned}
 a &= \frac{F}{m} \\
 &= \frac{115 \text{ N}}{90 \text{ kg}} \\
 &= 1.28 \frac{\text{m}}{\text{s}^2}
 \end{aligned}$$

The pair accelerates forward at 1.28 m/s^2 .

Section Summary

- **Newton's third law of motion** represents a basic symmetry in nature. It states: Whenever one body exerts a force on a second body, the first body experiences a force that is equal in magnitude and opposite in direction to the force that the first body exerts.
- A **thrust** is a reaction force that pushes a body forward in response to a backward force. Rockets, airplanes, and cars are pushed forward by a thrust reaction force.

Conceptual Questions

Exercise:

Problem:

When you take off in a jet aircraft, there is a sensation of being pushed back into the seat. Explain why you move backward in the seat—is there really a force backward on you? (The same reasoning explains whiplash injuries, in which the head is apparently thrown backward.)

Exercise:

Problem:

A device used since the 1940s to measure the kick or recoil of the body due to heart beats is the “ballistocardiograph.” What physics principle(s) are involved here to measure the force of cardiac contraction? How might we construct such a device?

Exercise:

Problem:

Describe a situation in which one system exerts a force on another and, as a consequence, experiences a force that is equal in magnitude and opposite in direction. Which of Newton's laws of motion apply?

Exercise:**Problem:**

Why does an ordinary rifle recoil (kick backward) when fired? The barrel of a recoilless rifle is open at both ends. Describe how Newton's third law applies when one is fired. Can you safely stand close behind one when it is fired?

Exercise:**Problem:**

An American football lineman reasons that it is senseless to try to out-push the opposing player, since no matter how hard he pushes he will experience an equal and opposite force from the other player. Use Newton's laws and draw a free-body diagram of an appropriate system to explain how he can still out-push the opposition if he is strong enough.

Exercise:**Problem:**

Newton's third law of motion tells us that forces always occur in pairs of equal and opposite magnitude. Explain how the choice of the "system of interest" affects whether one such pair of forces cancels.

Problem Exercises**Exercise:**

Problem:

What net external force is exerted on a 1100-kg artillery shell fired from a battleship if the shell is accelerated at $2.40 \times 10^4 \text{ m/s}^2$? What is the magnitude of the force exerted on the ship by the artillery shell?

Solution:

Force on shell: $2.64 \times 10^7 \text{ N}$

Force exerted on ship = $-2.64 \times 10^7 \text{ N}$, by Newton's third law

Exercise:**Problem:**

A brave but inadequate rugby player is being pushed backward by an opposing player who is exerting a force of 800 N on him. The mass of the losing player plus equipment is 90.0 kg, and he is accelerating at 1.20 m/s^2 backward. (a) What is the force of friction between the losing player's feet and the grass? (b) What force does the winning player exert on the ground to move forward if his mass plus equipment is 110 kg? (c) Draw a sketch of the situation showing the system of interest used to solve each part. For this situation, draw a free-body diagram and write the net force equation.

Glossary**Newton's third law of motion**

whenever one body exerts a force on a second body, the first body experiences a force that is equal in magnitude and opposite in direction to the force that the first body exerts

thrust

a reaction force that pushes a body forward in response to a backward force; rockets, airplanes, and cars are pushed forward by a thrust reaction force

Normal, Tension, and Other Examples of Force

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define normal and tension forces.
- Apply Newton's laws of motion to solve problems involving a variety of forces.
- Use trigonometric identities to resolve weight into components.

The information presented in this section supports the following AP[®] learning objectives and science practices:

- **2.B.1.1** The student is able to apply $F = mg$ to calculate the gravitational force on an object with mass m in a gravitational field of strength g in the context of the effects of a net force on objects and systems. **(S.P. 2.2, 7.2)**
- **3.A.2.1** The student is able to represent forces in diagrams or mathematically using appropriately labeled vectors with magnitude, direction, and units during the analysis of a situation. **(S.P. 1.1)**
- **3.A.3.1** The student is able to analyze a scenario and make claims (develop arguments, justify assertions) about the forces exerted on an object by other objects for different types of forces or components of forces. **(S.P. 6.4, 7.2)**
- **3.A.3.3** The student is able to describe a force as an interaction between two objects and identify both objects for any force. **(S.P. 1.4)**
- **3.A.4.1** The student is able to construct explanations of physical situations involving the interaction of bodies using

Newton's third law and the representation of action-reaction pairs of forces. (S.P. 1.4, 6.2)

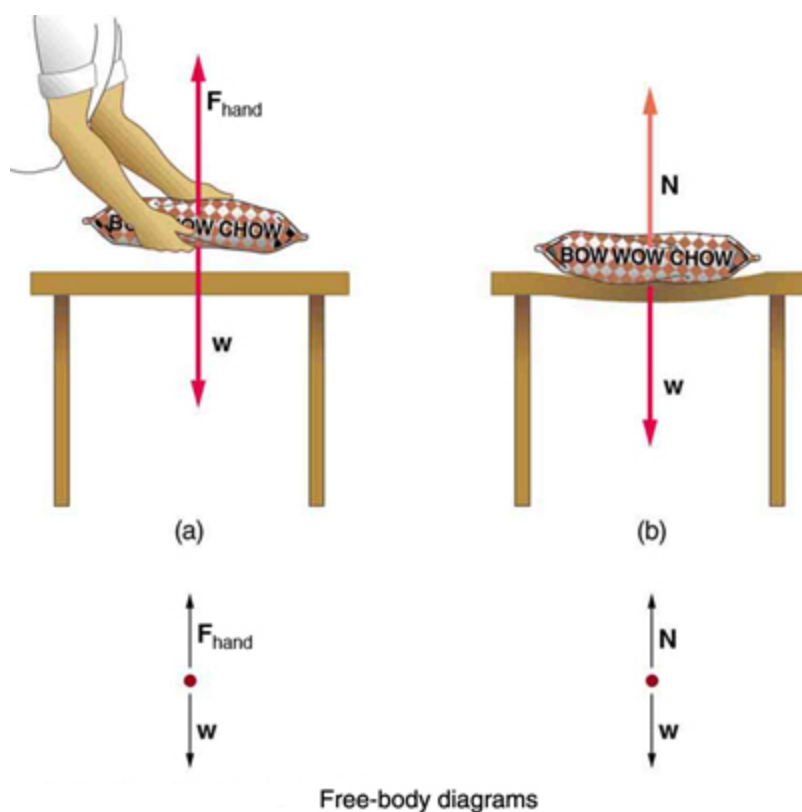
- **3.A.4.2** The student is able to use Newton's third law to make claims and predictions about the action-reaction pairs of forces when two objects interact. (S.P. 6.4, 7.2)
- **3.A.4.3** The student is able to analyze situations involving interactions among several objects by using free-body diagrams that include the application of Newton's third law to identify forces. (S.P. 1.4)
- **3.B.1.3** The student is able to re-express a free-body diagram representation into a mathematical representation and solve the mathematical representation for the acceleration of the object. (S.P. 1.5, 2.2)
- **3.B.2.1** The student is able to create and use free-body diagrams to analyze physical situations to solve problems with motion qualitatively and quantitatively. (S.P. 1.1, 1.4, 2.2)

Forces are given many names, such as push, pull, thrust, lift, weight, friction, and tension. Traditionally, forces have been grouped into several categories and given names relating to their source, how they are transmitted, or their effects. The most important of these categories are discussed in this section, together with some interesting applications. Further examples of forces are discussed later in this text.

Normal Force

Weight (also called force of gravity) is a pervasive force that acts at all times and must be counteracted to keep an object from falling. You definitely notice that you must support the weight of a heavy object by pushing up on it when you hold it stationary, as illustrated in [\[link\]](#)(a). But how do inanimate objects like a table support the weight of a mass placed on them, such as shown in [\[link\]](#)(b)? When the bag of dog food is placed on the table, the table actually sags slightly under the load. This would be

noticeable if the load were placed on a card table, but even rigid objects deform when a force is applied to them. Unless the object is deformed beyond its limit, it will exert a restoring force much like a deformed spring (or trampoline or diving board). The greater the deformation, the greater the restoring force. So when the load is placed on the table, the table sags until the restoring force becomes as large as the weight of the load. At this point the net external force on the load is zero. That is the situation when the load is stationary on the table. The table sags quickly, and the sag is slight so we do not notice it. But it is similar to the sagging of a trampoline when you climb onto it.



(a) The person holding the bag of dog food must supply an upward force \mathbf{F}_{hand} equal in magnitude and opposite in direction to the weight of the food \mathbf{w} . (b) The card table sags when the dog food is placed on it, much like a stiff trampoline. Elastic restoring

forces in the table grow as it sags until they supply a force \mathbf{N} equal in magnitude and opposite in direction to the weight of the load.

We must conclude that whatever supports a load, be it animate or not, must supply an upward force equal to the weight of the load, as we assumed in a few of the previous examples. If the force supporting a load is perpendicular to the surface of contact between the load and its support, this force is defined to be a **normal force** and here is given the symbol \mathbf{N} . (This is not the unit for force \mathbf{N} .) The word *normal* means perpendicular to a surface. The normal force can be less than the object's weight if the object is on an incline, as you will see in the next example.

Note:

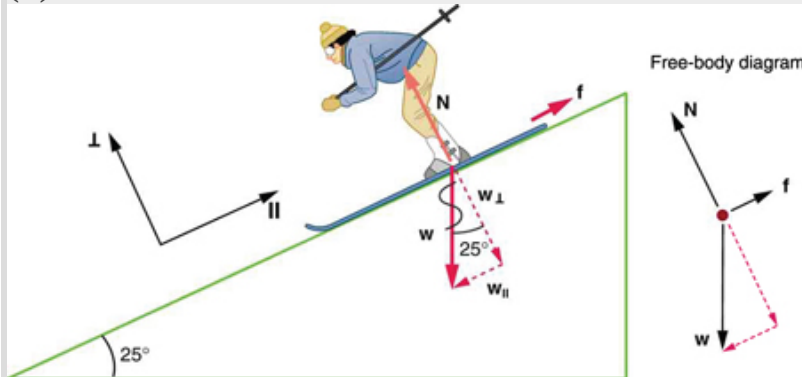
Common Misconception: Normal Force (\mathbf{N}) vs. Newton (\mathbf{N})

In this section we have introduced the quantity normal force, which is represented by the variable \mathbf{N} . This should not be confused with the symbol for the newton, which is also represented by the letter \mathbf{N} . These symbols are particularly important to distinguish because the units of a normal force (\mathbf{N}) happen to be newtons (\mathbf{N}). For example, the normal force \mathbf{N} that the floor exerts on a chair might be $\mathbf{N} = 100 \text{ N}$. One important difference is that normal force is a vector, while the newton is simply a unit. Be careful not to confuse these letters in your calculations! You will encounter more similarities among variables and units as you proceed in physics. Another example of this is the quantity work (W) and the unit watts (\mathbf{W}).

Example:

Weight on an Incline, a Two-Dimensional Problem

Consider the skier on a slope shown in [\[link\]](#). Her mass including equipment is 60.0 kg. (a) What is her acceleration if friction is negligible? (b) What is her acceleration if friction is known to be 45.0 N?



Since motion and friction are parallel to the slope, it is most convenient to project all forces onto a coordinate system where one axis is parallel to the slope and the other is perpendicular (axes shown to left of skier).

\mathbf{N} is perpendicular to the slope and \mathbf{f} is parallel to the slope, but \mathbf{w} has components along both axes, namely \mathbf{w}_{\perp} and \mathbf{w}_{\parallel} . \mathbf{N} is equal in magnitude to \mathbf{w}_{\perp} , so that there is no motion perpendicular to the slope, but f is less than w_{\parallel} , so that there is a downslope acceleration (along the parallel axis).

Strategy

This is a two-dimensional problem, since the forces on the skier (the system of interest) are not parallel. The approach we have used in two-dimensional kinematics also works very well here. Choose a convenient coordinate system and project the vectors onto its axes, creating *two* connected *one*-dimensional problems to solve. The most convenient coordinate system for motion on an incline is one that has one coordinate parallel to the slope and one perpendicular to the slope. (Remember that motions along mutually perpendicular axes are independent.) We use the symbols \perp and \parallel to represent perpendicular and parallel, respectively. This

choice of axes simplifies this type of problem, because there is no motion perpendicular to the slope and because friction is always parallel to the surface between two objects. The only external forces acting on the system are the skier's weight, friction, and the support of the slope, respectively labeled \mathbf{w} , \mathbf{f} , and \mathbf{N} in [\[link\]](#). \mathbf{N} is always perpendicular to the slope, and \mathbf{f} is parallel to it. But \mathbf{w} is not in the direction of either axis, and so the first step we take is to project it into components along the chosen axes, defining w_{\parallel} to be the component of weight parallel to the slope and w_{\perp} the component of weight perpendicular to the slope. Once this is done, we can consider the two separate problems of forces parallel to the slope and forces perpendicular to the slope.

Solution

The magnitude of the component of the weight parallel to the slope is $w_{\parallel} = w \sin (25^{\circ}) = mg \sin (25^{\circ})$, and the magnitude of the component of the weight perpendicular to the slope is $w_{\perp} = w \cos (25^{\circ}) = mg \cos (25^{\circ})$.

(a) Neglecting friction. Since the acceleration is parallel to the slope, we need only consider forces parallel to the slope. (Forces perpendicular to the slope add to zero, since there is no acceleration in that direction.) The forces parallel to the slope are the amount of the skier's weight parallel to the slope w_{\parallel} and friction f . Using Newton's second law, with subscripts to denote quantities parallel to the slope,

Equation:

$$a_{\parallel} = \frac{F_{\text{net}\parallel}}{m}$$

where $F_{\text{net}\parallel} = w_{\parallel} = mg \sin (25^{\circ})$, assuming no friction for this part, so that

Equation:

$$a_{\parallel} = \frac{F_{\text{net}\parallel}}{m} = \frac{mg \sin (25^{\circ})}{m} = g \sin (25^{\circ})$$

Equation:

$$(9.80 \text{ m/s}^2)(0.4226) = 4.14 \text{ m/s}^2$$

is the acceleration.

(b) Including friction. We now have a given value for friction, and we know its direction is parallel to the slope and it opposes motion between surfaces in contact. So the net external force is now

Equation:

$$F_{\text{net}\parallel} = w_{\parallel} - f,$$

and substituting this into Newton's second law, $a_{\parallel} = \frac{F_{\text{net}\parallel}}{m}$, gives

Equation:

$$a_{\parallel} = \frac{F_{\text{net}\parallel}}{m} = \frac{w_{\parallel} - f}{m} = \frac{mg \sin(25^\circ) - f}{m}.$$

We substitute known values to obtain

Equation:

$$a_{\parallel} = \frac{(60.0 \text{ kg})(9.80 \text{ m/s}^2)(0.4226) - 45.0 \text{ N}}{60.0 \text{ kg}},$$

which yields

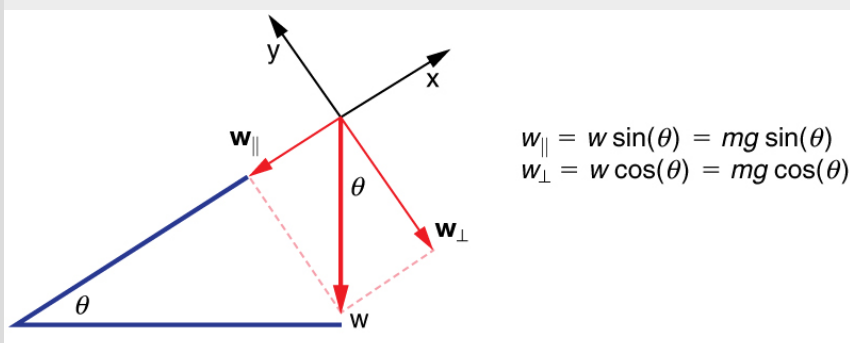
Equation:

$$a_{\parallel} = 3.39 \text{ m/s}^2,$$

which is the acceleration parallel to the incline when there is 45.0 N of opposing friction.

Discussion

Since friction always opposes motion between surfaces, the acceleration is smaller when there is friction than when there is none. In fact, it is a general result that if friction on an incline is negligible, then the acceleration down the incline is $a = g \sin\theta$, *regardless of mass*. This is related to the previously discussed fact that all objects fall with the same acceleration in the absence of air resistance. Similarly, all objects, regardless of mass, slide down a frictionless incline with the same acceleration (if the angle is the same).

Note:**Resolving Weight into Components**

An object rests on an incline that makes an angle θ with the horizontal.

When an object rests on an incline that makes an angle θ with the horizontal, the force of gravity acting on the object is divided into two components: a force acting perpendicular to the plane, w_{\perp} , and a force acting parallel to the plane, w_{\parallel} . The perpendicular force of weight, w_{\perp} , is typically equal in magnitude and opposite in direction to the normal force, \mathbf{N} . The force acting parallel to the plane, w_{\parallel} , causes the object to accelerate down the incline. The force of friction, \mathbf{f} , opposes the motion of the object, so it acts upward along the plane.

It is important to be careful when resolving the weight of the object into components. If the angle of the incline is at an angle θ to the horizontal, then the magnitudes of the weight components are

Equation:

$$w_{\parallel} = w \sin(\theta) = mg \sin(\theta)$$

and

Equation:

$$w_{\perp} = w \cos(\theta) = mg \cos(\theta).$$

Instead of memorizing these equations, it is helpful to be able to determine them from reason. To do this, draw the right triangle formed by the three weight vectors. Notice that the angle θ of the incline is the same as the angle formed between \mathbf{w} and \mathbf{w}_\perp . Knowing this property, you can use trigonometry to determine the magnitude of the weight components:

Equation:

$$\begin{aligned}\cos(\theta) &= \frac{w_\perp}{w} \\ w_\perp &= w \cos(\theta) = mg \cos(\theta)\end{aligned}$$

Equation:

$$\begin{aligned}\sin(\theta) &= \frac{w_\parallel}{w} \\ w_\parallel &= w \sin(\theta) = mg \sin(\theta)\end{aligned}$$

Note:

Take-Home Experiment: Force Parallel

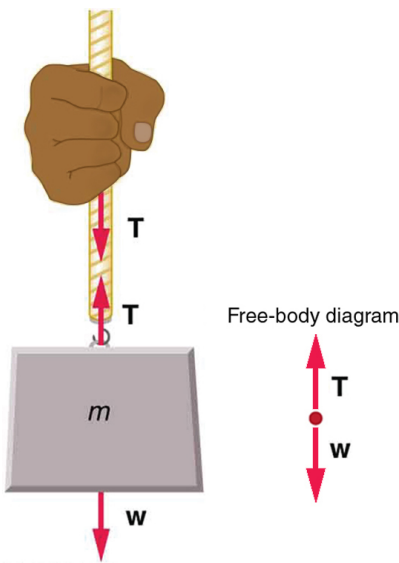
To investigate how a force parallel to an inclined plane changes, find a rubber band, some objects to hang from the end of the rubber band, and a board you can position at different angles. How much does the rubber band stretch when you hang the object from the end of the board? Now place the board at an angle so that the object slides off when placed on the board. How much does the rubber band extend if it is lined up parallel to the board and used to hold the object stationary on the board? Try two more angles. What does this show?

Tension

A **tension** is a force along the length of a medium, especially a force carried by a flexible medium, such as a rope or cable. The word “tension” comes from a Latin word meaning “to stretch.” Not coincidentally, the flexible cords that carry muscle forces to other parts of the body are called *tendons*.

Any flexible connector, such as a string, rope, chain, wire, or cable, can exert pulls only parallel to its length; thus, a force carried by a flexible connector is a tension with direction parallel to the connector. It is important to understand that tension is a pull in a connector. In contrast, consider the phrase: “You can’t push a rope.” The tension force pulls outward along the two ends of a rope.

Consider a person holding a mass on a rope as shown in [\[link\]](#).



When a perfectly flexible connector (one requiring no force to bend it) such as this rope transmits a force \mathbf{T} , that force must be parallel to the length of the rope, as shown. The pull such a flexible connector exerts is a tension. Note that

the rope pulls with equal force but in opposite directions on the hand and the supported mass (neglecting the weight of the rope). This is an example of Newton's third law. The rope is the medium that carries the equal and opposite forces between the two objects. The tension anywhere in the rope between the hand and the mass is equal. Once you have determined the tension in one location, you have determined the tension at all locations along the rope.

Tension in the rope must equal the weight of the supported mass, as we can prove using Newton's second law. If the 5.00-kg mass in the figure is stationary, then its acceleration is zero, and thus $\mathbf{F}_{\text{net}} = 0$. The only external forces acting on the mass are its weight \mathbf{w} and the tension \mathbf{T} supplied by the rope. Thus,

Equation:

$$F_{\text{net}} = T - w = 0,$$

where T and w are the magnitudes of the tension and weight and their signs indicate direction, with up being positive here. Thus, just as you would expect, the tension equals the weight of the supported mass:

Equation:

$$T = w = mg.$$

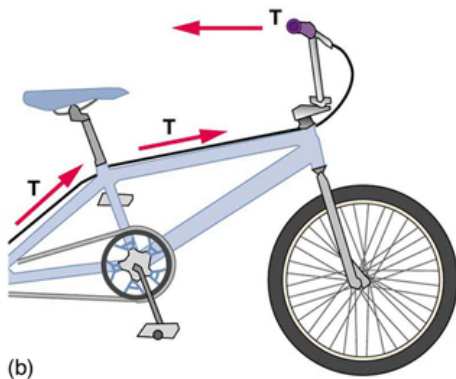
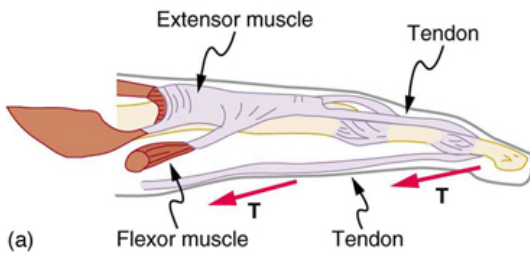
For a 5.00-kg mass, then (neglecting the mass of the rope) we see that

Equation:

$$T = mg = (5.00 \text{ kg})(9.80 \text{ m/s}^2) = 49.0 \text{ N}.$$

If we cut the rope and insert a spring, the spring would extend a length corresponding to a force of 49.0 N, providing a direct observation and measure of the tension force in the rope.

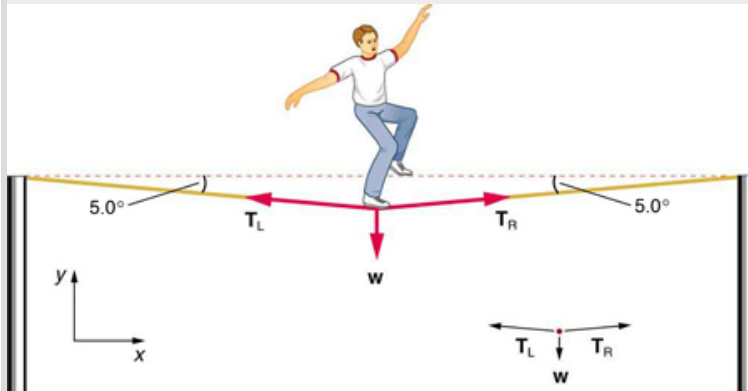
Flexible connectors are often used to transmit forces around corners, such as in a hospital traction system, a finger joint, or a bicycle brake cable. If there is no friction, the tension is transmitted undiminished. Only its direction changes, and it is always parallel to the flexible connector. This is illustrated in [\[link\]](#) (a) and (b).



(a) Tendons in the finger carry force T from the muscles to other parts of the finger, usually changing the force's direction, but not its magnitude (the tendons are relatively friction free). (b) The brake cable on a bicycle carries the tension T from the handlebars to the brake mechanism. Again, the direction but not the magnitude of T is changed.

Example:
What Is the Tension in a Tightrope?

Calculate the tension in the wire supporting the 70.0-kg tightrope walker shown in [\[link\]](#).



The weight of a tightrope walker causes a wire to sag by 5.0° . The system of interest here is the point in the wire at which the tightrope walker is standing.

Strategy

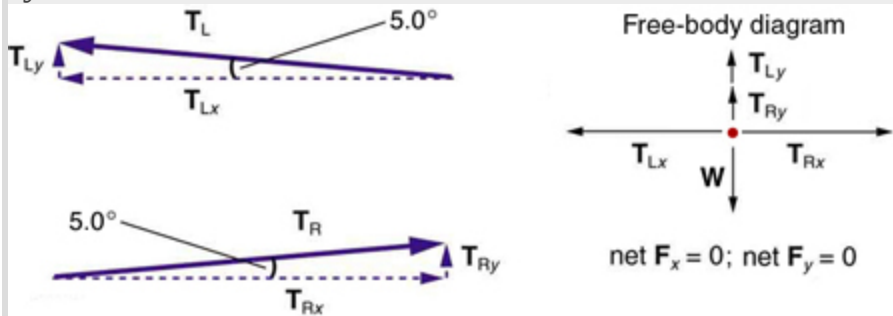
As you can see in the figure, the wire is not perfectly horizontal (it cannot be!), but is bent under the person's weight. Thus, the tension on either side of the person has an upward component that can support his weight. As usual, forces are vectors represented pictorially by arrows having the same directions as the forces and lengths proportional to their magnitudes. The system is the tightrope walker, and the only external forces acting on him are his weight w and the two tensions T_L (left tension) and T_R (right tension), as illustrated. It is reasonable to neglect the weight of the wire itself. The net external force is zero since the system is stationary. A little trigonometry can now be used to find the tensions. One conclusion is possible at the outset—we can see from part (b) of the figure that the magnitudes of the tensions T_L and T_R must be equal. This is because there is no horizontal acceleration in the rope, and the only forces acting to the left and right are T_L and T_R . Thus, the magnitude of those forces must be equal so that they cancel each other out.

Whenever we have two-dimensional vector problems in which no two vectors are parallel, the easiest method of solution is to pick a convenient coordinate system and project the vectors onto its axes. In this case the best

coordinate system has one axis horizontal and the other vertical. We call the horizontal the x -axis and the vertical the y -axis.

Solution

First, we need to resolve the tension vectors into their horizontal and vertical components. It helps to draw a new free-body diagram showing all of the horizontal and vertical components of each force acting on the system.



When the vectors are projected onto vertical and horizontal axes, their components along those axes must add to zero, since the tightrope walker is stationary. The small angle results in T being much greater than w .

Consider the horizontal components of the forces (denoted with a subscript x):

Equation:

$$F_{\text{net}x} = T_{Lx} - T_{Rx}.$$

The net external horizontal force $F_{\text{net}x} = 0$, since the person is stationary. Thus,

Equation:

$$\begin{aligned} F_{\text{net}x} = 0 &= T_{Lx} - T_{Rx} \\ T_{Lx} &= T_{Rx}. \end{aligned}$$

Now, observe [\[link\]](#). You can use trigonometry to determine the magnitude of T_L and T_R . Notice that:

Equation:

$$\begin{aligned}\cos (5.0^{\circ}) &= \frac{T_{Lx}}{T_L} \\ T_{Lx} &= T_L \cos (5.0^{\circ}) \\ \cos (5.0^{\circ}) &= \frac{T_{Rx}}{T_R} \\ T_{Rx} &= T_R \cos (5.0^{\circ}).\end{aligned}$$

Equating T_{Lx} and T_{Rx} :

Equation:

$$T_L \cos (5.0^{\circ}) = T_R \cos (5.0^{\circ}).$$

Thus,

Equation:

$$T_L = T_R = T,$$

as predicted. Now, considering the vertical components (denoted by a subscript y), we can solve for T . Again, since the person is stationary, Newton's second law implies that net $F_y = 0$. Thus, as illustrated in the free-body diagram in [\[link\]](#),

Equation:

$$F_{\text{net}y} = T_{Ly} + T_{Ry} - w = 0.$$

Observing [\[link\]](#), we can use trigonometry to determine the relationship between T_{Ly} , T_{Ry} , and T . As we determined from the analysis in the horizontal direction, $T_L = T_R = T$:

Equation:

$$\begin{aligned}\sin (5.0^{\circ}) &= \frac{T_{Ly}}{T_L} \\ T_{Ly} = T_L \sin (5.0^{\circ}) &= T \sin (5.0^{\circ}) \\ \sin (5.0^{\circ}) &= \frac{T_{Ry}}{T_R} \\ T_{Ry} = T_R \sin (5.0^{\circ}) &= T \sin (5.0^{\circ}).\end{aligned}$$

Now, we can substitute the values for T_{Ly} and T_{Ry} , into the net force equation in the vertical direction:

Equation:

$$\begin{aligned}F_{\text{net}y} &= T_{Ly} + T_{Ry} - w = 0 \\F_{\text{net}y} &= T \sin (5.0^\circ) + T \sin (5.0^\circ) - w = 0 \\2 T \sin (5.0^\circ) - w &= 0 \\2 T \sin (5.0^\circ) &= w\end{aligned}$$

and

Equation:

$$T = \frac{w}{2 \sin (5.0^\circ)} = \frac{mg}{2 \sin (5.0^\circ)},$$

so that

Equation:

$$T = \frac{(70.0 \text{ kg})(9.80 \text{ m/s}^2)}{2(0.0872)},$$

and the tension is

Equation:

$$T = 3900 \text{ N}.$$

Discussion

Note that the vertical tension in the wire acts as a normal force that supports the weight of the tightrope walker. The tension is almost six times the 686-N weight of the tightrope walker. Since the wire is nearly horizontal, the vertical component of its tension is only a small fraction of the tension in the wire. The large horizontal components are in opposite directions and cancel, and so most of the tension in the wire is not used to support the weight of the tightrope walker.

If we wish to *create* a very large tension, all we have to do is exert a force perpendicular to a flexible connector, as illustrated in [\[link\]](#). As we saw in the last example, the weight of the tightrope walker acted as a force perpendicular to the rope. We saw that the tension in the roped related to the weight of the tightrope walker in the following way:

Equation:

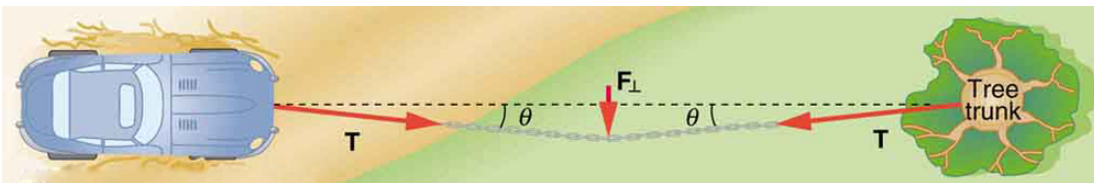
$$T = \frac{w}{2 \sin (\theta)} .$$

We can extend this expression to describe the tension T created when a perpendicular force (\mathbf{F}_{\perp}) is exerted at the middle of a flexible connector:

Equation:

$$T = \frac{F_{\perp}}{2 \sin (\theta)} .$$

Note that θ is the angle between the horizontal and the bent connector. In this case, T becomes very large as θ approaches zero. Even the relatively small weight of any flexible connector will cause it to sag, since an infinite tension would result if it were horizontal (i.e., $\theta = 0$ and $\sin \theta = 0$). (See [\[link\]](#).)



We can create a very large tension in the chain by pushing on it perpendicular to its length, as shown. Suppose we wish to pull a car out of the mud when no tow truck is available. Each time the car moves forward, the chain is tightened to keep it as nearly straight as possible. The tension in the chain is given by $T = \frac{F_{\perp}}{2 \sin (\theta)}$; since θ is small, T is very large. This situation is

analogous to the tightrope walker shown in [\[link\]](#), except that the tensions shown here are those transmitted to the car and the tree rather than those acting at the point where \mathbf{F}_\perp is applied.



Unless an infinite tension is exerted, any flexible connector—such as the chain at the bottom of the picture—will sag under its own weight, giving a characteristic curve when the weight is evenly distributed along the length.

Suspension bridges—such as the Golden Gate Bridge shown in this image—are essentially very heavy flexible connectors. The weight of the bridge is evenly distributed along the length of flexible connectors, usually cables, which take on the characteristic shape.

(credit: Leaflet, Wikimedia Commons)

Extended Topic: Real Forces and Inertial Frames

There is another distinction among forces in addition to the types already mentioned. Some forces are real, whereas others are not. *Real forces* are those that have some physical origin, such as the gravitational pull.

Contrastingly, *fictitious forces* are those that arise simply because an observer is in an accelerating frame of reference, such as one that rotates (like a merry-go-round) or undergoes linear acceleration (like a car slowing down). For example, if a satellite is heading due north above Earth's northern hemisphere, then to an observer on Earth it will appear to experience a force to the west that has no physical origin. Of course, what is happening here is that Earth is rotating toward the east and moves east under the satellite. In Earth's frame this looks like a westward force on the satellite, or it can be interpreted as a violation of Newton's first law (the law of inertia). An **inertial frame of reference** is one in which all forces are real and, equivalently, one in which Newton's laws have the simple forms given in this chapter.

Earth's rotation is slow enough that Earth is nearly an inertial frame. You ordinarily must perform precise experiments to observe fictitious forces and the slight departures from Newton's laws, such as the effect just described. On the large scale, such as for the rotation of weather systems and ocean currents, the effects can be easily observed.

The crucial factor in determining whether a frame of reference is inertial is whether it accelerates or rotates relative to a known inertial frame. Unless stated otherwise, all phenomena discussed in this text are considered in inertial frames.

All the forces discussed in this section are real forces, but there are a number of other real forces, such as lift and thrust, that are not discussed in this section. They are more specialized, and it is not necessary to discuss every type of force. It is natural, however, to ask where the basic simplicity we seek to find in physics is in the long list of forces. Are some more basic than others? Are some different manifestations of the same underlying

force? The answer to both questions is yes, as will be seen in the next (extended) section and in the treatment of modern physics later in the text.

Note:

PhET Explorations: Forces in 1 Dimension

Explore the forces at work when you try to push a filing cabinet. Create an applied force and see the resulting friction force and total force acting on the cabinet. Charts show the forces, position, velocity, and acceleration vs. time. View a free-body diagram of all the forces (including gravitational and normal forces).

[Forces in
1
Dimensio
n](#)

Test Prep for AP Courses

Exercise:

Problem:

An archer shoots an arrow straight up with a force of 24.5 N. The arrow has a mass of 0.4 kg. What is the force of gravity on the arrow?

- a. 9.8 m/s^2
- b. 9.8 N
- c. 61.25 N
- d. 3.9 N

Exercise:

Problem:

A cable raises a mass of 120.0 kg with an acceleration of 1.3 m/s². What force of tension is in the cable?

Solution:

The force of tension must equal the force of gravity plus the force necessary to accelerate the mass. $F = mg$ can be used to calculate the first, and $F = ma$ can be used to calculate the second.

For gravity:

$$\begin{aligned} F &= mg \\ &= (120.0 \text{ kg})(9.8 \text{ m/s}^2) \\ &= 1205.4 \text{ N} \end{aligned}$$

For acceleration:

$$\begin{aligned} F &= ma \\ &= (120.0 \text{ kg})(1.3 \text{ m/s}^2) \\ &= 159.9 \text{ N} \end{aligned}$$

The total force of tension in the cable is 1176 N + 156 N = 1332 N.

Exercise:**Problem:**

A child pulls a wagon along a grassy field. Define the system, the pairs of forces at work, and the results.

Exercise:**Problem:**

Two teams are engaging in a tug-of-war. The rope suddenly snaps. Which statement is true about the forces involved?

- a. The forces exerted by the two teams are no longer equal; the teams will accelerate in opposite directions as a result.
 - b. The forces exerted by the players are no longer balanced by the force of tension in the rope; the teams will accelerate in opposite directions as a result.
 - c. The force of gravity balances the forces exerted by the players; the teams will fall as a result.
 - d. The force of tension in the rope is transferred to the players; the teams will accelerate in opposite directions as a result.
-

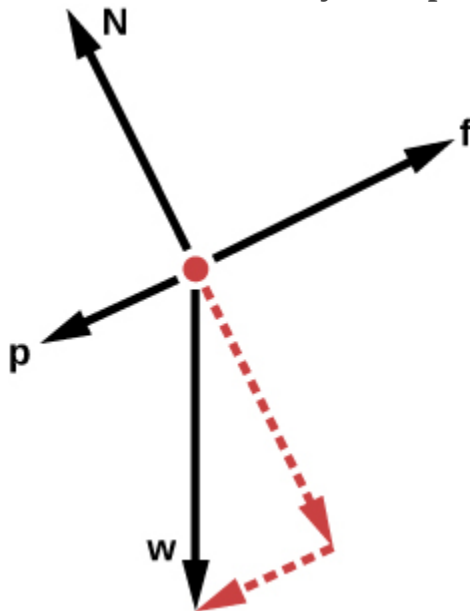
Solution:

(b)

Exercise:

Problem:

The following free-body diagram represents a toboggan on a hill. What acceleration would you expect, and why?



- a. Acceleration down the hill; the force due to being pushed, together with the downhill component of gravity, overcomes the opposing force of friction.

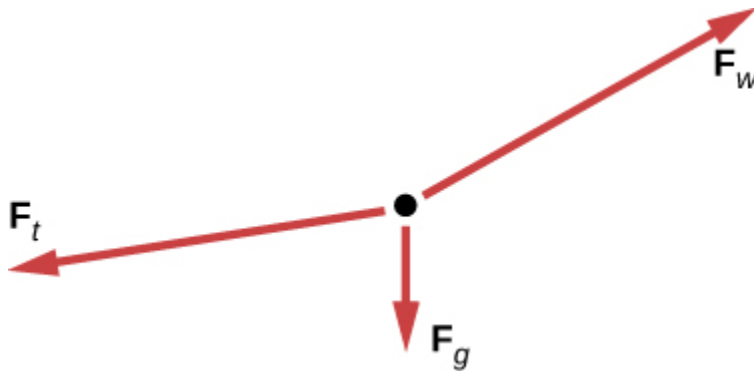
- b. Acceleration down the hill; friction is less than the opposing component of force due to gravity.
- c. No movement; friction is greater than the force due to being pushed.
- d. It depends on how strong the force due to friction is. p

Exercise:

Problem:

Draw a free-body diagram to represent the forces acting on a kite on a string that is floating stationary in the air. Label the forces in your diagram.

Solution:



The diagram has a black dot and three solid red arrows pointing away from the dot. Arrow F_t is long and pointing to the left and slightly down. Arrow F_w is also long and is a bit below a diagonal line halfway between pointing up and pointing to the right. A short arrow F_g is pointing down.

F_g is the force on the kite due to gravity.

F_w is the force exerted on the kite by the wind.

F_t is the force of tension in the string holding the kite. It must balance the vector sum of the other two forces for the kite to float stationary in the air.

Exercise:

Problem:

A car is sliding down a hill with a slope of 20° . The mass of the car is 965 kg. When a cable is used to pull the car up the slope, a force of 4215 N is applied. What is the car's acceleration, ignoring friction?

Section Summary

- When objects rest on a surface, the surface applies a force to the object that supports the weight of the object. This supporting force acts perpendicular to and away from the surface. It is called a normal force, N .
- When objects rest on a non-accelerating horizontal surface, the magnitude of the normal force is equal to the weight of the object:

Equation:

$$N = mg.$$

- When objects rest on an inclined plane that makes an angle θ with the horizontal surface, the weight of the object can be resolved into components that act perpendicular (w_\perp) and parallel (w_\parallel) to the surface of the plane. These components can be calculated using:

Equation:

$$w_\parallel = w \sin (\theta) = mg \sin (\theta)$$

Equation:

$$w_\perp = w \cos (\theta) = mg \cos (\theta).$$

- The pulling force that acts along a stretched flexible connector, such as a rope or cable, is called tension, \mathbf{T} . When a rope supports the weight of an object that is at rest, the tension in the rope is equal to the weight of the object:

Equation:

$$T = mg.$$

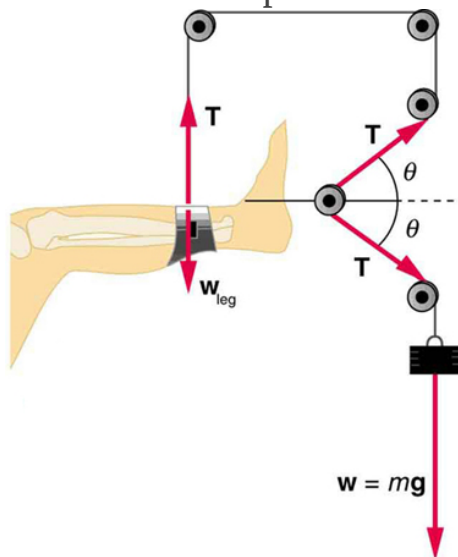
- In any inertial frame of reference (one that is not accelerated or rotated), Newton's laws have the simple forms given in this chapter and all forces are real forces having a physical origin.

Conceptual Questions

Exercise:

Problem:

If a leg is suspended by a traction setup as shown in [\[link\]](#), what is the tension in the rope?



A leg is suspended by a traction system in which wires are used to transmit forces. Frictionless pulleys change the direction of the force T without changing its magnitude.

Exercise:**Problem:**

In a traction setup for a broken bone, with pulleys and rope available, how might we be able to increase the force along the tibia using the same weight? (See [\[link\]](#).) (Note that the tibia is the shin bone shown in this image.)

Problem Exercises**Exercise:****Problem:**

Two teams of nine members each engage in a tug of war. Each of the first team's members has an average mass of 68 kg and exerts an average force of 1350 N horizontally. Each of the second team's members has an average mass of 73 kg and exerts an average force of 1365 N horizontally. (a) What is magnitude of the acceleration of the two teams? (b) What is the tension in the section of rope between the teams?

Solution:

- a. 0.11 m/s^2
- b. $1.2 \times 10^4 \text{ N}$

Exercise:**Problem:**

What force does a trampoline have to apply to a 45.0-kg gymnast to accelerate her straight up at 7.50 m/s^2 ? Note that the answer is independent of the velocity of the gymnast—she can be moving either up or down, or be stationary.

Exercise:

Problem:

(a) Calculate the tension in a vertical strand of spider web if a spider of mass 8.00×10^{-5} kg hangs motionless on it. (b) Calculate the tension in a horizontal strand of spider web if the same spider sits motionless in the middle of it much like the tightrope walker in [\[link\]](#). The strand sags at an angle of 12° below the horizontal. Compare this with the tension in the vertical strand (find their ratio).

Solution:

(a) 7.84×10^{-4} N

(b) 1.89×10^{-3} N . This is 2.41 times the tension in the vertical strand.

Exercise:**Problem:**

Suppose a 60.0-kg gymnast climbs a rope. (a) What is the tension in the rope if he climbs at a constant speed? (b) What is the tension in the rope if he accelerates upward at a rate of 1.50 m/s^2 ?

Exercise:**Problem:**

Show that, as stated in the text, a force \mathbf{F}_\perp exerted on a flexible medium at its center and perpendicular to its length (such as on the tightrope wire in [\[link\]](#)) gives rise to a tension of magnitude

$$T = \frac{F_\perp}{2 \sin(\theta)}.$$

Solution:

Newton's second law applied in vertical direction gives

Equation:

$$F_y = F - 2T \sin \theta = 0$$

Equation:

$$F = 2T \sin \theta$$

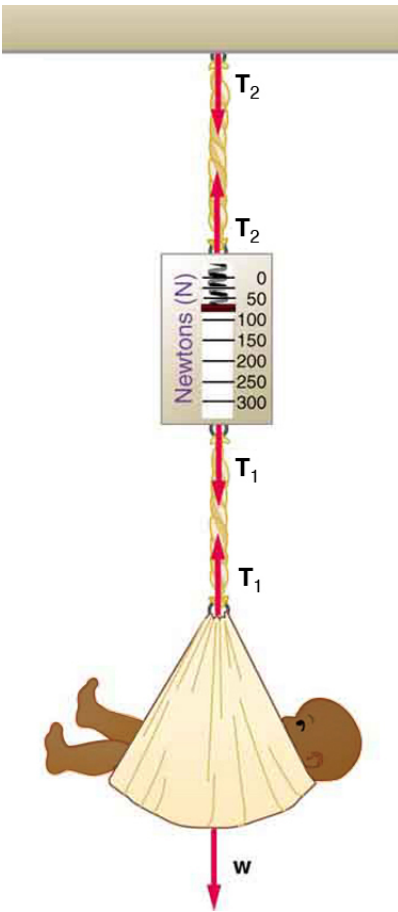
Equation:

$$T = \frac{F}{2 \sin \theta}.$$

Exercise:

Problem:

Consider the baby being weighed in [\[link\]](#). (a) What is the mass of the child and basket if a scale reading of 55 N is observed? (b) What is the tension T_1 in the cord attaching the baby to the scale? (c) What is the tension T_2 in the cord attaching the scale to the ceiling, if the scale has a mass of 0.500 kg? (d) Draw a sketch of the situation indicating the system of interest used to solve each part. The masses of the cords are negligible.



A baby is weighed
using a spring
scale.

Glossary

inertial frame of reference

a coordinate system that is not accelerating; all forces acting in an inertial frame of reference are real forces, as opposed to fictitious forces that are observed due to an accelerating frame of reference

normal force

the force that a surface applies to an object to support the weight of the object; acts perpendicular to the surface on which the object rests

tension

the pulling force that acts along a medium, especially a stretched flexible connector, such as a rope or cable; when a rope supports the weight of an object, the force on the object due to the rope is called a tension force

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Apply a problem-solving procedure to solve problems using Newton's laws of motion

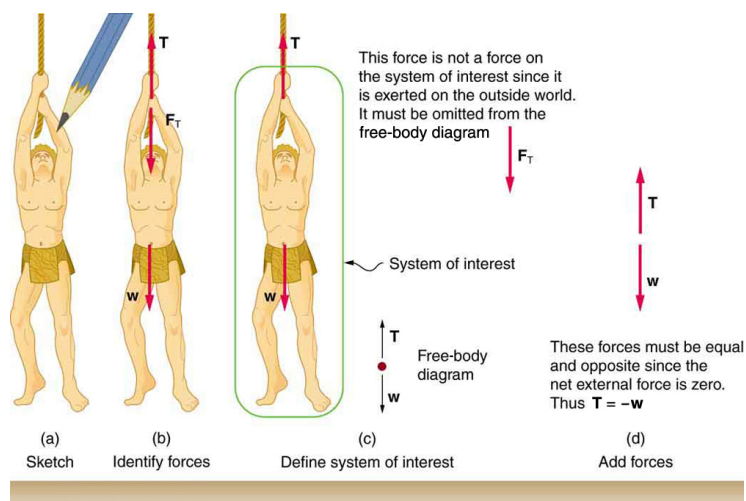
The information presented in this section supports the following AP® learning objectives and science practices:

- **3.A.2.1** The student is able to represent forces in diagrams or mathematically using appropriately labeled vectors with magnitude, direction, and units during the analysis of a situation. **(S.P. 1.1)**
- **3.A.3.3** The student is able to describe a force as an interaction between two objects and identify both objects for any force. **(S.P. 1.4)**
- **3.B.1.1** The student is able to predict the motion of an object subject to forces exerted by several objects using an application of Newton's second law in a variety of physical situations with acceleration in one dimension. **(S.P. 6.4, 7.2)**
- **3.B.1.3** The student is able to re-express a free-body diagram representation into a mathematical representation and solve the mathematical representation for the acceleration of the object. **(S.P. 1.5, 2.2)**
- **3.B.2.1** The student is able to create and use free-body diagrams to analyze physical situations to solve problems with motion qualitatively and quantitatively. **(S.P. 1.1, 1.4, 2.2)**

Success in problem solving is obviously necessary to understand and apply physical principles, not to mention the more immediate need of passing exams. The basics of problem solving, presented earlier in this text, are followed here, but specific strategies useful in applying Newton's laws of motion are emphasized. These techniques also reinforce concepts that are useful in many other areas of physics. Many problem-solving strategies are stated outright in the worked examples, and so the following techniques should reinforce skills you have already begun to develop.

Problem-Solving Strategy for Newton's Laws of Motion

Step 1. As usual, it is first necessary to identify the physical principles involved. *Once it is determined that Newton's laws of motion are involved (if the problem involves forces), it is particularly important to draw a careful sketch of the situation.* Such a sketch is shown in [\[link\]\(a\)](#). Then, as in [\[link\]\(b\)](#), use arrows to represent all forces, label them carefully, and make their lengths and directions correspond to the forces they represent (whenever sufficient information exists).



(a) A sketch of Tarzan hanging from a vine. (b) Arrows are used to represent all forces. \mathbf{T} is the tension in the vine above Tarzan, \mathbf{F}_T is the force he exerts on the vine, and \mathbf{w} is his weight. All other forces, such as the nudge of a breeze, are assumed negligible. (c) Suppose we are given the ape man's mass and asked to find the tension in the vine. We then define the system of interest as shown and draw a free-body diagram. \mathbf{F}_T is no longer shown, because it is not a force acting on the system of interest; rather, \mathbf{F}_T acts on the outside world. (d) Showing only the arrows, the head-to-tail method of addition is used. It is apparent that $\mathbf{T} = -\mathbf{w}$, if Tarzan is stationary.

Step 2. Identify what needs to be determined and what is known or can be inferred from the problem as stated. That is, make a list of knowns and unknowns. *Then carefully determine the system of interest.* This decision is a crucial step, since Newton's second law involves only external forces. Once the system of interest has been identified, it becomes possible to determine which forces are external and which are internal, a necessary step to employ Newton's second law. (See [\[link\]\(c\)](#).) Newton's third law may be used to identify whether forces are exerted between components of a system (internal) or between the system and something outside (external). As illustrated earlier in this chapter, the system of interest depends on what question we need to answer. This choice becomes easier with practice, eventually developing into an almost unconscious process. Skill in clearly defining systems will be beneficial in later chapters as well.

A diagram showing the system of interest and all of the external forces is called a **free-body diagram**. Only forces are shown on free-body diagrams, not acceleration or velocity. We have drawn several of these in worked examples. [\[link\]\(c\)](#) shows a free-body diagram for the system of interest. Note that no internal forces are shown in a free-body diagram.

Step 3. Once a free-body diagram is drawn, *Newton's second law can be applied to solve the problem.* This is done in [\[link\]\(d\)](#) for a particular situation. In general, once external forces are clearly identified in free-body diagrams, it should be a straightforward task to put them into equation form and solve for the unknown, as done in all previous examples. If the problem is one-dimensional—that is, if all forces are parallel—then they add like scalars. If the problem is two-dimensional, then it must be broken down into a pair of one-dimensional problems. This is done by projecting the force vectors onto a set of axes chosen for convenience. As seen in previous examples, the choice of axes can simplify the problem. For example, when an incline is involved, a set of axes with one axis parallel to the incline and one perpendicular to it is most convenient. It is almost always convenient to make one axis parallel to the direction of motion, if this is known.

Note:**Applying Newton's Second Law**

Before you write net force equations, it is critical to determine whether the system is accelerating in a particular direction. If the acceleration is zero in a particular direction, then the net force is zero in that direction. Similarly, if the acceleration is nonzero in a particular direction, then the net force is described by the equation: $F_{\text{net}} = ma$. For example, if the system is accelerating in the horizontal direction, but it is not accelerating in the vertical direction, then you will have the following conclusions:

Equation:

$$F_{\text{net } x} = ma,$$

Equation:

$$F_{\text{net } y} = 0.$$

You will need this information in order to determine unknown forces acting in a system.

Step 4. As always, *check the solution to see whether it is reasonable*. In some cases, this is obvious. For example, it is reasonable to find that friction causes an object to slide down an incline more slowly than when no friction exists. In practice, intuition develops gradually through problem solving, and with experience it becomes progressively easier to judge whether an answer is reasonable. Another way to check your solution is to check the units. If you are solving for force and end up with units of m/s, then you have made a mistake.

Test Prep for AP Courses**Exercise:****Problem:**

A toboggan with two riders has a total mass of 85.0 kg. A third person is pushing the toboggan with a force of 42.5 N at the top of a hill with an angle of 15° . The force of friction on the toboggan is 31.0 N. Which statement describes an accurate free-body diagram to represent the situation?

- a. An arrow of magnitude 10.5 N points down the slope of the hill.
- b. An arrow of magnitude 833 N points straight down.
- c. An arrow of magnitude 833 N points perpendicular to the slope of the hill.
- d. An arrow of magnitude 73.5 N points down the slope of the hill.

Solution:

(b)

Exercise:**Problem:**

A mass of 2.0 kg is suspended from the ceiling of an elevator by a rope. What is the tension in the rope when the elevator (i) accelerates upward at 1.5 m/s^2 ? (ii) accelerates downward at 1.5 m/s^2 ?

- a. (i) 22.6 N; (ii) 16.6 N
- b. Because the mass is hanging from the elevator itself, the tension in the rope will not change in either case.
- c. (i) 22.6 N; (ii) 19.6 N
- d. (i) 16.6 N; (ii) 19.6 N

Exercise:

Problem: Which statement is true about drawing free-body diagrams?

- a. Drawing a free-body diagram should be the last step in solving a problem about forces.
- b. Drawing a free-body diagram helps you compare forces quantitatively.
- c. The forces in a free-body diagram should always balance.
- d. Drawing a free-body diagram can help you determine the net force.

Solution:

(d)

Section Summary

- To solve problems involving Newton's laws of motion, follow the procedure described:
 1. Draw a sketch of the problem.
 2. Identify known and unknown quantities, and identify the system of interest. Draw a free-body diagram, which is a sketch showing all of the forces acting on an object. The object is represented by a dot, and the forces are represented by vectors extending in different directions from the dot. If vectors act in directions that are not horizontal or vertical, resolve the vectors into horizontal and vertical components and draw them on the free-body diagram.
 3. Write Newton's second law in the horizontal and vertical directions and add the forces acting on the object. If the object does not accelerate in a particular direction (for example, the x -direction) then $F_{\text{net } x} = 0$. If the object does accelerate in that direction, $F_{\text{net } x} = ma$.
 4. Check your answer. Is the answer reasonable? Are the units correct?

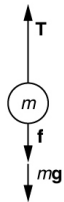
Problem Exercises

Exercise:

Problem:

A 5.00×10^5 -kg rocket is accelerating straight up. Its engines produce 1.250×10^7 N of thrust, and air resistance is 4.50×10^6 N. What is the rocket's acceleration? Explicitly show how you follow the steps in the Problem-Solving Strategy for Newton's laws of motion.

Solution:



Using the free-body diagram:

$$F_{\text{net}} = T - f - mg = ma,$$

so that

$$a = \frac{T - f - mg}{m} = \frac{1.250 \times 10^7 \text{ N} - 4.50 \times 10^6 \text{ N} - (5.00 \times 10^5 \text{ kg})(9.80 \text{ m/s}^2)}{5.00 \times 10^5 \text{ kg}} = 6.20 \text{ m/s}^2.$$

Exercise:

Problem:

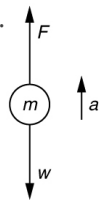
The wheels of a midsize car exert a force of 2100 N backward on the road to accelerate the car in the forward direction. If the force of friction including air resistance is 250 N and the acceleration of the car is 1.80 m/s^2 , what is the mass of the car plus its occupants? Explicitly show how you follow the steps in the Problem-Solving Strategy for Newton's laws of motion. For this situation, draw a free-body diagram and write the net force equation.

Exercise:**Problem:**

Calculate the force a 70.0-kg high jumper must exert on the ground to produce an upward acceleration 4.00 times the acceleration due to gravity. Explicitly show how you follow the steps in the Problem-Solving Strategy for Newton's laws of motion.

Solution:

Use Newton's laws of motion.



Given : $a = 4.00g = (4.00)(9.80 \text{ m/s}^2) = 39.2 \text{ m/s}^2$; $m = 70.0 \text{ kg}$,

Find: F .

$$\sum F = +F - w = ma, \text{ so } F = ma + w = ma + mg = m(a + g).$$

that

$F = (70.0 \text{ kg})[(39.2 \text{ m/s}^2) + (9.80 \text{ m/s}^2)] = 3.43 \times 10^3 \text{ N}$. The force exerted by the high-jumper is actually down on the ground, but F is up from the ground and makes him jump.

This result is reasonable, since it is quite possible for a person to exert a force of the magnitude of 10^3 N .

Exercise:**Problem:**

When landing after a spectacular somersault, a 40.0-kg gymnast decelerates by pushing straight down on the mat. Calculate the force she must exert if her deceleration is 7.00 times the acceleration due to gravity. Explicitly show how you follow the steps in the Problem-Solving Strategy for Newton's laws of motion.

Exercise:**Problem:**

A freight train consists of two 8.00×10^4 -kg engines and 45 cars with average masses of $5.50 \times 10^4 \text{ kg}$. (a) What force must each engine exert backward on the track to accelerate the train at a rate of $5.00 \times 10^{-2} \text{ m/s}^2$ if the force of friction is $7.50 \times 10^5 \text{ N}$, assuming the engines exert identical forces? This is not a large frictional force for such a massive system. Rolling friction for trains is small, and consequently trains are very energy-efficient transportation systems. (b) What is the force in the coupling between the 37th and 38th cars (this is the force each exerts on the other), assuming all cars have the same mass and that friction is evenly distributed among all of the cars and engines?

Solution:

(a) $4.41 \times 10^5 \text{ N}$

(b) $1.50 \times 10^5 \text{ N}$

Exercise:

Problem:

Commercial airplanes are sometimes pushed out of the passenger loading area by a tractor. (a) An 1800-kg tractor exerts a force of $1.75 \times 10^4 \text{ N}$ backward on the pavement, and the system experiences forces resisting motion that total 2400 N. If the acceleration is 0.150 m/s^2 , what is the mass of the airplane? (b) Calculate the force exerted by the tractor on the airplane, assuming 2200 N of the friction is experienced by the airplane. (c) Draw two sketches showing the systems of interest used to solve each part, including the free-body diagrams for each.

Exercise:

Problem:

A 1100-kg car pulls a boat on a trailer. (a) What total force resists the motion of the car, boat, and trailer, if the car exerts a 1900-N force on the road and produces an acceleration of 0.550 m/s^2 ? The mass of the boat plus trailer is 700 kg. (b) What is the force in the hitch between the car and the trailer if 80% of the resisting forces are experienced by the boat and trailer?

Solution:

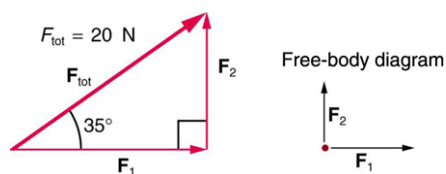
(a) 910 N

(b) $1.11 \times 10^3 \text{ N}$

Exercise:

Problem:

(a) Find the magnitudes of the forces \mathbf{F}_1 and \mathbf{F}_2 that add to give the total force \mathbf{F}_{tot} shown in [\[link\]](#). This may be done either graphically or by using trigonometry. (b) Show graphically that the same total force is obtained independent of the order of addition of \mathbf{F}_1 and \mathbf{F}_2 . (c) Find the direction and magnitude of some other pair of vectors that add to give \mathbf{F}_{tot} . Draw these to scale on the same drawing used in part (b) or a similar picture.



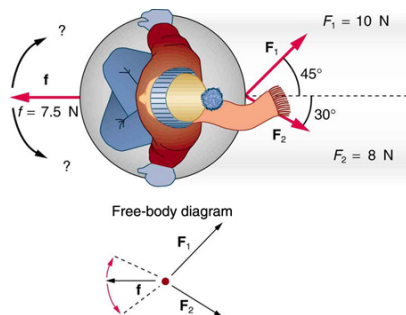
Exercise:

Problem:

Two children pull a third child on a snow saucer sled exerting forces \mathbf{F}_1 and \mathbf{F}_2 as shown from above in [\[link\]](#). Find the acceleration of the 49.00-kg sled and child system. Note that the direction of the frictional force is unspecified; it will be in the opposite direction of the sum of \mathbf{F}_1 and \mathbf{F}_2 .

Solution:

$a = 0.139 \text{ m/s}$, $\theta = 12.4^\circ$ north of east

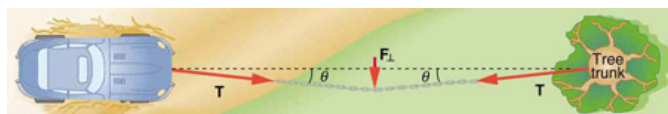


An overhead view of the horizontal forces acting on a child's snow saucer sled.

Exercise:

Problem:

Suppose your car was mired deeply in the mud and you wanted to use the method illustrated in [\[link\]](#) to pull it out. (a) What force would you have to exert perpendicular to the center of the rope to produce a force of 12,000 N on the car if the angle is 2.00° ? In this part, explicitly show how you follow the steps in the Problem-Solving Strategy for Newton's laws of motion. (b) Real ropes stretch under such forces. What force would be exerted on the car if the angle increases to 7.00° and you still apply the force found in part (a) to its center?



Exercise:

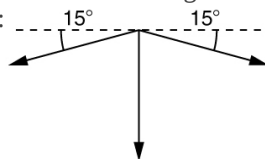
Problem:

What force is exerted on the tooth in [\[link\]](#) if the tension in the wire is 25.0 N? Note that the force applied to the tooth is smaller than the tension in the wire, but this is necessitated by practical considerations of how force can be applied in the mouth. Explicitly show how you follow steps in the Problem-Solving Strategy for Newton's laws of motion.

Solution:

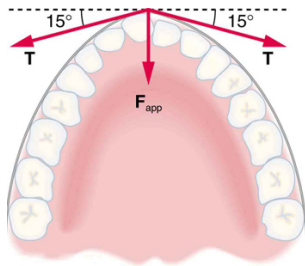
Use Newton's laws since we are looking for forces.

Draw a force diagram:



The tension is given as $T = 25.0 \text{ N}$. Find F_{app} . Using Newton's laws gives: $\Sigma F_x = 0$, so that y-components of the two tensions: $F_{app} = 2 T \sin \theta = 2(25.0 \text{ N}) \sin(15^\circ) =$

This seems reasonable, since the applied tensions should be greater than the force applied to the tooth.



Braces are used to apply forces to teeth to realign them. Shown in this figure are the tensions applied by the wire to the protruding tooth. The total force applied to the tooth by the wire, \mathbf{F}_{app} , points straight toward the back of the mouth.

Exercise:

Problem:

[\[link\]](#) shows Superhero and Trusty Sidekick hanging motionless from a rope. Superhero's mass is 90.0 kg, while Trusty Sidekick's is 55.0 kg, and the mass of the rope is negligible. (a) Draw a free-body diagram of the situation showing all forces acting on Superhero, Trusty Sidekick, and the rope. (b) Find the tension in the rope above Superhero. (c) Find the tension in the rope between Superhero and Trusty Sidekick. Indicate on your free-body diagram the system of interest used to solve each part.



Superhero
and Trusty
Sidekick

hang
motionless
on a rope
as they try
to figure
out what to
do next.
Will the
tension be
the same
everywher
e in the
rope?

Exercise:

Problem:

A nurse pushes a cart by exerting a force on the handle at a downward angle 35.0° below the horizontal. The loaded cart has a mass of 28.0 kg, and the force of friction is 60.0 N. (a) Draw a free-body diagram for the system of interest. (b) What force must the nurse exert to move at a constant velocity?

Exercise:

Problem:

Construct Your Own Problem Consider the tension in an elevator cable during the time the elevator starts from rest and accelerates its load upward to some cruising velocity. Taking the elevator and its load to be the system of interest, draw a free-body diagram. Then calculate the tension in the cable. Among the things to consider are the mass of the elevator and its load, the final velocity, and the time taken to reach that velocity.

Exercise:

Problem:

Construct Your Own Problem Consider two people pushing a toboggan with four children on it up a snow-covered slope. Construct a problem in which you calculate the acceleration of the toboggan and its load. Include a free-body diagram of the appropriate system of interest as the basis for your analysis. Show vector forces and their components and explain the choice of coordinates. Among the things to be considered are the forces exerted by those pushing, the angle of the slope, and the masses of the toboggan and children.

Exercise:

Problem:

Unreasonable Results (a) Repeat [\[link\]](#), but assume an acceleration of 1.20 m/s^2 is produced. (b) What is unreasonable about the result? (c) Which premise is unreasonable, and why is it unreasonable?

Exercise:

Problem:

Unreasonable Results (a) What is the initial acceleration of a rocket that has a mass of $1.50 \times 10^6 \text{ kg}$ at takeoff, the engines of which produce a thrust of $2.00 \times 10^6 \text{ N}$? Do not neglect gravity. (b) What is unreasonable about the result? (This result has been unintentionally achieved by several real rockets.) (c) Which premise is unreasonable, or which premises are inconsistent? (You may find it useful to compare this problem to the rocket problem earlier in this section.)

Further Applications of Newton's Laws of Motion

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Apply problem-solving techniques to solve for quantities in more complex systems of forces.
- Integrate concepts from kinematics to solve problems using Newton's laws of motion.

The information presented in this section supports the following AP® learning objectives and science practices:

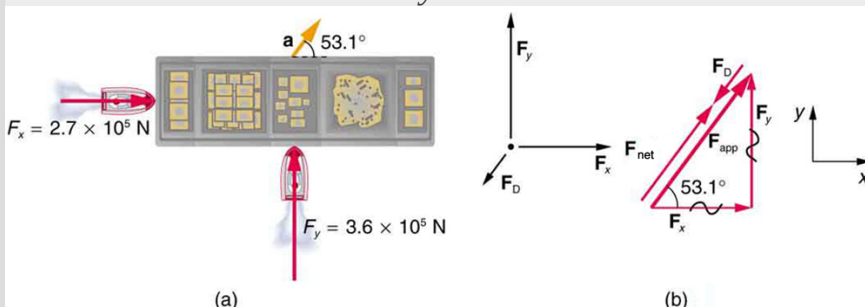
- **3.A.2.1** The student is able to represent forces in diagrams or mathematically using appropriately labeled vectors with magnitude, direction, and units during the analysis of a situation. **(S.P. 1.1)**
- **3.A.3.1** The student is able to analyze a scenario and make claims (develop arguments, justify assertions) about the forces exerted on an object by other objects for different types of forces or components of forces. **(S.P. 6.4, 7.2)**
- **3.A.3.3** The student is able to describe a force as an interaction between two objects and identify both objects for any force. **(S.P. 1.4)**
- **3.B.1.1** The student is able to predict the motion of an object subject to forces exerted by several objects using an application of Newton's second law in a variety of physical situations with acceleration in one dimension. **(S.P. 6.4, 7.2)**
- **3.B.1.3** The student is able to re-express a free-body diagram representation into a mathematical representation and solve the mathematical representation for the acceleration of the object. **(S.P. 1.5, 2.2)**
- **3.B.2.1** The student is able to create and use free-body diagrams to analyze physical situations to solve problems with motion qualitatively and quantitatively. **(S.P. 1.1, 1.4, 2.2)**

There are many interesting applications of Newton's laws of motion, a few more of which are presented in this section. These serve also to illustrate some further subtleties of physics and to help build problem-solving skills.

Example:

Drag Force on a Barge

Suppose two tugboats push on a barge at different angles, as shown in [\[link\]](#). The first tugboat exerts a force of $2.7 \times 10^5 \text{ N}$ in the x -direction, and the second tugboat exerts a force of $3.6 \times 10^5 \text{ N}$ in the y -direction.



(a) A view from above of two tugboats pushing on a barge. (b) The free-body diagram for the ship contains only forces acting in the plane of the water. It omits the two vertical forces—the weight of the barge and the buoyant force of the water supporting it cancel and are not shown. Since the applied forces are perpendicular, the x - and y -axes are in the same direction as \mathbf{F}_x and \mathbf{F}_y .

The problem quickly becomes a one-dimensional problem along the direction of \mathbf{F}_{app} , since friction is in the direction opposite to \mathbf{F}_{app} .

If the mass of the barge is $5.0 \times 10^6 \text{ kg}$ and its acceleration is observed to be $7.5 \times 10^{-2} \text{ m/s}^2$ in the direction shown, what is the drag force of the water on the barge resisting the motion? (Note: drag force is a frictional force exerted by fluids, such as air or water. The drag force opposes the motion of the object.)

Strategy

The directions and magnitudes of acceleration and the applied forces are given in [\[link\]\(a\)](#). We will define the total force of the tugboats on the barge as \mathbf{F}_{app} so that:

Equation:

$$\mathbf{F}_{\text{app}} = \mathbf{F}_x + \mathbf{F}_y$$

Since the barge is flat bottomed, the drag of the water \mathbf{F}_D will be in the direction opposite to \mathbf{F}_{app} , as shown in the free-body diagram in [\[link\]](#)(b). The system of interest here is the barge, since the forces on *it* are given as well as its acceleration. Our strategy is to find the magnitude and direction of the net applied force \mathbf{F}_{app} , and then apply Newton's second law to solve for the drag force \mathbf{F}_D .

Solution

Since \mathbf{F}_x and \mathbf{F}_y are perpendicular, the magnitude and direction of \mathbf{F}_{app} are easily found. First, the resultant magnitude is given by the Pythagorean theorem:

Equation:

$$F_{\text{app}} = \sqrt{F_x^2 + F_y^2}$$

$$F_{\text{app}} = \sqrt{(2.7 \times 10^5 \text{ N})^2 + (3.6 \times 10^5 \text{ N})^2} = 4.5 \times 10^5 \text{ N}.$$

The angle is given by

Equation:

$$\theta = \tan^{-1}\left(\frac{F_y}{F_x}\right)$$

$$\theta = \tan^{-1}\left(\frac{3.6 \times 10^5 \text{ N}}{2.7 \times 10^5 \text{ N}}\right) = 53^\circ,$$

which we know, because of Newton's first law, is the same direction as the acceleration. \mathbf{F}_D is in the opposite direction of \mathbf{F}_{app} , since it acts to slow down the acceleration. Therefore, the net external force is in the same direction as \mathbf{F}_{app} , but its magnitude is slightly less than \mathbf{F}_{app} . The problem is now one-dimensional. From [\[link\]](#)(b), we can see that

Equation:

$$F_{\text{net}} = F_{\text{app}} - F_D.$$

But Newton's second law states that

Equation:

$$F_{\text{net}} = ma.$$

Thus,

Equation:

$$F_{\text{app}} - F_D = ma.$$

This can be solved for the magnitude of the drag force of the water F_D in terms of known quantities:

Equation:

$$F_D = F_{\text{app}} - ma.$$

Substituting known values gives

Equation:

$$F_D = (4.5 \times 10^5 \text{ N}) - (5.0 \times 10^6 \text{ kg})(7.5 \times 10^{-2} \text{ m/s}^2) = 7.5 \times 10^4 \text{ N}.$$

The direction of \mathbf{F}_D has already been determined to be in the direction opposite to \mathbf{F}_{app} , or at an angle of 53° south of west.

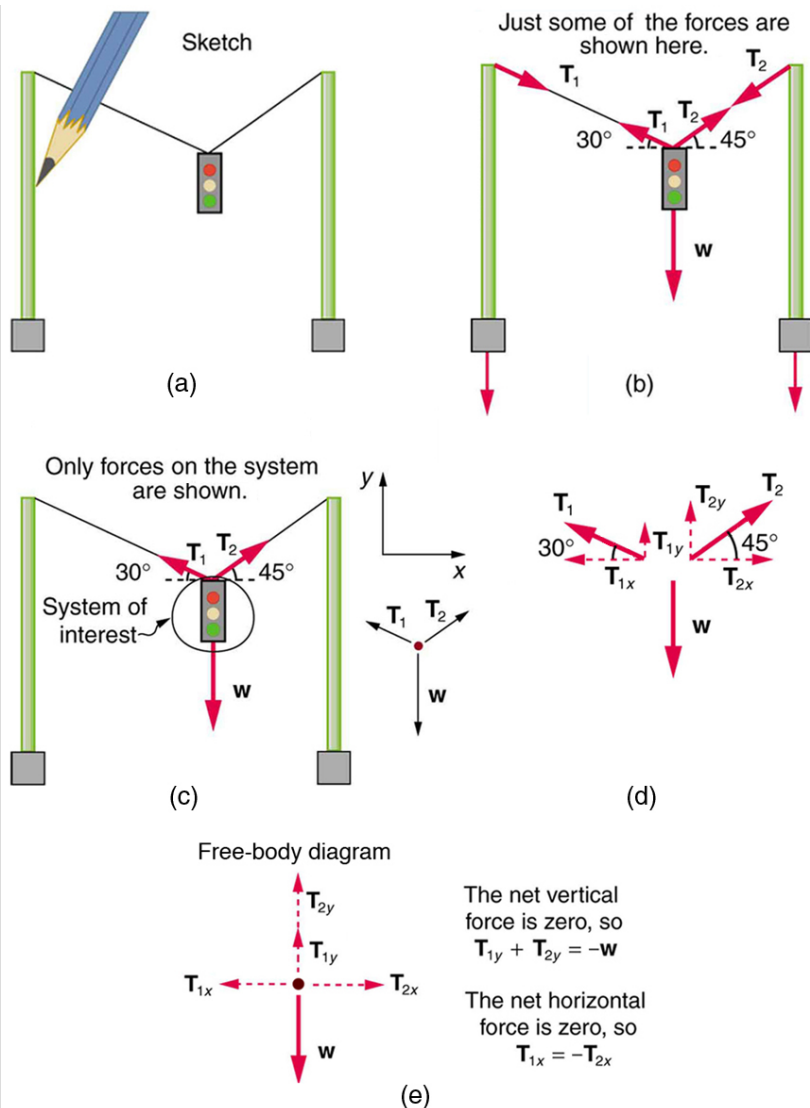
Discussion

The numbers used in this example are reasonable for a moderately large barge. It is certainly difficult to obtain larger accelerations with tugboats, and small speeds are desirable to avoid running the barge into the docks. Drag is relatively small for a well-designed hull at low speeds, consistent with the answer to this example, where F_D is less than 1/600th of the weight of the ship.

In the earlier example of a tightrope walker we noted that the tensions in wires supporting a mass were equal only because the angles on either side were equal. Consider the following example, where the angles are not equal; slightly more trigonometry is involved.

Example:**Different Tensions at Different Angles**

Consider the traffic light (mass 15.0 kg) suspended from two wires as shown in [\[link\]](#). Find the tension in each wire, neglecting the masses of the wires.



A traffic light is suspended from two wires. (b) Some of the forces involved. (c) Only forces acting on the system are shown here. The free-body diagram for the traffic light is also shown. (d) The forces projected onto vertical (y) and horizontal (x) axes. The horizontal components of the tensions must cancel, and the sum of the vertical components of the tensions must equal the weight of the traffic light. (e) The free-body diagram shows the vertical and horizontal forces acting on the traffic light.

Strategy

The system of interest is the traffic light, and its free-body diagram is shown in [\[link\]](#) (c). The three forces involved are not parallel, and so they must be projected onto a coordinate system. The most convenient coordinate system has one axis vertical and one horizontal, and the vector projections on it are shown in part (d) of the figure. There are two unknowns in this problem (T_1 and T_2), so two equations are needed to find them. These two equations come from applying Newton's second law along the vertical and horizontal axes, noting that the net external force is zero along each axis because acceleration is zero.

Solution

First consider the horizontal or x -axis:

Equation:

$$F_{\text{net}x} = T_{2x} - T_{1x} = 0.$$

Thus, as you might expect,

Equation:

$$T_{1x} = T_{2x}.$$

This gives us the following relationship between T_1 and T_2 :

Equation:

$$T_1 \cos(30^\circ) = T_2 \cos(45^\circ).$$

Thus,

Equation:

$$T_2 = (1.225)T_1.$$

Note that T_1 and T_2 are not equal in this case, because the angles on either side are not equal. It is reasonable that T_2 ends up being greater than T_1 , because it is exerted more vertically than T_1 .

Now consider the force components along the vertical or y -axis:

Equation:

$$F_{\text{net}y} = T_{1y} + T_{2y} - w = 0.$$

This implies

Equation:

$$T_{1y} + T_{2y} = w.$$

Substituting the expressions for the vertical components gives

Equation:

$$T_1 \sin (30^\circ) + T_2 \sin (45^\circ) = w.$$

There are two unknowns in this equation, but substituting the expression for T_2 in terms of T_1 reduces this to one equation with one unknown:

Equation:

$$T_1(0.500) + (1.225T_1)(0.707) = w = mg,$$

which yields

Equation:

$$(1.366)T_1 = (15.0 \text{ kg})(9.80 \text{ m/s}^2).$$

Solving this last equation gives the magnitude of T_1 to be

Equation:

$$T_1 = 108 \text{ N}.$$

Finally, the magnitude of T_2 is determined using the relationship between them, $T_2 = 1.225 T_1$, found above. Thus we obtain

Equation:

$$T_2 = 132 \text{ N}.$$

Discussion

Both tensions would be larger if both wires were more horizontal, and they will be equal if and only if the angles on either side are the same (as they were in the earlier example of a tightrope walker).

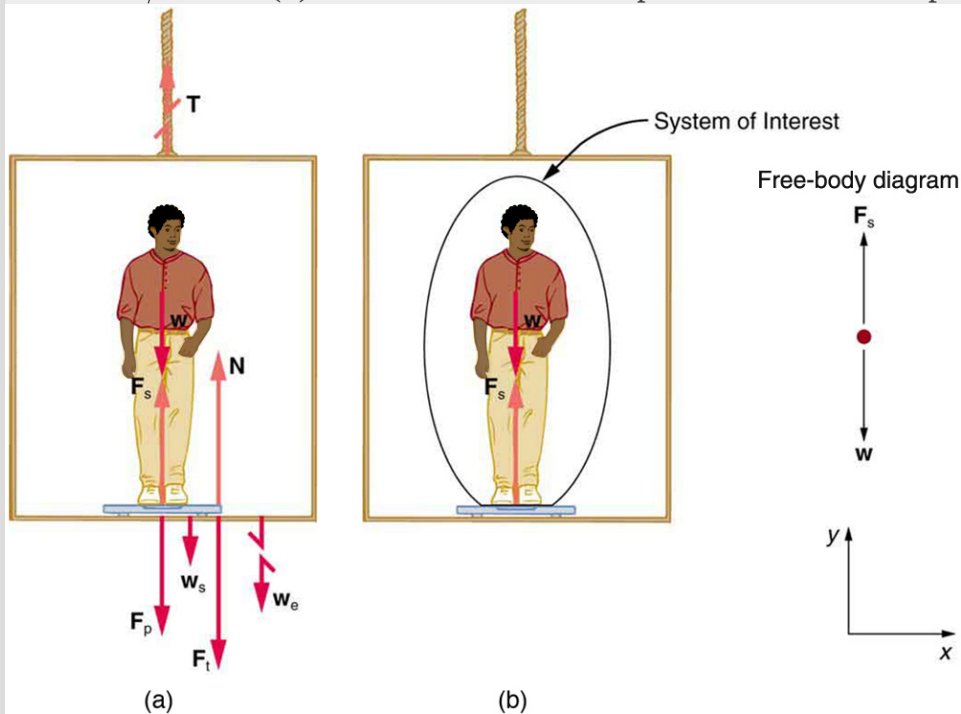
The bathroom scale is an excellent example of a normal force acting on a body. It provides a quantitative reading of how much it must push upward to support the weight of an object. But can you predict what you would see on the dial of a bathroom scale if you stood on it during an elevator ride? Will you see a value greater than your weight when the elevator starts up? What about when the elevator moves upward at a constant speed: will the scale still read more than your weight at rest? Consider the following example.

Example:

What Does the Bathroom Scale Read in an Elevator?

[\[link\]](#) shows a 75.0-kg man (weight of about 165 lb) standing on a bathroom scale in an elevator. Calculate the scale reading: (a) if the elevator accelerates upward at a rate

of 1.20 m/s^2 , and (b) if the elevator moves upward at a constant speed of 1 m/s .



(a) The various forces acting when a person stands on a bathroom scale in an elevator. The arrows are approximately correct for when the elevator is accelerating upward—broken arrows represent forces too large to be drawn to scale. \mathbf{T} is the tension in the supporting cable, \mathbf{w} is the weight of the person, \mathbf{w}_s is the weight of the scale, \mathbf{w}_e is the weight of the elevator, \mathbf{F}_s is the force of the scale on the person, \mathbf{F}_p is the force of the person on the scale, \mathbf{F}_t is the force of the scale on the floor of the elevator, and \mathbf{N} is the force of the floor upward on the scale. (b) The free-body diagram shows only the external forces acting on the designated system of interest—the person.

Strategy

If the scale is accurate, its reading will equal F_p , the magnitude of the force the person exerts downward on it. [\[link\]](#)(a) shows the numerous forces acting on the elevator, scale, and person. It makes this one-dimensional problem look much more formidable than if the person is chosen to be the system of interest and a free-body diagram is drawn as in [\[link\]](#)(b). Analysis of the free-body diagram using Newton's laws can produce answers to both parts (a) and (b) of this example, as well as some other questions that might arise. The only forces acting on the person are his weight \mathbf{w} and the upward force of the scale \mathbf{F}_s . According to Newton's third law \mathbf{F}_p and \mathbf{F}_s are

equal in magnitude and opposite in direction, so that we need to find F_s in order to find what the scale reads. We can do this, as usual, by applying Newton's second law,
Equation:

$$F_{\text{net}} = ma.$$

From the free-body diagram we see that $F_{\text{net}} = F_s - w$, so that
Equation:

$$F_s - w = ma.$$

Solving for F_s gives an equation with only one unknown:
Equation:

$$F_s = ma + w,$$

or, because $w = mg$, simply
Equation:

$$F_s = ma + mg.$$

No assumptions were made about the acceleration, and so this solution should be valid for a variety of accelerations in addition to the ones in this exercise.

Solution for (a)

In this part of the problem, $a = 1.20 \text{ m/s}^2$, so that

Equation:

$$F_s = (75.0 \text{ kg})(1.20 \text{ m/s}^2) + (75.0 \text{ kg})(9.80 \text{ m/s}^2),$$

yielding

Equation:

$$F_s = 825 \text{ N}.$$

Discussion for (a)

This is about 185 lb. What would the scale have read if he were stationary? Since his acceleration would be zero, the force of the scale would be equal to his weight:

Equation:

$$\begin{aligned} F_{\text{net}} &= ma = 0 = F_s - w \\ F_s &= w = mg \\ F_s &= (75.0 \text{ kg})(9.80 \text{ m/s}^2) \\ F_s &= 735 \text{ N}. \end{aligned}$$

So, the scale reading in the elevator is greater than his 735-N (165 lb) weight. This means that the scale is pushing up on the person with a force greater than his weight, as it must in order to accelerate him upward. Clearly, the greater the acceleration of the elevator, the greater the scale reading, consistent with what you feel in rapidly accelerating versus slowly accelerating elevators.

Solution for (b)

Now, what happens when the elevator reaches a constant upward velocity? Will the scale still read more than his weight? For any constant velocity—up, down, or stationary—acceleration is zero because $a = \frac{\Delta v}{\Delta t}$, and $\Delta v = 0$.

Thus,

Equation:

$$F_s = ma + mg = 0 + mg.$$

Now

Equation:

$$F_s = (75.0 \text{ kg})(9.80 \text{ m/s}^2),$$

which gives

Equation:

$$F_s = 735 \text{ N}.$$

Discussion for (b)

The scale reading is 735 N, which equals the person's weight. This will be the case whenever the elevator has a constant velocity—moving up, moving down, or stationary.

The solution to the previous example also applies to an elevator accelerating downward, as mentioned. When an elevator accelerates downward, a is negative, and the scale reading is *less* than the weight of the person, until a constant downward velocity is reached, at which time the scale reading again becomes equal to the person's weight. If the elevator is in free-fall and accelerating downward at g , then the scale reading will be zero and the person will *appear* to be weightless.

Integrating Concepts: Newton's Laws of Motion and Kinematics

Physics is most interesting and most powerful when applied to general situations that involve more than a narrow set of physical principles. Newton's laws of motion can also be integrated with other concepts that have been discussed previously in this text to

solve problems of motion. For example, forces produce accelerations, a topic of kinematics, and hence the relevance of earlier chapters. When approaching problems that involve various types of forces, acceleration, velocity, and/or position, use the following steps to approach the problem:

Problem-Solving Strategy

- Step 1. *Identify which physical principles are involved.* Listing the givens and the quantities to be calculated will allow you to identify the principles involved.
- Step 2. *Solve the problem using strategies outlined in the text.* If these are available for the specific topic, you should refer to them. You should also refer to the sections of the text that deal with a particular topic. The following worked example illustrates how these strategies are applied to an integrated concept problem.

Example:

What Force Must a Soccer Player Exert to Reach Top Speed?

A soccer player starts from rest and accelerates forward, reaching a velocity of 8.00 m/s in 2.50 s. (a) What was his average acceleration? (b) What average force did he exert backward on the ground to achieve this acceleration? The player’s mass is 70.0 kg, and air resistance is negligible.

Strategy

To solve an integrated concept problem, first identify the physical principles involved and identify the chapters in which they are found. Part (a) of this example considers acceleration along a straight line. This is a topic of kinematics. Part (b) deals with force, a dynamics topic found in this chapter.

The following solutions to each part of the example illustrate how the specific

problem-solving strategies are applied. These involve identifying knowns and unknowns, checking to see if the answer is reasonable, and so forth.

Solution for (a)

We are given the initial and final velocities (zero and 8.00 m/s forward); thus, the change in velocity is $\Delta v = 8.00 \text{ m/s}$. We are given the elapsed time, and so $\Delta t = 2.50 \text{ s}$. The unknown is acceleration, which can be found from its definition:

Equation:

$$a = \frac{\Delta v}{\Delta t}.$$

Substituting the known values yields

Equation:

$$\begin{aligned} a &= \frac{8.00 \text{ m/s}}{2.50 \text{ s}} \\ &= 3.20 \text{ m/s}^2. \end{aligned}$$

Discussion for (a)

This is an attainable acceleration for an athlete in good condition.

Solution for (b)

Here we are asked to find the average force the player exerts backward to achieve this forward acceleration. Neglecting air resistance, this would be equal in magnitude to the net external force on the player, since this force causes his acceleration. Since we now know the player's acceleration and are given his mass, we can use Newton's second law to find the force exerted. That is,

Equation:

$$F_{\text{net}} = ma.$$

Substituting the known values of m and a gives

Equation:

$$\begin{aligned} F_{\text{net}} &= (70.0 \text{ kg})(3.20 \text{ m/s}^2) \\ &= 224 \text{ N}. \end{aligned}$$

Discussion for (b)

This is about 50 pounds, a reasonable average force.

This worked example illustrates how to apply problem-solving strategies to situations that include topics from different chapters. The first step is to identify the physical principles involved in the problem. The second step is to solve for the unknown using familiar problem-solving strategies. These strategies are found throughout the text, and many worked examples show how to use them for single topics. You will find these

techniques for integrated concept problems useful in applications of physics outside of a physics course, such as in your profession, in other science disciplines, and in everyday life. The following problems will build your skills in the broad application of physical principles.

Test Prep for AP Courses

Exercise:

Problem:

A basketball player jumps as he shoots the ball. Describe the forces that are acting on the ball and on the basketball player. What are the results?

Exercise:

Problem:

Two people push on a boulder to try to move it. The mass of the boulder is 825 kg. One person pushes north with a force of 64 N. The other pushes west with a force of 38 N. Predict the magnitude of the acceleration of the boulder. Assume that friction is negligible.

Solution:

A free-body diagram would show a northward force of 64 N and a westward force of 38 N. The net force is equal to the sum of the two applied forces. It can be found using the Pythagorean theorem:

$$\begin{aligned} F_{\text{net}} &= \sqrt{F_x^2 + F_y^2} \\ &= \sqrt{(38 \text{ N})^2 + (64 \text{ N})^2} \\ &= 74.4 \text{ N} \end{aligned}$$

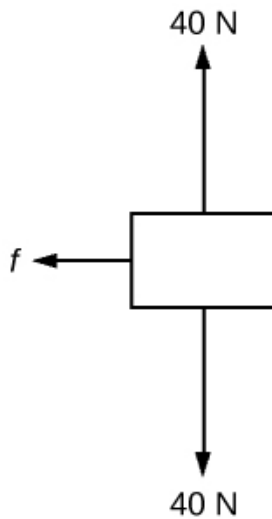
$$\text{Since } a = \frac{F}{m},$$

$$\begin{aligned} a &= \frac{74.4 \text{ N}}{825 \text{ kg}} \\ &= 0.09 \text{ m/s}^2 \end{aligned}$$

The boulder will accelerate at 0.09 m/s².

Exercise:

Problem:



The figure shows the forces exerted on a block that is sliding on a horizontal surface: the gravitational force of 40 N, the 40 N normal force exerted by the surface, and a frictional force exerted to the left. The coefficient of friction between the block and the surface is 0.20. The acceleration of the block is most nearly

- a. 1.0 m/s^2 to the right
- b. 1.0 m/s^2 to the left
- c. 2.0 m/s^2 to the right
- d. 2.0 m/s^2 to the left

Summary

- Newton's laws of motion can be applied in numerous situations to solve problems of motion.
- Some problems will contain multiple force vectors acting in different directions on an object. Be sure to draw diagrams, resolve all force vectors into horizontal and vertical components, and draw a free-body diagram. Always analyze the direction in which an object accelerates so that you can determine whether $F_{\text{net}} = ma$ or $F_{\text{net}} = 0$.
- The normal force on an object is not always equal in magnitude to the weight of the object. If an object is accelerating, the normal force will be less than or greater than the weight of the object. Also, if the object is on an inclined plane, the normal force will always be less than the full weight of the object.
- Some problems will contain various physical quantities, such as forces, acceleration, velocity, or position. You can apply concepts from kinematics and

dynamics in order to solve these problems of motion.

Conceptual Questions

Exercise:

Problem:

To simulate the apparent weightlessness of space orbit, astronauts are trained in the hold of a cargo aircraft that is accelerating downward at g . Why will they appear to be weightless, as measured by standing on a bathroom scale, in this accelerated frame of reference? Is there any difference between their apparent weightlessness in orbit and in the aircraft?

Exercise:

Problem:

A cartoon shows the toupee coming off the head of an elevator passenger when the elevator rapidly stops during an upward ride. Can this really happen without the person being tied to the floor of the elevator? Explain your answer.

Problem Exercises

Exercise:

Problem:

A flea jumps by exerting a force of 1.20×10^{-5} N straight down on the ground. A breeze blowing on the flea parallel to the ground exerts a force of 0.500×10^{-6} N on the flea. Find the direction and magnitude of the acceleration of the flea if its mass is 6.00×10^{-7} kg. Do not neglect the gravitational force.

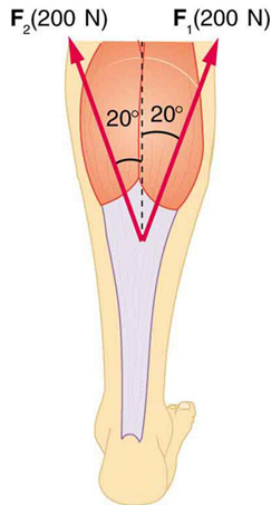
Solution:

10.2 m/s^2 , 4.67° from vertical

Exercise:

Problem:

Two muscles in the back of the leg pull upward on the Achilles tendon, as shown in [\[link\]](#). (These muscles are called the medial and lateral heads of the gastrocnemius muscle.) Find the magnitude and direction of the total force on the Achilles tendon. What type of movement could be caused by this force?



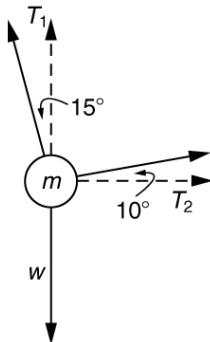
Achilles
tendon

Exercise:

Problem:

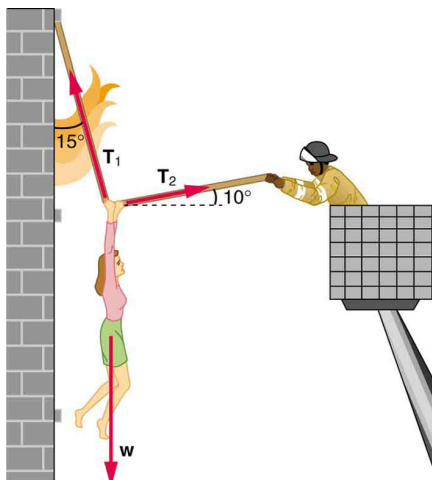
A 76.0-kg person is being pulled away from a burning building as shown in [\[link\]](#). Calculate the tension in the two ropes if the person is momentarily motionless. Include a free-body diagram in your solution.

Solution:



$$T_1 = 736 \text{ N}$$

$$T_2 = 194 \text{ N}$$



The force \mathbf{T}_2 needed to hold steady the person being rescued from the fire is less than her weight and less than the force \mathbf{T}_1 in the other rope, since the more vertical rope supports a greater part of her weight (a vertical force).

Exercise:

Problem:

Integrated Concepts A 35.0-kg dolphin decelerates from 12.0 to 7.50 m/s in 2.30 s to join another dolphin in play. What average force was exerted to slow him if he was moving horizontally? (The gravitational force is balanced by the buoyant force of the water.)

Exercise:

Problem:

Integrated Concepts When starting a foot race, a 70.0-kg sprinter exerts an average force of 650 N backward on the ground for 0.800 s. (a) What is his final speed? (b) How far does he travel?

Solution:

(a) 7.43 m/s

(b) 2.97 m

Exercise:

Problem:

Integrated Concepts A large rocket has a mass of 2.00×10^6 kg at takeoff, and its engines produce a thrust of 3.50×10^7 N. (a) Find its initial acceleration if it takes off vertically. (b) How long does it take to reach a velocity of 120 km/h straight up, assuming constant mass and thrust? (c) In reality, the mass of a rocket decreases significantly as its fuel is consumed. Describe qualitatively how this affects the acceleration and time for this motion.

Exercise:

Problem:

Integrated Concepts A basketball player jumps straight up for a ball. To do this, he lowers his body 0.300 m and then accelerates through this distance by forcefully straightening his legs. This player leaves the floor with a vertical velocity sufficient to carry him 0.900 m above the floor. (a) Calculate his velocity when he leaves the floor. (b) Calculate his acceleration while he is straightening his legs. He goes from zero to the velocity found in part (a) in a distance of 0.300 m. (c) Calculate the force he exerts on the floor to do this, given that his mass is 110 kg.

Solution:

(a) 4.20 m/s

(b) 29.4 m/s^2

(c) $4.31 \times 10^3 \text{ N}$

Exercise:

Problem:

Integrated Concepts A 2.50-kg fireworks shell is fired straight up from a mortar and reaches a height of 110 m. (a) Neglecting air resistance (a poor assumption, but we will make it for this example), calculate the shell's velocity when it leaves the mortar. (b) The mortar itself is a tube 0.450 m long. Calculate the average acceleration of the shell in the tube as it goes from zero to the velocity found in (a). (c) What is the average force on the shell in the mortar? Express your answer in newtons and as a ratio to the weight of the shell.

Exercise:

Problem:

Integrated Concepts Repeat [\[link\]](#) for a shell fired at an angle 10.0° from the vertical.

Solution:

(a) 47.1 m/s

(b) $2.47 \times 10^3 \text{ m/s}^2$

(c) $6.18 \times 10^3 \text{ N}$. The average force is 252 times the shell's weight.

Exercise:**Problem:**

Integrated Concepts An elevator filled with passengers has a mass of 1700 kg.

(a) The elevator accelerates upward from rest at a rate of 1.20 m/s^2 for 1.50 s.

Calculate the tension in the cable supporting the elevator. (b) The elevator continues upward at constant velocity for 8.50 s. What is the tension in the cable during this time?

(c) The elevator decelerates at a rate of 0.600 m/s^2 for 3.00 s.

What is the tension in the cable during deceleration? (d) How high has the elevator moved above its original starting point, and what is its final velocity?

Exercise:**Problem:**

Unreasonable Results (a) What is the final velocity of a car originally traveling at 50.0 km/h that decelerates at a rate of 0.400 m/s^2 for 50.0 s? (b) What is unreasonable about the result? (c) Which premise is unreasonable, or which premises are inconsistent?

Exercise:**Problem:**

Unreasonable Results A 75.0-kg man stands on a bathroom scale in an elevator that accelerates from rest to 30.0 m/s in 2.00 s. (a) Calculate the scale reading in newtons and compare it with his weight. (The scale exerts an upward force on him equal to its reading.) (b) What is unreasonable about the result? (c) Which premise is unreasonable, or which premises are inconsistent?

Extended Topic: The Four Basic Forces—An Introduction

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Understand the four basic forces that underlie the processes in nature.

The information presented in this section supports the following AP® learning objectives and science practices:

- **3.C.4.1** The student is able to make claims about various contact forces between objects based on the microscopic cause of those forces. **(S.P. 6.1)**
- **3.C.4.2** The student is able to explain contact forces (tension, friction, normal, buoyant, spring) as arising from interatomic electric forces and that they therefore have certain directions. **(S.P. 6.2)**
- **3.G.1.1** The student is able to articulate situations when the gravitational force is the dominant force and when the electromagnetic, weak, and strong forces can be ignored. **(S.P. 7.1)**

One of the most remarkable simplifications in physics is that only four distinct forces account for all known phenomena. In fact, nearly all of the forces we experience directly are due to only one basic force, called the electromagnetic force. (The gravitational force is the only force we experience directly that is not electromagnetic.) This is a tremendous simplification of the myriad of *apparently* different forces we can list, only a few of which were discussed in the previous section. As we will see, the basic forces are all thought to act through the exchange of microscopic carrier particles, and the characteristics of the basic forces are determined by the types of particles exchanged. Action at a distance, such as the gravitational force of Earth on the Moon, is explained by the existence of a **force field** rather than by “physical contact.”

The *four basic forces* are the gravitational force, the electromagnetic force, the weak nuclear force, and the strong nuclear force. Their properties are summarized in [\[link\]](#). Since the weak and strong nuclear forces act over an extremely short range, the size of a nucleus or less, we do not experience them directly, although they are crucial to the very structure of matter. These forces determine which nuclei are stable and which decay, and they are the basis of the release of energy in certain nuclear reactions. Nuclear forces determine not only the stability of nuclei, but also the relative abundance of elements in nature. The properties of the nucleus of an atom determine the number of electrons it has and, thus, indirectly determine the chemistry of the atom. More will be said of all of these topics in later chapters.

Note:**Concept Connections: The Four Basic Forces**

The four basic forces will be encountered in more detail as you progress through the text. The gravitational force is defined in [Uniform Circular Motion and Gravitation](#), electric force in [Electric Charge and Electric Field](#), magnetic force in [Magnetism](#), and nuclear forces in [Radioactivity and Nuclear Physics](#). On a macroscopic scale, electromagnetism and gravity are the basis for all forces. The nuclear forces are vital to the substructure of matter, but they are not directly experienced on the macroscopic scale.

Force	Approximate Relative Strengths	Range	Attraction/Repulsion	Carrier Particle
Gravitational	10^{-38}	∞	attractive only	Graviton
Electromagnetic	10^{-2}	∞	attractive and repulsive	Photon
Weak nuclear	10^{-13}	$< 10^{-18}\text{m}$	attractive and repulsive	W^+ , W^- , Z^0
Strong nuclear	1	$< 10^{-15}\text{m}$	attractive and repulsive	gluons

Properties of the Four Basic Forces[\[footnote\]](#)

The graviton is a proposed particle, though it has not yet been observed by scientists. See the discussion of gravitational waves later in this section. The particles W^+ , W^- , and Z^0 are called vector bosons; these were predicted by theory and first observed in 1983. There are eight types of gluons proposed by scientists, and their existence is indicated by meson exchange in the nuclei of atoms.

The gravitational force is surprisingly weak—it is only because gravity is always attractive that we notice it at all. Our weight is the gravitational force due to the *entire* Earth acting on us. On the very large scale, as in astronomical systems, the gravitational force is the dominant force determining the motions of moons, planets, stars, and galaxies. The gravitational force also affects the nature of space and time. As we shall see later in the study of general relativity, space is curved in the vicinity of very massive bodies, such as the Sun, and time actually slows down near massive bodies.

Take a good look at the ranges for the four fundamental forces listed in [\[link\]](#). The range of the strong nuclear force, 10^{-15} m, is approximately the size of the nucleus of an atom; the weak nuclear force has an even shorter range. At scales on the order of 10^{-10} m, approximately the size of an atom, both nuclear forces are completely dominated by the electromagnetic force. Notice that this scale is still utterly tiny compared to our everyday experience. At scales that we do experience daily, electromagnetism tends to be negligible, due to its attractive and repulsive properties canceling each other out. That leaves gravity, which is usually the strongest of the forces at scales above $\sim 10^{-4}$ m, and hence includes our everyday activities, such as throwing, climbing stairs, and walking.

Electromagnetic forces can be either attractive or repulsive. They are long-range forces, which act over extremely large distances, and they nearly cancel for macroscopic objects. (Remember that it is the *net* external force that is important.) If they did not cancel, electromagnetic forces would completely overwhelm the gravitational force. The electromagnetic force is a combination of electrical forces (such as those that cause static electricity) and magnetic forces (such as those that affect a compass needle). These two forces were thought to be quite distinct until early in the 19th century, when scientists began to discover that they are different manifestations of the same force. This discovery is a classical case of the *unification of forces*. Similarly, friction, tension, and all of the other classes of forces we experience directly (except gravity, of course) are due to electromagnetic interactions of atoms and molecules. It is still convenient to consider these forces separately in specific applications, however, because of the ways they manifest themselves.

Note:

Concept Connections: Unifying Forces

Attempts to unify the four basic forces are discussed in relation to elementary particles later in this text. By “unify” we mean finding connections between the forces that show that they are different manifestations of a single force. Even if such unification is achieved, the forces will retain their separate characteristics on the macroscopic scale and may be identical only under extreme conditions such as those existing in the early universe.

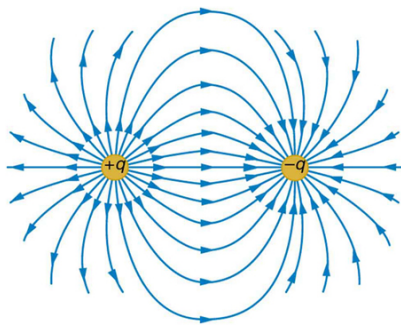
Physicists are now exploring whether the four basic forces are in some way related. Attempts to unify all forces into one come under the rubric of Grand Unified Theories (GUTs), with which there has been some success in recent years. It is now known that under conditions of extremely high density and temperature, such as existed in the early universe, the electromagnetic and weak nuclear forces are indistinguishable. They can now be considered to be different manifestations of one force, called the *electroweak* force. So the list of four has been reduced in a sense to only three. Further progress in unifying all forces is proving difficult—especially the inclusion of the gravitational force, which has the special characteristics of affecting the space and time in which the other forces exist.

While the unification of forces will not affect how we discuss forces in this text, it is fascinating that such underlying simplicity exists in the face of the overt complexity of the universe. There

is no reason that nature must be simple—it simply is.

Action at a Distance: Concept of a Field

All forces act at a distance. This is obvious for the gravitational force. Earth and the Moon, for example, interact without coming into contact. It is also true for all other forces. Friction, for example, is an electromagnetic force between atoms that may not actually touch. What is it that carries forces between objects? One way to answer this question is to imagine that a **force field** surrounds whatever object creates the force. A second object (often called a *test object*) placed in this field will experience a force that is a function of location and other variables. The field itself is the “thing” that carries the force from one object to another. The field is defined so as to be a characteristic of the object creating it; the field does not depend on the test object placed in it. Earth’s gravitational field, for example, is a function of the mass of Earth and the distance from its center, independent of the presence of other masses. The concept of a field is useful because equations can be written for force fields surrounding objects (for gravity, this yields $w = mg$ at Earth’s surface), and motions can be calculated from these equations. (See [\[link\]](#).)



The electric force field between a positively charged particle and a negatively charged particle. When a positive test charge is placed in the field, the charge will experience a force in the direction of the force field lines.

Note:

Concept Connections: Force Fields

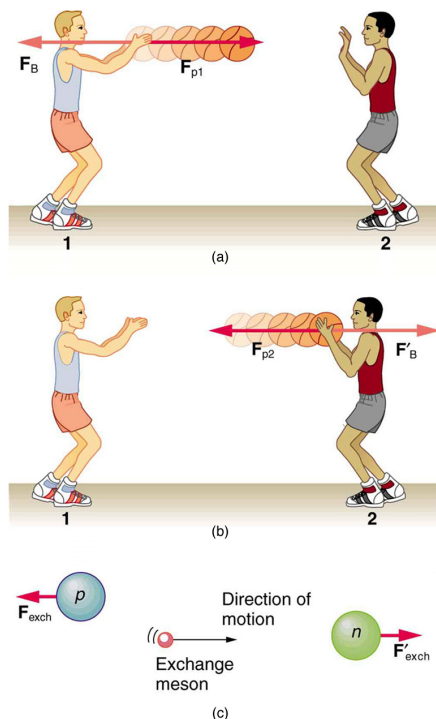
The concept of a *force field* is also used in connection with electric charge and is presented in [Electric Charge and Electric Field](#). It is also a useful idea for all the basic forces, as will be seen in [Particle Physics](#). Fields help us to visualize forces and how they are transmitted, as well as to describe them with precision and to link forces with subatomic carrier particles.

Note:

Making Connections: Vector and Scalar Fields

These fields may be either scalar or vector fields. Gravity and electromagnetism are examples of vector fields. A test object placed in such a field will have both the magnitude and direction of the resulting force on the test object completely defined by the object's location in the field. We will later cover examples of scalar fields, which have a magnitude but no direction.

The field concept has been applied very successfully; we can calculate motions and describe nature to high precision using field equations. As useful as the field concept is, however, it leaves unanswered the question of what carries the force. It has been proposed in recent decades, starting in 1935 with Hideki Yukawa's (1907–1981) work on the strong nuclear force, that all forces are transmitted by the exchange of elementary particles. We can visualize particle exchange as analogous to macroscopic phenomena such as two people passing a basketball back and forth, thereby exerting a repulsive force without touching one another. (See [link](#).)

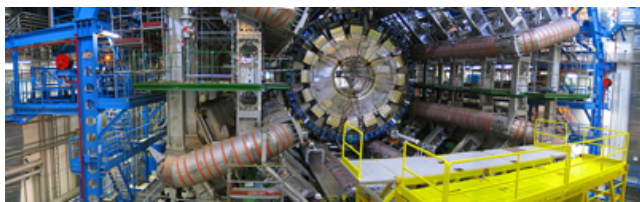


The exchange of masses

resulting in repulsive forces.

(a) The person throwing the basketball exerts a force \mathbf{F}_{p1} on it toward the other person and feels a reaction force \mathbf{F}_B away from the second person. (b) The person catching the basketball exerts a force \mathbf{F}_{p2} on it to stop the ball and feels a reaction force \mathbf{F}'_B away from the first person. (c) The analogous exchange of a meson between a proton and a neutron carries the strong nuclear forces \mathbf{F}_{exch} and $\mathbf{F}'_{\text{exch}}$ between them. An attractive force can also be exerted by the exchange of a mass—if person 2 pulled the basketball away from the first person as he tried to retain it, then the force between them would be attractive.

This idea of particle exchange deepens rather than contradicts field concepts. It is more satisfying philosophically to think of something physical actually moving between objects acting at a distance. [\[link\]](#) lists the exchange or **carrier particles**, both observed and proposed, that carry the four forces. But the real fruit of the particle-exchange proposal is that searches for Yukawa's proposed particle found it *and* a number of others that were completely unexpected, stimulating yet more research. All of this research eventually led to the proposal of quarks as the underlying substructure of matter, which is a basic tenet of GUTs. If successful, these theories will explain not only forces, but also the structure of matter itself. Yet physics is an experimental science, so the test of these theories must lie in the domain of the real world. As of this writing, scientists at the CERN laboratory in Switzerland are starting to test these theories using the world's largest particle accelerator: the Large Hadron Collider. This accelerator (27 km in circumference) allows two high-energy proton beams, traveling in opposite directions, to collide. An energy of 14 trillion electron volts will be available. It is anticipated that some new particles, possibly force carrier particles, will be found. (See [\[link\]](#).) One of the force carriers of high interest that researchers hope to detect is the Higgs boson. The observation of its properties might tell us why different particles have different masses.



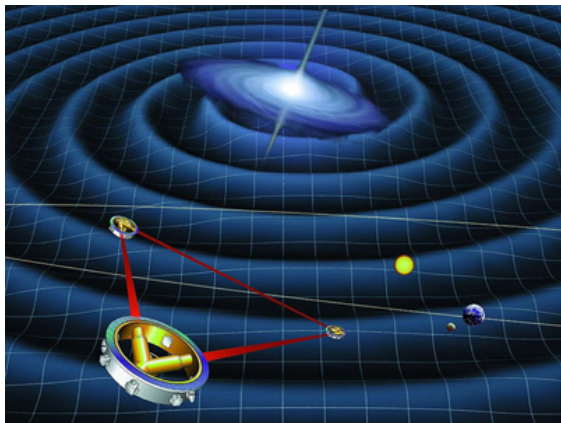
The world's largest particle accelerator spans the border between Switzerland and France. Two beams, traveling in opposite directions close to the speed of light, collide in a tube similar to the central tube shown here. External magnets determine the beam's path. Special detectors will analyze particles created in these collisions. Questions as broad as what is the origin of mass and what was matter like the first few seconds of our universe will be explored. This accelerator began preliminary operation in 2008. (credit: Frank Hommes)

Tiny particles also have wave-like behavior, something we will explore more in a later chapter. To better understand force-carrier particles from another perspective, let us consider gravity. The search for gravitational waves has been going on for a number of years. Almost 100 years ago, Einstein predicted the existence of these waves as part of his general theory of relativity. Gravitational waves are created during the collision of massive stars, in black holes, or in supernova explosions—like shock waves. These gravitational waves will travel through space from such sites much like a pebble dropped into a pond sends out ripples—except these waves move at the speed of light. A detector apparatus has been built in the U.S., consisting of two large installations nearly 3000 km apart—one in Washington state and one in Louisiana! The facility is called the Laser Interferometer Gravitational-Wave Observatory (LIGO). Each installation is designed to use optical lasers to examine any slight shift in the relative positions of two masses due to the effect of gravity waves. The two sites allow simultaneous measurements of these small effects to be separated from other natural phenomena, such as earthquakes. Initial operation of the detectors began in 2002, and work is proceeding on increasing their sensitivity. Similar installations have been built in Italy (VIRGO), Germany (GEO600), and Japan (TAMA300) to provide a worldwide network of gravitational wave detectors.

International collaboration in this area is moving into space with the joint EU/US project LISA (Laser Interferometer Space Antenna). Earthquakes and other Earthly noises will be no problem for these monitoring spacecraft. LISA will complement LIGO by looking at much more massive black holes through the observation of gravitational-wave sources emitting much larger wavelengths. Three satellites will be placed in space above Earth in an equilateral triangle (with 5,000,000-km sides) ([link](#)). The system will measure the relative positions of each satellite to

detect passing gravitational waves. Accuracy to within 10% of the size of an atom will be needed to detect any waves. The launch of this project might be as early as 2018.

“I’m sure LIGO will tell us something about the universe that we didn’t know before. The history of science tells us that any time you go where you haven’t been before, you usually find something that really shakes the scientific paradigms of the day. Whether gravitational wave astrophysics will do that, only time will tell.” —David Reitze, LIGO Input Optics Manager, University of Florida



Space-based future experiments for the measurement of gravitational waves. Shown here is a drawing of LISA’s orbit. Each satellite of LISA will consist of a laser source and a mass. The lasers will transmit a signal to measure the distance between each satellite’s test mass. The relative motion of these masses will provide information about passing gravitational waves. (credit: NASA)

The ideas presented in this section are but a glimpse into topics of modern physics that will be covered in much greater depth in later chapters.

Test Prep for AP Courses

Exercise:

Problem:

Which phenomenon correctly describes the direction and magnitude of normal forces?

- a. electromagnetic attraction
- b. electromagnetic repulsion
- c. gravitational attraction
- d. gravitational repulsion

Solution:

(b)

Exercise:

Problem:

Explain which of the four fundamental forces is responsible for a ball bouncing off the ground after it hits, and why this force has this effect.

Exercise:

Problem:

Which of the basic forces best explains tension in a rope being pulled between two people? Is the acting force causing attraction or repulsion in this instance?

- a. gravity; attraction
- b. electromagnetic; attraction
- c. weak and strong nuclear; attraction
- d. weak and strong nuclear; repulsion

Solution:

(b)

Exercise:

Problem:

Explain how interatomic electric forces produce the normal force, and why it has the direction it does.

Exercise:

Problem:

The gravitational force is the weakest of the four basic forces. In which case can the electromagnetic, strong, and weak forces be ignored because the gravitational force is so strongly dominant?

- a. a person jumping on a trampoline
- b. a rocket blasting off from Earth
- c. a log rolling down a hill
- d. all of the above

Solution:

(d)

Exercise:**Problem:**

Describe a situation in which gravitational force is the dominant force. Why can the other three basic forces be ignored in the situation you described?

Summary

- The various types of forces that are categorized for use in many applications are all manifestations of the *four basic forces* in nature.
- The properties of these forces are summarized in [\[link\]](#).
- Everything we experience directly without sensitive instruments is due to either electromagnetic forces or gravitational forces. The nuclear forces are responsible for the submicroscopic structure of matter, but they are not directly sensed because of their short ranges. Attempts are being made to show all four forces are different manifestations of a single unified force.
- A force field surrounds an object creating a force and is the carrier of that force.

Conceptual Questions**Exercise:****Problem:**

Explain, in terms of the properties of the four basic forces, why people notice the gravitational force acting on their bodies if it is such a comparatively weak force.

Exercise:**Problem:**

What is the dominant force between astronomical objects? Why are the other three basic forces less significant over these very large distances?

Exercise:**Problem:**

Give a detailed example of how the exchange of a particle can result in an *attractive* force. (For example, consider one child pulling a toy out of the hands of another.)

Problem Exercises

Exercise:**Problem:**

(a) What is the strength of the weak nuclear force relative to the strong nuclear force? (b) What is the strength of the weak nuclear force relative to the electromagnetic force? Since the weak nuclear force acts at only very short distances, such as inside nuclei, where the strong and electromagnetic forces also act, it might seem surprising that we have any knowledge of it at all. We have such knowledge because the weak nuclear force is responsible for beta decay, a type of nuclear decay not explained by other forces.

Solution:

(a) 1×10^{-13}

(b) 1×10^{-11}

Exercise:**Problem:**

(a) What is the ratio of the strength of the gravitational force to that of the strong nuclear force? (b) What is the ratio of the strength of the gravitational force to that of the weak nuclear force? (c) What is the ratio of the strength of the gravitational force to that of the electromagnetic force? What do your answers imply about the influence of the gravitational force on atomic nuclei?

Exercise:**Problem:**

What is the ratio of the strength of the strong nuclear force to that of the electromagnetic force? Based on this ratio, you might expect that the strong force dominates the nucleus, which is true for small nuclei. Large nuclei, however, have sizes greater than the range of the strong nuclear force. At these sizes, the electromagnetic force begins to affect nuclear stability. These facts will be used to explain nuclear fusion and fission later in this text.

Solution:

10^2

Glossary

carrier particle

a fundamental particle of nature that is surrounded by a characteristic force field; photons are carrier particles of the electromagnetic force

force field

a region in which a test particle will experience a force

Connection for AP® Courses
class="introduction"

Total hip
replacemen
t surgery
has become
a common
procedure.

The head
(or ball) of
the patient's
femur fits
into a cup
that has a
hard
plastic-like
inner
lining.

(credit:
National
Institutes of
Health, via
Wikimedia
Commons)



Have you ever wondered why it is difficult to walk on a smooth surface like ice? The interaction between you and the surface is a result of forces that affect your motion. In the previous chapter, you learned Newton's laws of motion and examined how net force affects the motion, position and shape of an object. Now we will look at some interesting and common forces that will provide further applications of Newton's laws of motion.

The information presented in this chapter supports learning objectives covered under Big Idea 3 of the AP Physics Curriculum Framework, which refer to the nature of forces and their roles in interactions among objects. The chapter discusses examples of specific contact forces, such as friction, air or liquid drag, and elasticity that may affect the motion or shape of an object. It also discusses the nature of forces on both macroscopic and microscopic levels (Enduring Understanding 3.C and Essential Knowledge 3.C.4). In addition, Newton's laws are applied to describe the motion of an

object (Enduring Understanding 3.B) and to examine relationships between contact forces and other forces exerted on an object (Enduring Understanding 3.A, 3.A.3 and Essential Knowledge 3.A.4). The examples in this chapter give you practice in using vector properties of forces (Essential Knowledge 3.A.2) and free-body diagrams (Essential Knowledge 3.B.2) to determine net force (Essential Knowledge 3.B.1).

Big Idea 3 The interactions of an object with other objects can be described by forces.

Enduring Understanding 3.A All forces share certain common characteristics when considered by observers in inertial reference frames.

Essential Knowledge 3.A.2 Forces are described by vectors.

Essential Knowledge 3.A.3 A force exerted on an object is always due to the interaction of that object with another object.

Essential Knowledge 3.A.4 If one object exerts a force on a second object, the second object always exerts a force of equal magnitude on the first object in the opposite direction.

Enduring Understanding 3.B Classically, the acceleration of an object interacting with other objects can be predicted by using $\vec{a} = \frac{\vec{F}_{\text{net}}}{m}$.

Essential Knowledge 3.B.1 If an object of interest interacts with several other objects, the net force is the vector sum of the individual forces.

Essential Knowledge 3.B.2 Free-body diagrams are useful tools for visualizing forces being exerted on a single object and writing the equations that represent a physical situation.

Enduring Understanding 3.C At the macroscopic level, forces can be categorized as either long-range (action-at-a-distance) forces or contact forces.

Essential Knowledge 3.C.4 Contact forces result from the interaction of one object touching another object, and they arise from interatomic electric forces. These forces include tension, friction, normal, spring (Physics 1), and buoyant (Physics 2).

Friction

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Discuss the general characteristics of friction.
- Describe the various types of friction.
- Calculate the magnitudes of static and kinetic frictional forces.

The information presented in this section supports the following AP® learning objectives and science practices:

- **3.C.4.1** The student is able to make claims about various contact forces between objects based on the microscopic cause of those forces. **(S.P. 6.1)**
- **3.C.4.2** The student is able to explain contact forces (tension, friction, normal, buoyant, spring) as arising from interatomic electric forces and that they therefore have certain directions. **(S.P. 6.2)**

Friction is a force that is around us all the time that opposes relative motion between systems in contact but also allows us to move (which you have discovered if you have ever tried to walk on ice). While a common force, the behavior of friction is actually very complicated and is still not completely understood. We have to rely heavily on observations for whatever understandings we can gain. However, we can still deal with its more elementary general characteristics and understand the circumstances in which it behaves.

Note:**Friction**

Friction is a force that opposes relative motion between systems in contact.

One of the simpler characteristics of friction is that it is parallel to the contact surface between systems and always in a direction that opposes motion or attempted motion of the systems relative to each other. If two systems are in contact and moving relative to one another, then the friction between them is called **kinetic friction**. For example, friction slows a hockey puck sliding on ice. But when objects are stationary, **static friction** can act between them; the static friction is usually greater than the kinetic friction between the objects.

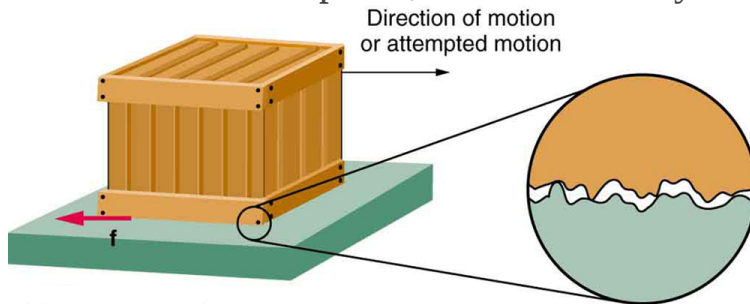
Note:**Kinetic Friction**

If two systems are in contact and moving relative to one another, then the friction between them is called kinetic friction.

Imagine, for example, trying to slide a heavy crate across a concrete floor—you may push harder and harder on the crate and not move it at all. This means that the static friction responds to what you do—it increases to be equal to and in the opposite direction of your push. But if you finally push hard enough, the crate seems to slip suddenly and starts to move. Once in motion it is easier to keep it in motion than it was to get it started, indicating that the kinetic friction force is less than the static friction force. If you add mass to the crate, say by placing a box on top of it, you need to push even harder to get it started and also to keep it moving. Furthermore, if you oiled the concrete you would find it to be easier to get the crate started and keep it going (as you might expect).

[\[link\]](#) is a crude pictorial representation of how friction occurs at the interface between two objects. Close-up inspection of these surfaces shows

them to be rough. So when you push to get an object moving (in this case, a crate), you must raise the object until it can skip along with just the tips of the surface hitting, break off the points, or do both. A considerable force can be resisted by friction with no apparent motion. The harder the surfaces are pushed together (such as if another box is placed on the crate), the more force is needed to move them. Part of the friction is due to adhesive forces between the surface molecules of the two objects, which explain the dependence of friction on the nature of the substances. Adhesion varies with substances in contact and is a complicated aspect of surface physics. Once an object is moving, there are fewer points of contact (fewer molecules adhering), so less force is required to keep the object moving. At small but nonzero speeds, friction is nearly independent of speed.



Frictional forces, such as f , always oppose motion or attempted motion between objects in contact. Friction arises in part because of the roughness of the surfaces in contact, as seen in the expanded view. In order for the object to move, it must rise to where the peaks can skip along the bottom surface. Thus a force is required just to set the object in motion. Some of the peaks will be broken off, also requiring a force to maintain motion. Much of the friction is actually due to attractive forces between molecules making up the two objects, so that even perfectly smooth surfaces are not friction-free. Such adhesive forces also depend on the substances the

surfaces are made of, explaining, for example, why rubber-soled shoes slip less than those with leather soles.

The magnitude of the frictional force has two forms: one for static situations (static friction), the other for when there is motion (kinetic friction).

When there is no motion between the objects, the **magnitude of static friction f_s** is

Equation:

$$f_s \leq \mu_s N,$$

where μ_s is the coefficient of static friction and N is the magnitude of the normal force (the force perpendicular to the surface).

Note:

Magnitude of Static Friction

Magnitude of static friction f_s is

Equation:

$$f_s \leq \mu_s N,$$

where μ_s is the coefficient of static friction and N is the magnitude of the normal force.

The symbol \leq means *less than or equal to*, implying that static friction can have a minimum and a maximum value of $\mu_s N$. Static friction is a responsive force that increases to be equal and opposite to whatever force is

exerted, up to its maximum limit. Once the applied force exceeds $f_{s(\max)}$, the object will move. Thus

Equation:

$$f_{s(\max)} = \mu_s N.$$

Once an object is moving, the **magnitude of kinetic friction** f_k is given by **Equation:**

$$f_k = \mu_k N,$$

where μ_k is the coefficient of kinetic friction. A system in which $f_k = \mu_k N$ is described as a system in which *friction behaves simply*.

Note:

Magnitude of Kinetic Friction

The magnitude of kinetic friction f_k is given by

Equation:

$$f_k = \mu_k N,$$

where μ_k is the coefficient of kinetic friction.

As seen in [\[link\]](#), the coefficients of kinetic friction are less than their static counterparts. That values of μ in [\[link\]](#) are stated to only one or, at most, two digits is an indication of the approximate description of friction given by the above two equations.

System	Static friction μ_s	Kinetic friction μ_k
Rubber on dry concrete	1.0	0.7
Rubber on wet concrete	0.7	0.5
Wood on wood	0.5	0.3
Waxed wood on wet snow	0.14	0.1
Metal on wood	0.5	0.3
Steel on steel (dry)	0.6	0.3
Steel on steel (oiled)	0.05	0.03
Teflon on steel	0.04	0.04
Bone lubricated by synovial fluid	0.016	0.015
Shoes on wood	0.9	0.7
Shoes on ice	0.1	0.05
Ice on ice	0.1	0.03
Steel on ice	0.04	0.02

Coefficients of Static and Kinetic Friction

The equations given earlier include the dependence of friction on materials and the normal force. The direction of friction is always opposite that of

motion, parallel to the surface between objects, and perpendicular to the normal force. For example, if the crate you try to push (with a force parallel to the floor) has a mass of 100 kg, then the normal force would be equal to its weight, $W = mg = (100 \text{ kg})(9.80 \text{ m/s}^2) = 980 \text{ N}$, perpendicular to the floor. If the coefficient of static friction is 0.45, you would have to exert a force parallel to the floor greater than

$f_{s(\text{max})} = \mu_s N = (0.45)(980 \text{ N}) = 440 \text{ N}$ to move the crate. Once there is motion, friction is less and the coefficient of kinetic friction might be 0.30, so that a force of only 290 N ($f_k = \mu_k N = (0.30)(980 \text{ N}) = 290 \text{ N}$) would keep it moving at a constant speed. If the floor is lubricated, both coefficients are considerably less than they would be without lubrication. Coefficient of friction is a unit less quantity with a magnitude usually between 0 and 1.0. The coefficient of the friction depends on the two surfaces that are in contact.

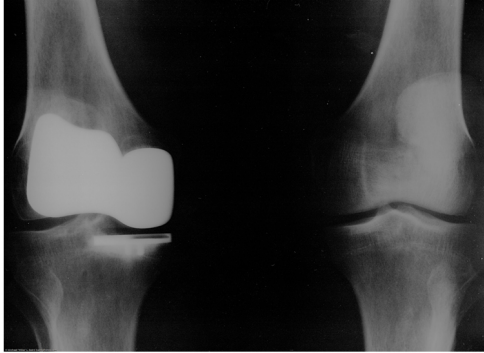
Note:

Take-Home Experiment

Find a small plastic object (such as a food container) and slide it on a kitchen table by giving it a gentle tap. Now spray water on the table, simulating a light shower of rain. What happens now when you give the object the same-sized tap? Now add a few drops of (vegetable or olive) oil on the surface of the water and give the same tap. What happens now? This latter situation is particularly important for drivers to note, especially after a light rain shower. Why?

Many people have experienced the slipperiness of walking on ice. However, many parts of the body, especially the joints, have much smaller coefficients of friction—often three or four times less than ice. A joint is formed by the ends of two bones, which are connected by thick tissues. The knee joint is formed by the lower leg bone (the tibia) and the thighbone (the femur). The hip is a ball (at the end of the femur) and socket (part of the pelvis) joint. The ends of the bones in the joint are covered by cartilage, which provides a smooth, almost glassy surface. The joints also produce a

fluid (synovial fluid) that reduces friction and wear. A damaged or arthritic joint can be replaced by an artificial joint ([link](#)). These replacements can be made of metals (stainless steel or titanium) or plastic (polyethylene), also with very small coefficients of friction.



Artificial knee replacement is a procedure that has been performed for more than 20 years. In this figure, we see the post-op x rays of the right knee joint replacement. (credit: Mike Baird, Flickr)

Other natural lubricants include saliva produced in our mouths to aid in the swallowing process, and the slippery mucus found between organs in the body, allowing them to move freely past each other during heartbeats, during breathing, and when a person moves. Artificial lubricants are also

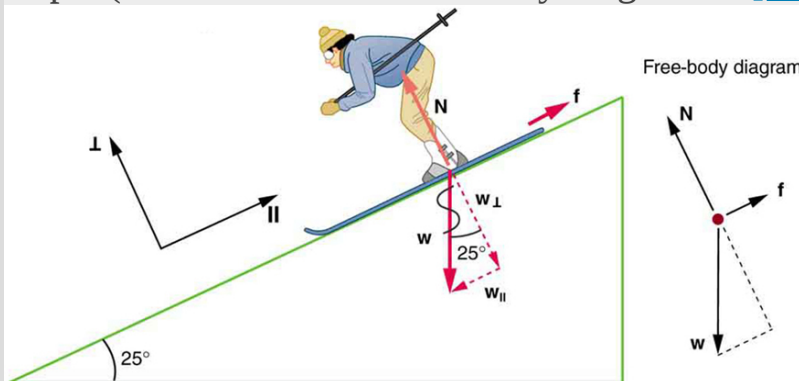
common in hospitals and doctor's clinics. For example, when ultrasonic imaging is carried out, the gel that couples the transducer to the skin also serves to lubricate the surface between the transducer and the skin—thereby reducing the coefficient of friction between the two surfaces. This allows the transducer to move freely over the skin.

Example:
Skiing Exercise

A skier with a mass of 62 kg is sliding down a snowy slope. Find the coefficient of kinetic friction for the skier if friction is known to be 45.0 N.

Strategy

The magnitude of kinetic friction was given in to be 45.0 N. Kinetic friction is related to the normal force N as $f_k = \mu_k N$; thus, the coefficient of kinetic friction can be found if we can find the normal force of the skier on a slope. The normal force is always perpendicular to the surface, and since there is no motion perpendicular to the surface, the normal force should equal the component of the skier's weight perpendicular to the slope. (See the skier and free-body diagram in [\[link\]](#).)



The motion of the skier and friction are parallel to the slope and so it is most convenient to project all forces onto a coordinate system where one axis is parallel to the slope and the other is perpendicular (axes shown to left of skier). N (the normal force) is perpendicular to the slope, and f (the friction) is parallel to the slope, but w

(the skier's weight) has components along both axes, namely \mathbf{w}_\perp and $\mathbf{W}_{//}$. \mathbf{N} is equal in magnitude to \mathbf{w}_\perp , so there is no motion perpendicular to the slope. However, \mathbf{f} is less than $\mathbf{W}_{//}$ in magnitude, so there is acceleration down the slope (along the x -axis).

That is,

Equation:

$$N = w_\perp = w \cos 25^\circ = mg \cos 25^\circ.$$

Substituting this into our expression for kinetic friction, we get

Equation:

$$f_k = \mu_k mg \cos 25^\circ,$$

which can now be solved for the coefficient of kinetic friction μ_k .

Solution

Solving for μ_k gives

Equation:

$$\mu_k = \frac{f_k}{N} = \frac{f_k}{w \cos 25^\circ} = \frac{f_k}{mg \cos 25^\circ}.$$

Substituting known values on the right-hand side of the equation,

Equation:

$$\mu_k = \frac{45.0 \text{ N}}{(62 \text{ kg})(9.80 \text{ m/s}^2)(0.906)} = 0.082.$$

Discussion

This result is a little smaller than the coefficient listed in [\[link\]](#) for waxed wood on snow, but it is still reasonable since values of the coefficients of friction can vary greatly. In situations like this, where an object of mass m

slides down a slope that makes an angle θ with the horizontal, friction is given by $f_k = \mu_k mg \cos \theta$. All objects will slide down a slope with constant acceleration under these circumstances. Proof of this is left for this chapter's Problems and Exercises.

Note:

Take-Home Experiment

An object will slide down an inclined plane at a constant velocity if the net force on the object is zero. We can use this fact to measure the coefficient of kinetic friction between two objects. As shown in [\[link\]](#), the kinetic friction on a slope $f_k = \mu_k mg \cos \theta$. The component of the weight down the slope is equal to $mg \sin \theta$ (see the free-body diagram in [\[link\]](#)). These forces act in opposite directions, so when they have equal magnitude, the acceleration is zero. Writing these out:

Equation:

$$f_k = F g_x$$

Equation:

$$\mu_k mg \cos \theta = mg \sin \theta.$$

Solving for μ_k , we find that

Equation:

$$\mu_k = \frac{mg \sin \theta}{mg \cos \theta} = \tan \theta.$$

Put a coin on a book and tilt it until the coin slides at a constant velocity down the book. You might need to tap the book lightly to get the coin to move. Measure the angle of tilt relative to the horizontal and find μ_k . Note that the coin will not start to slide at all until an angle greater than θ is attained, since the coefficient of static friction is larger than the coefficient of kinetic friction. Discuss how this may affect the value for μ_k and its uncertainty.

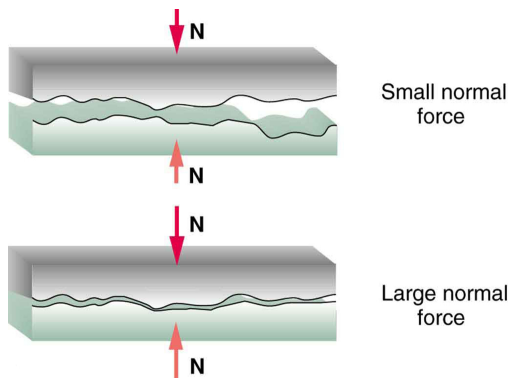
We have discussed that when an object rests on a horizontal surface, there is a normal force supporting it equal in magnitude to its weight. Furthermore, simple friction is always proportional to the normal force.

Note:

Making Connections: Submicroscopic Explanations of Friction

The simpler aspects of friction dealt with so far are its macroscopic (large-scale) characteristics. Great strides have been made in the atomic-scale explanation of friction during the past several decades. Researchers are finding that the atomic nature of friction seems to have several fundamental characteristics. These characteristics not only explain some of the simpler aspects of friction—they also hold the potential for the development of nearly friction-free environments that could save hundreds of billions of dollars in energy which is currently being converted (unnecessarily) to heat.

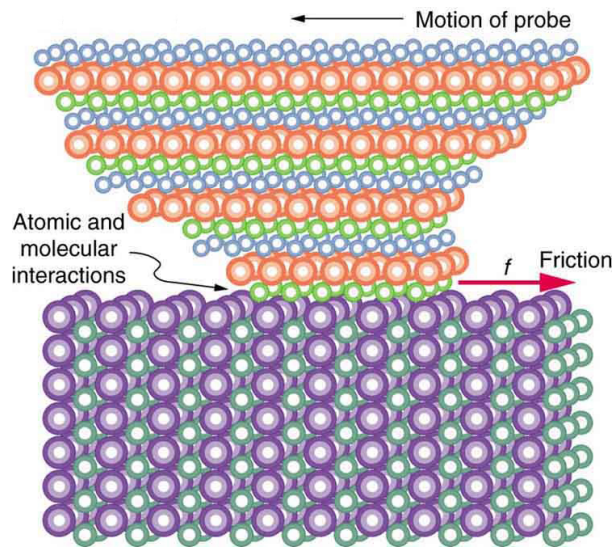
[\[link\]](#) illustrates one macroscopic characteristic of friction that is explained by microscopic (small-scale) research. We have noted that friction is proportional to the normal force, but not to the area in contact, a somewhat counterintuitive notion. When two rough surfaces are in contact, the actual contact area is a tiny fraction of the total area since only high spots touch. When a greater normal force is exerted, the actual contact area increases, and it is found that the friction is proportional to this area.



Two rough surfaces in

contact have a much smaller area of actual contact than their total area. When there is a greater normal force as a result of a greater applied force, the area of actual contact increases as does friction.

But the atomic-scale view promises to explain far more than the simpler features of friction. The mechanism for how heat is generated is now being determined. In other words, why do surfaces get warmer when rubbed? Essentially, atoms are linked with one another to form lattices. When surfaces rub, the surface atoms adhere and cause atomic lattices to vibrate—essentially creating sound waves that penetrate the material. The sound waves diminish with distance and their energy is converted into heat. Chemical reactions that are related to frictional wear can also occur between atoms and molecules on the surfaces. [\[link\]](#) shows how the tip of a probe drawn across another material is deformed by atomic-scale friction. The force needed to drag the tip can be measured and is found to be related to shear stress, which will be discussed later in this chapter. The variation in shear stress is remarkable (more than a factor of 10^{12}) and difficult to predict theoretically, but shear stress is yielding a fundamental understanding of a large-scale phenomenon known since ancient times—friction.



The tip of a probe is deformed sideways by frictional force as the probe is dragged across a surface. Measurements of how the force varies for different materials are yielding fundamental insights into the atomic nature of friction.

Note:

PhET Explorations: Forces and Motion

Explore the forces at work when you try to push a filing cabinet. Create an applied force and see the resulting friction force and total force acting on the cabinet. Charts show the forces, position, velocity, and acceleration vs. time. Draw a free-body diagram of all the forces (including gravitational and normal forces).

[Forces](#)
[and](#)

Test Prep for AP Courses

Exercise:

Problem:

When a force of 20 N is applied to a stationary box weighing 40 N, the box does not move. This means the coefficient of static friction

- a. is equal to 0.5.
- b. is greater than 0.5.
- c. is less than 0.5.
- d. cannot be determined.

Solution:

(b)

Exercise:

Problem:

A 2-kg block slides down a ramp which is at an incline of 25° . If the frictional force is 4.86 N, what is the coefficient of friction? At what incline will the box slide at a constant velocity? Assume $g = 10 \text{ m/s}^2$.

Exercise:

Problem:

A block is given a short push and then slides with constant friction across a horizontal floor. Which statement best explains the direction of the force that friction applies on the moving block?

- a. Friction will be in the same direction as the block's motion because molecular interactions between the block and the floor will deform the block in the direction of motion.
 - b. Friction will be in the same direction as the block's motion because thermal energy generated at the interface between the block and the floor adds kinetic energy to the block.
 - c. Friction will be in the opposite direction of the block's motion because molecular interactions between the block and the floor will deform the block in the opposite direction of motion.
 - d. Friction will be in the opposite direction of the block's motion because thermal energy generated at the interface between the block and the floor converts some of the block's kinetic energy to potential energy.
-

Solution:

(c)

Exercise:

Problem:

A student pushes a cardboard box across a carpeted floor and afterwards notices that the bottom of the box feels warm. Explain how interactions between molecules in the cardboard and molecules in the carpet produced this heat.

Section Summary

- Friction is a contact force between systems that opposes the motion or attempted motion between them. Simple friction is proportional to the normal force N pushing the systems together. (A normal force is always perpendicular to the contact surface between systems.) Friction depends on both of the materials involved. The magnitude of static friction f_s between systems stationary relative to one another is given by

Equation:

$$f_s \leq \mu_s N,$$

where μ_s is the coefficient of static friction, which depends on both of the materials.

- The kinetic friction force f_k between systems moving relative to one another is given by

Equation:

$$f_k = \mu_k N,$$

where μ_k is the coefficient of kinetic friction, which also depends on both materials.

Conceptual Questions

Exercise:

Problem:

Define normal force. What is its relationship to friction when friction behaves simply?

Exercise:

Problem:

The glue on a piece of tape can exert forces. Can these forces be a type of simple friction? Explain, considering especially that tape can stick to vertical walls and even to ceilings.

Exercise:

Problem:

When you learn to drive, you discover that you need to let up slightly on the brake pedal as you come to a stop or the car will stop with a jerk. Explain this in terms of the relationship between static and kinetic friction.

Exercise:

Problem:

When you push a piece of chalk across a chalkboard, it sometimes screeches because it rapidly alternates between slipping and sticking to the board. Describe this process in more detail, in particular explaining how it is related to the fact that kinetic friction is less than static friction. (The same slip-grab process occurs when tires screech on pavement.)

Problems & Exercises**Exercise:****Problem:**

A physics major is cooking breakfast when he notices that the frictional force between his steel spatula and his Teflon frying pan is only 0.200 N. Knowing the coefficient of kinetic friction between the two materials, he quickly calculates the normal force. What is it?

Solution:

5.00 N

Exercise:**Problem:**

(a) When rebuilding her car's engine, a physics major must exert 300 N of force to insert a dry steel piston into a steel cylinder. What is the magnitude of the normal force between the piston and cylinder? (b) What is the magnitude of the force would she have to exert if the steel parts were oiled?

Exercise:

Problem:

(a) What is the maximum frictional force in the knee joint of a person who supports 66.0 kg of her mass on that knee? (b) During strenuous exercise it is possible to exert forces to the joints that are easily ten times greater than the weight being supported. What is the maximum force of friction under such conditions? The frictional forces in joints are relatively small in all circumstances except when the joints deteriorate, such as from injury or arthritis. Increased frictional forces can cause further damage and pain.

Exercise:**Problem:**

Suppose you have a 120-kg wooden crate resting on a wood floor. (a) What maximum force can you exert horizontally on the crate without moving it? (b) If you continue to exert this force once the crate starts to slip, what will the magnitude of its acceleration then be?

Solution:

(a) 588 N

(b) 1.96 m/s^2

Exercise:**Problem:**

(a) If half of the weight of a small $1.00 \times 10^3 \text{ kg}$ utility truck is supported by its two drive wheels, what is the magnitude of the maximum acceleration it can achieve on dry concrete? (b) Will a metal cabinet lying on the wooden bed of the truck slip if it accelerates at this rate? (c) Solve both problems assuming the truck has four-wheel drive.

Exercise:

Problem:

A team of eight dogs pulls a sled with waxed wood runners on wet snow (mush!). The dogs have average masses of 19.0 kg, and the loaded sled with its rider has a mass of 210 kg. (a) Calculate the magnitude of the acceleration starting from rest if each dog exerts an average force of 185 N backward on the snow. (b) What is the magnitude of the acceleration once the sled starts to move? (c) For both situations, calculate the magnitude of the force in the coupling between the dogs and the sled.

Solution:

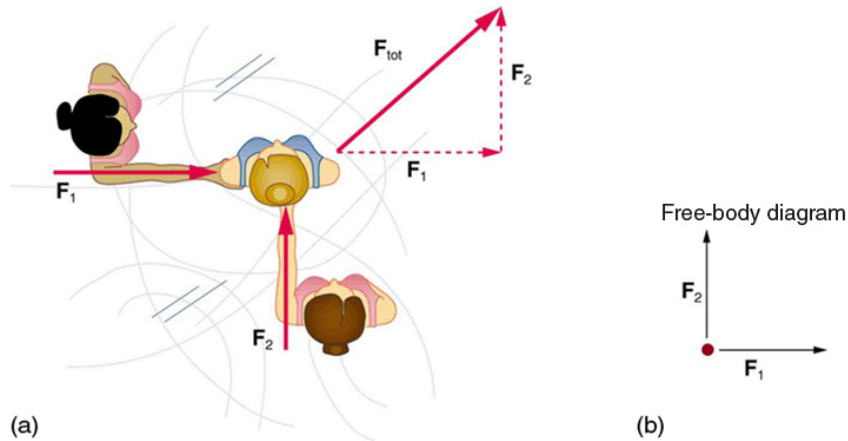
(a) 3.29 m/s^2

(b) 3.52 m/s^2

(c) 980 N; 945 N

Exercise:**Problem:**

Consider the 65.0-kg ice skater being pushed by two others shown in [\[link\]](#). (a) Find the direction and magnitude of \mathbf{F}_{tot} , the total force exerted on her by the others, given that the magnitudes F_1 and F_2 are 26.4 N and 18.6 N, respectively. (b) What is her initial acceleration if she is initially stationary and wearing steel-bladed skates that point in the direction of \mathbf{F}_{tot} ? (c) What is her acceleration assuming she is already moving in the direction of \mathbf{F}_{tot} ? (Remember that friction always acts in the direction opposite that of motion or attempted motion between surfaces in contact.)



Exercise:

Problem:

Show that the acceleration of any object down a frictionless incline that makes an angle θ with the horizontal is $a = g \sin \theta$. (Note that this acceleration is independent of mass.)

Exercise:

Problem:

Show that the acceleration of any object down an incline where friction behaves simply (that is, where $f_k = \mu_k N$) is $a = g(\sin \theta - \mu_k \cos \theta)$. Note that the acceleration is independent of mass and reduces to the expression found in the previous problem when friction becomes negligibly small ($\mu_k = 0$).

Exercise:

Problem:

Calculate the deceleration of a snow boarder going up a 5.0° slope assuming the coefficient of friction for waxed wood on wet snow. The result of [\[link\]](#) may be useful, but be careful to consider the fact that the snow boarder is going uphill. Explicitly show how you follow the steps in [Problem-Solving Strategies](#).

Solution:

$$1.83 \text{ m/s}^2$$

Exercise:**Problem:**

(a) Calculate the acceleration of a skier heading down a 10.0° slope, assuming the coefficient of friction for waxed wood on wet snow. (b) Find the angle of the slope down which this skier could coast at a constant velocity. You can neglect air resistance in both parts, and you will find the result of [\[link\]](#) to be useful. Explicitly show how you follow the steps in the [Problem-Solving Strategies](#).

Exercise:**Problem:**

If an object is to rest on an incline without slipping, then friction must equal the component of the weight of the object parallel to the incline. This requires greater and greater friction for steeper slopes. Show that the maximum angle of an incline above the horizontal for which an object will not slide down is $\theta = \tan^{-1} \mu_s$. You may use the result of the previous problem. Assume that $a = 0$ and that static friction has reached its maximum value.

Exercise:**Problem:**

Calculate the maximum deceleration of a car that is heading down a 6° slope (one that makes an angle of 6° with the horizontal) under the following road conditions. You may assume that the weight of the car is evenly distributed on all four tires and that the coefficient of static friction is involved—that is, the tires are not allowed to slip during the deceleration. (Ignore rolling.) Calculate for a car: (a) On dry concrete. (b) On wet concrete. (c) On ice, assuming that $\mu_s = 0.100$, the same as for shoes on ice.

Exercise:

Problem:

Calculate the maximum acceleration of a car that is heading up a 4° slope (one that makes an angle of 4° with the horizontal) under the following road conditions. Assume that only half the weight of the car is supported by the two drive wheels and that the coefficient of static friction is involved—that is, the tires are not allowed to slip during the acceleration. (Ignore rolling.) (a) On dry concrete. (b) On wet concrete. (c) On ice, assuming that $\mu_s = 0.100$, the same as for shoes on ice.

Solution:

(a) 4.20 m/s^2

(b) 2.74 m/s^2

(c) -0.195 m/s^2

Exercise:

Problem: Repeat [\[link\]](#) for a car with four-wheel drive.

Exercise:**Problem:**

A freight train consists of two 8.00×10^5 -kg engines and 45 cars with average masses of 5.50×10^5 kg. (a) What force must each engine exert backward on the track to accelerate the train at a rate of $5.00 \times 10^{-2} \text{ m/s}^2$ if the force of friction is $7.50 \times 10^5 \text{ N}$, assuming the engines exert identical forces? This is not a large frictional force for such a massive system. Rolling friction for trains is small, and consequently trains are very energy-efficient transportation systems. (b) What is the magnitude of the force in the coupling between the 37th and 38th cars (this is the force each exerts on the other), assuming all cars have the same mass and that friction is evenly distributed among all of the cars and engines?

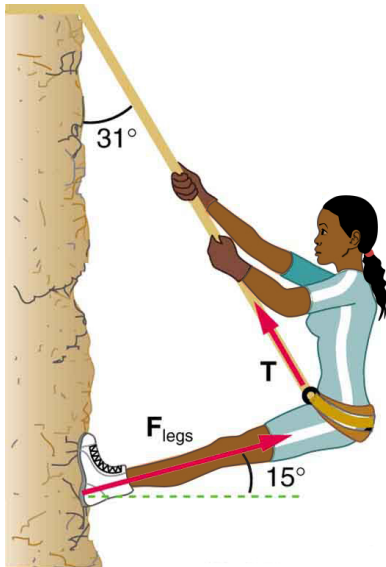
Solution:

(a) $1.03 \times 10^6 \text{ N}$

(b) $3.48 \times 10^5 \text{ N}$

Exercise:**Problem:**

Consider the 52.0-kg mountain climber in [\[link\]](#). (a) Find the tension in the rope and the force that the mountain climber must exert with her feet on the vertical rock face to remain stationary. Assume that the force is exerted parallel to her legs. Also, assume negligible force exerted by her arms. (b) What is the minimum coefficient of friction between her shoes and the cliff?



Part of the climber's weight is supported by her rope and part by friction between her feet and the rock face.

Exercise:**Problem:**

A contestant in a winter sporting event pushes a 45.0-kg block of ice across a frozen lake as shown in [\[link\]](#)(a). (a) Calculate the minimum force F he must exert to get the block moving. (b) What is the magnitude of its acceleration once it starts to move, if that force is maintained?

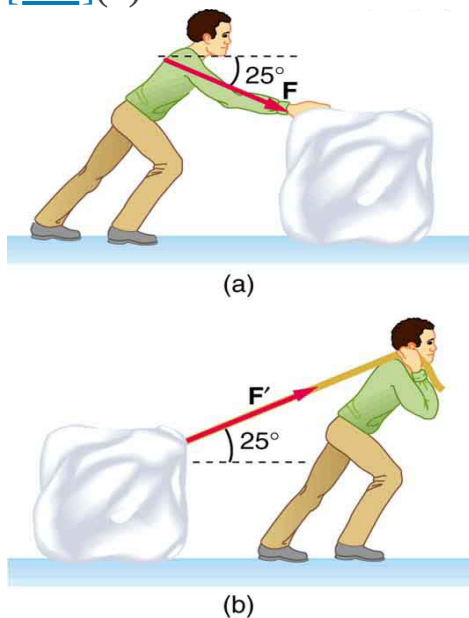
Solution:

(a) 51.0 N

(b) 0.720 m/s^2

Exercise:**Problem:**

Repeat [\[link\]](#) with the contestant pulling the block of ice with a rope over his shoulder at the same angle above the horizontal as shown in [\[link\]](#)(b).



Which method of

sliding a block of ice
requires less force—(a)
pushing or (b) pulling
at the same angle above
the horizontal?

Glossary

friction

a force that opposes relative motion or attempts at motion between systems in contact

kinetic friction

a force that opposes the motion of two systems that are in contact and moving relative to one another

static friction

a force that opposes the motion of two systems that are in contact and are not moving relative to one another

magnitude of static friction

$f_s \leq \mu_s N$, where μ_s is the coefficient of static friction and N is the magnitude of the normal force

magnitude of kinetic friction

$f_k = \mu_k N$, where μ_k is the coefficient of kinetic friction

Drag Forces

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define drag force and model it mathematically.
- Discuss the applications of drag force.
- Define terminal velocity.
- Perform calculations to find terminal velocity.

Another interesting force in everyday life is the force of drag on an object when it is moving in a fluid (either a gas or a liquid). You feel the drag force when you move your hand through water. You might also feel it if you move your hand during a strong wind. The faster you move your hand, the harder it is to move. You feel a smaller drag force when you tilt your hand so only the side goes through the air—you have decreased the area of your hand that faces the direction of motion. Like friction, the **drag force** always opposes the motion of an object. Unlike simple friction, the drag force is proportional to some function of the velocity of the object in that fluid. This functionality is complicated and depends upon the shape of the object, its size, its velocity, and the fluid it is in. For most large objects such as bicyclists, cars, and baseballs not moving too slowly, the magnitude of the drag force F_D is found to be proportional to the square of the speed of the object. We can write this relationship mathematically as $F_D \propto v^2$. When taking into account other factors, this relationship becomes

Equation:

$$F_D = \frac{1}{2} C \rho A v^2,$$

where C is the drag coefficient, A is the area of the object facing the fluid, and ρ is the density of the fluid. (Recall that density is mass per unit volume.) This equation can also be written in a more generalized fashion as $F_D = b v^2$, where b is a constant equivalent to $0.5 C \rho A$. We have set the exponent for these equations as 2 because, when an object is moving at high velocity through air, the magnitude of the drag force is proportional to the square of the speed. As we shall see in a few pages on fluid dynamics, for small particles moving at low speeds in a fluid, the exponent is equal to 1.

Note:

Drag Force

Drag force F_D is found to be proportional to the square of the speed of the object. Mathematically

Equation:

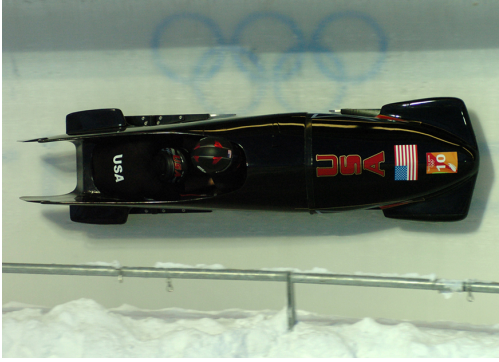
$$F_D \propto v^2$$

Equation:

$$F_D = \frac{1}{2} C \rho A v^2,$$

where C is the drag coefficient, A is the area of the object facing the fluid, and ρ is the density of the fluid.

Athletes as well as car designers seek to reduce the drag force to lower their race times. (See [\[link\]](#)). “Aerodynamic” shaping of an automobile can reduce the drag force and so increase a car's gas mileage.



From racing cars to bobsled racers, aerodynamic shaping is crucial to achieving top speeds. Bobsleds are designed for speed. They are shaped like a bullet with tapered fins. (credit: U.S. Army, via Wikimedia Commons)

The value of the drag coefficient, C , is determined empirically, usually with the use of a wind tunnel. (See [\[link\]](#)).



NASA researchers

test a model plane
in a wind tunnel.
(credit:
NASA/Ames)

The drag coefficient can depend upon velocity, but we will assume that it is a constant here. [\[link\]](#) lists some typical drag coefficients for a variety of objects. Notice that the drag coefficient is a dimensionless quantity. At highway speeds, over 50% of the power of a car is used to overcome air drag. The most fuel-efficient cruising speed is about 70–80 km/h (about 45–50 mi/h). For this reason, during the 1970s oil crisis in the United States, maximum speeds on highways were set at about 90 km/h (55 mi/h).

Object	C
Airfoil	0.05
Toyota Camry	0.28
Ford Focus	0.32
Honda Civic	0.36
Ferrari Testarossa	0.37
Dodge Ram pickup	0.43
Sphere	0.45
Hummer H2 SUV	0.64

Object	C
Skydiver (feet first)	0.70
Bicycle	0.90
Skydiver (horizontal)	1.0
Circular flat plate	1.12

Typical values of Drag Coefficient C

Substantial research is under way in the sporting world to minimize drag. The dimples on golf balls are being redesigned as are the clothes that athletes wear. Bicycle racers and some swimmers and runners wear full bodysuits. Australian Cathy Freeman wore a full body suit in the 2000 Sydney Olympics, and won the gold medal for the 400 m race. Many swimmers in the 2008 Beijing Olympics wore (Speedo) body suits; it might have made a difference in breaking many world records (See [\[link\]](#)). Most elite swimmers (and cyclists) shave their body hair. Such innovations can have the effect of slicing away milliseconds in a race, sometimes making the difference between a gold and a silver medal. One consequence is that careful and precise guidelines must be continuously developed to maintain the integrity of the sport.



Body suits, such as this LZR Racer Suit, have

been credited with many world records after their release in 2008. Smoother “skin” and more compression forces on a swimmer's body provide at least 10% less drag.
(credit: NASA/Kathy Barnstorff)

Some interesting situations connected to Newton's second law occur when considering the effects of drag forces upon a moving object. For instance, consider a skydiver falling through air under the influence of gravity. The two forces acting on him are the force of gravity and the drag force (ignoring the buoyant force). The downward force of gravity remains constant regardless of the velocity at which the person is moving. However, as the person's velocity increases, the magnitude of the drag force increases until the magnitude of the drag force is equal to the gravitational force, thus producing a net force of zero. A zero net force means that there is no acceleration, as given by Newton's second law. At this point, the person's velocity remains constant and we say that the person has reached his *terminal velocity* (v_t). Since F_D is proportional to the speed, a heavier skydiver must go faster for F_D to equal his weight. Let's see how this works out more quantitatively.

At the terminal velocity,

Equation:

$$F_{\text{net}} = mg - F_D = ma = 0.$$

Thus,

Equation:

$$mg = F_D.$$

Using the equation for drag force, we have

Equation:

$$mg = \frac{1}{2} \rho C A v^2.$$

Solving for the velocity, we obtain

Equation:

$$v = \sqrt{\frac{2mg}{\rho C A}}.$$

Assume the density of air is $\rho = 1.21 \text{ kg/m}^3$. A 75-kg skydiver descending head first will have an area approximately $A = 0.18 \text{ m}^2$ and a drag coefficient of approximately $C = 0.70$. We find that

Equation:

$$\begin{aligned} v &= \sqrt{\frac{2(75 \text{ kg})(9.80 \text{ m/s}^2)}{(1.21 \text{ kg/m}^3)(0.70)(0.18 \text{ m}^2)}} \\ &= 98 \text{ m/s} \\ &= 350 \text{ km/h.} \end{aligned}$$

This means a skydiver with a mass of 75 kg achieves a maximum terminal velocity of about 350 km/h while traveling in a pike (head first) position, minimizing the area and his drag. In a spread-eagle position, that terminal velocity may decrease to about 200 km/h as the area increases. This terminal velocity becomes much smaller after the parachute opens.

Note:

Take-Home Experiment

This interesting activity examines the effect of weight upon terminal velocity. Gather together some nested coffee filters. Leaving them in their

original shape, measure the time it takes for one, two, three, four, and five nested filters to fall to the floor from the same height (roughly 2 m). (Note that, due to the way the filters are nested, drag is constant and only mass varies.) They obtain terminal velocity quite quickly, so find this velocity as a function of mass. Plot the terminal velocity v versus mass. Also plot v^2 versus mass. Which of these relationships is more linear? What can you conclude from these graphs?

Example:

A Terminal Velocity

Find the terminal velocity of an 85-kg skydiver falling in a spread-eagle position.

Strategy

At terminal velocity, $F_{\text{net}} = 0$. Thus the drag force on the skydiver must equal the force of gravity (the person's weight). Using the equation of drag force, we find $mg = \frac{1}{2}\rho CA v^2$.

Thus the terminal velocity v_t can be written as

Equation:

$$v_t = \sqrt{\frac{2mg}{\rho CA}}.$$

Solution

All quantities are known except the person's projected area. This is an adult (85 kg) falling spread eagle. We can estimate the frontal area as

Equation:

$$A = (2 \text{ m})(0.35 \text{ m}) = 0.70 \text{ m}^2.$$

Using our equation for v_t , we find that

Equation:

$$\begin{aligned}
 v_t &= \sqrt{\frac{2(85 \text{ kg})(9.80 \text{ m/s}^2)}{(1.21 \text{ kg/m}^3)(1.0)(0.70 \text{ m}^2)}} \\
 &= 44 \text{ m/s.}
 \end{aligned}$$

Discussion

This result is consistent with the value for v_t mentioned earlier. The 75-kg skydiver going feet first had a $v = 98 \text{ m/s}$. He weighed less but had a smaller frontal area and so a smaller drag due to the air.

The size of the object that is falling through air presents another interesting application of air drag. If you fall from a 5-m high branch of a tree, you will likely get hurt—possibly fracturing a bone. However, a small squirrel does this all the time, without getting hurt. You don't reach a terminal velocity in such a short distance, but the squirrel does.

The following interesting quote on animal size and terminal velocity is from a 1928 essay by a British biologist, J.B.S. Haldane, titled “On Being the Right Size.”

To the mouse and any smaller animal, [gravity] presents practically no dangers. You can drop a mouse down a thousand-yard mine shaft; and, on arriving at the bottom, it gets a slight shock and walks away, provided that the ground is fairly soft. A rat is killed, a man is broken, and a horse splashes. For the resistance presented to movement by the air is proportional to the surface of the moving object. Divide an animal's length, breadth, and height each by ten; its weight is reduced to a thousandth, but its surface only to a hundredth. So the resistance to falling in the case of the small animal is relatively ten times greater than the driving force.

The above quadratic dependence of air drag upon velocity does not hold if the object is very small, is going very slow, or is in a denser medium than air. Then we find that the drag force is proportional just to the velocity. This relationship is given by **Stokes' law**, which states that

Equation:

$$F_s = 6\pi r\eta v,$$

where r is the radius of the object, η is the viscosity of the fluid, and v is the object's velocity.

Note:

Stokes' Law

Equation:

$$F_s = 6\pi r\eta v,$$

where r is the radius of the object, η is the viscosity of the fluid, and v is the object's velocity.

Good examples of this law are provided by microorganisms, pollen, and dust particles. Because each of these objects is so small, we find that many of these objects travel unaided only at a constant (terminal) velocity.

Terminal velocities for bacteria (size about $1\text{ }\mu\text{m}$) can be about $2\text{ }\mu\text{m/s}$. To move at a greater speed, many bacteria swim using flagella (organelles shaped like little tails) that are powered by little motors embedded in the cell. Sediment in a lake can move at a greater terminal velocity (about $5\text{ }\mu\text{m/s}$), so it can take days to reach the bottom of the lake after being deposited on the surface.

If we compare animals living on land with those in water, you can see how drag has influenced evolution. Fishes, dolphins, and even massive whales are streamlined in shape to reduce drag forces. Birds are streamlined and migratory species that fly large distances often have particular features such as long necks. Flocks of birds fly in the shape of a spear head as the flock forms a streamlined pattern (see [\[link\]](#)). In humans, one important example of streamlining is the shape of sperm, which need to be efficient in their use of energy.



Geese fly in a V formation during their long migratory travels. This shape reduces drag and energy consumption for individual birds, and also allows them a better way to communicate.
(credit: Julo, Wikimedia Commons)

Note:

Galileo's Experiment

Galileo is said to have dropped two objects of different masses from the Tower of Pisa. He measured how long it took each to reach the ground. Since stopwatches weren't readily available, how do you think he measured their fall time? If the objects were the same size, but with different masses, what do you think he should have observed? Would this result be different if done on the Moon?

Note:

PhET Explorations: Masses & Springs

A realistic mass and spring laboratory. Hang masses from springs and adjust the spring stiffness and damping. You can even slow time. Transport the lab to different planets. A chart shows the kinetic, potential, and thermal energy for each spring.

[Masses
&
Spring
s](#)

Section Summary

- Drag forces acting on an object moving in a fluid oppose the motion. For larger objects (such as a baseball) moving at a velocity v in air, the drag force is given by

Equation:

$$F_D = \frac{1}{2}C\rho Av^2,$$

where C is the drag coefficient (typical values are given in [\[link\]](#)), A is the area of the object facing the fluid, and ρ is the fluid density.

- For small objects (such as a bacterium) moving in a denser medium (such as water), the drag force is given by Stokes' law,

Equation:

$$F_s = 6\pi\eta rv,$$

where r is the radius of the object, η is the fluid viscosity, and v is the object's velocity.

Conceptual Questions

Exercise:

Problem:

Athletes such as swimmers and bicyclists wear body suits in competition. Formulate a list of pros and cons of such suits.

Exercise:

Problem:

Two expressions were used for the drag force experienced by a moving object in a liquid. One depended upon the speed, while the other was proportional to the square of the speed. In which types of motion would each of these expressions be more applicable than the other one?

Exercise:

Problem:

As cars travel, oil and gasoline leaks onto the road surface. If a light rain falls, what does this do to the control of the car? Does a heavy rain make any difference?

Exercise:

Problem:

Why can a squirrel jump from a tree branch to the ground and run away undamaged, while a human could break a bone in such a fall?

Problems & Exercise

Exercise:

Problem:

The terminal velocity of a person falling in air depends upon the weight and the area of the person facing the fluid. Find the terminal velocity (in meters per second and kilometers per hour) of an 80.0-kg skydiver falling in a pike (headfirst) position with a surface area of 0.140 m^2 .

Solution:

115 m/s; 414 km/hr

Exercise:**Problem:**

A 60-kg and a 90-kg skydiver jump from an airplane at an altitude of 6000 m, both falling in the pike position. Make some assumption on their frontal areas and calculate their terminal velocities. How long will it take for each skydiver to reach the ground (assuming the time to reach terminal velocity is small)? Assume all values are accurate to three significant digits.

Exercise:**Problem:**

A 560-g squirrel with a surface area of 930 cm^2 falls from a 5.0-m tree to the ground. Estimate its terminal velocity. (Use a drag coefficient for a horizontal skydiver.) What will be the velocity of a 56-kg person hitting the ground, assuming no drag contribution in such a short distance?

Solution:

25 m/s; 9.9 m/s

Exercise:

Problem:

To maintain a constant speed, the force provided by a car's engine must equal the drag force plus the force of friction of the road (the rolling resistance). (a) What are the magnitudes of drag forces at 70 km/h and 100 km/h for a Toyota Camry? (Drag area is 0.70 m^2) (b) What is the magnitude of drag force at 70 km/h and 100 km/h for a Hummer H2? (Drag area is 2.44 m^2) Assume all values are accurate to three significant digits.

Exercise:**Problem:**

By what factor does the drag force on a car increase as it goes from 65 to 110 km/h?

Solution:

2.9

Exercise:**Problem:**

Calculate the speed a spherical rain drop would achieve falling from 5.00 km (a) in the absence of air drag (b) with air drag. Take the size across of the drop to be 4 mm, the density to be $1.00 \times 10^3 \text{ kg/m}^3$, and the surface area to be πr^2 .

Exercise:**Problem:**

Using Stokes' law, verify that the units for viscosity are kilograms per meter per second.

Solution:

$$[\eta] = \frac{[F_s]}{[r][v]} = \frac{\text{kg}\cdot\text{m}/\text{s}^2}{\text{m}\cdot\text{m}/\text{s}} = \frac{\text{kg}}{\text{m}\cdot\text{s}}$$

Exercise:**Problem:**

Find the terminal velocity of a spherical bacterium (diameter $2.00\text{ }\mu\text{m}$) falling in water. You will first need to note that the drag force is equal to the weight at terminal velocity. Take the density of the bacterium to be $1.10 \times 10^3\text{ kg/m}^3$.

Exercise:**Problem:**

Stokes' law describes sedimentation of particles in liquids and can be used to measure viscosity. Particles in liquids achieve terminal velocity quickly. One can measure the time it takes for a particle to fall a certain distance and then use Stokes' law to calculate the viscosity of the liquid. Suppose a steel ball bearing (density $7.8 \times 10^3\text{ kg/m}^3$, diameter 3.0 mm) is dropped in a container of motor oil. It takes 12 s to fall a distance of 0.60 m . Calculate the viscosity of the oil.

Solution:

$0.76\text{ kg/m} \cdot \text{s}$

Glossary

drag force

F_D , found to be proportional to the square of the speed of the object; mathematically

Equation:

$$F_D \propto v^2$$

Equation:

$$F_D = \frac{1}{2} C \rho A v^2,$$

where C is the drag coefficient, A is the area of the object facing the fluid, and ρ is the density of the fluid

Stokes' law

$F_s = 6\pi r\eta v$, where r is the radius of the object, η is the viscosity of the fluid, and v is the object's velocity

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- State Hooke's law.
- Explain Hooke's law using graphical representation between deformation and applied force.
- Discuss the three types of deformations such as changes in length, sideways shear, and changes in volume.
- Describe with examples the Young's modulus, shear modulus, and bulk modulus.
- Determine the change in length given mass, length, and radius.

We now move from consideration of forces that affect the motion of an object (such as friction and drag) to those that affect an object's shape. If a bulldozer pushes a car into a wall, the car will not move but it will noticeably change shape. A change in shape due to the application of a force is a **deformation**. Even very small forces are known to cause some deformation. For small deformations, two important characteristics are observed. First, the object returns to its original shape when the force is removed—that is, the deformation is elastic for small deformations. Second, the size of the deformation is proportional to the force—that is, for small deformations, Hooke's law is obeyed. In equation form, **Hooke's law** is given by

Equation:

$$F = k\Delta L,$$

where ΔL is the amount of deformation (the change in length, for example) produced by the force F , and k is a proportionality constant that depends on the shape and composition of the object and the direction of the force. Note that this force is a function of the deformation ΔL —it is not constant as a kinetic friction force is. Rearranging this to

Equation:

$$\Delta L = \frac{F}{k}$$

makes it clear that the deformation is proportional to the applied force. [\[link\]](#) shows the Hooke's law relationship between the extension ΔL of a spring or of a human bone. For metals or springs, the straight line region in which Hooke's law pertains is much larger. Bones are brittle and the elastic region is small and the fracture abrupt. Eventually a large enough stress to the material will cause it to break or fracture.

Note:

Hooke's Law

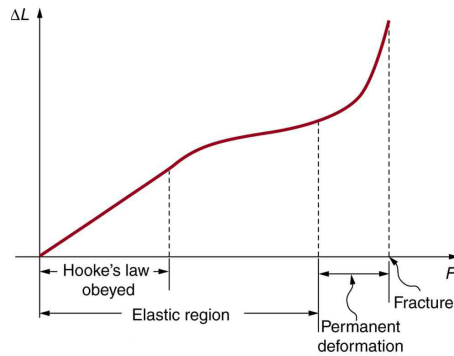
Equation:

$$F = k\Delta L,$$

where ΔL is the amount of deformation (the change in length, for example) produced by the force F , and k is a proportionality constant that depends on the shape and composition of the object and the direction of the force.

Equation:

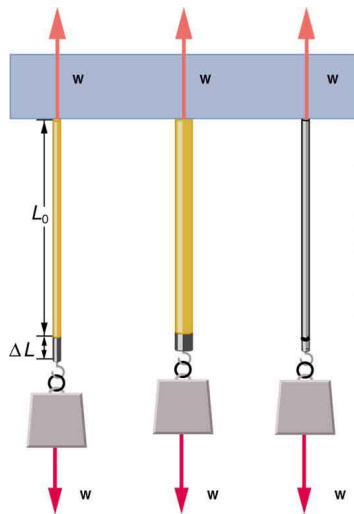
$$\Delta L = \frac{F}{k}$$



A graph of deformation ΔL versus applied force F . The straight segment is the linear region where Hooke's law is obeyed. The slope of the straight region is $\frac{1}{k}$. For larger forces, the graph is curved but the deformation is still elastic— ΔL will return to zero if the force is removed. Still greater forces permanently deform the object until it finally fractures. The shape of the curve near fracture depends on several factors, including how the force F is applied. Note that in this graph the slope increases just before fracture, indicating that a small increase in F is

producing a large
increase in L near the
fracture.

The proportionality constant k depends upon a number of factors for the material. For example, a guitar string made of nylon stretches when it is tightened, and the elongation ΔL is proportional to the force applied (at least for small deformations). Thicker nylon strings and ones made of steel stretch less for the same applied force, implying they have a larger k (see [\[link\]](#)). Finally, all three strings return to their normal lengths when the force is removed, provided the deformation is small. Most materials will behave in this manner if the deformation is less than about 0.1% or about 1 part in 10^3 .



The same force,
in this case a
weight (w),
applied to three
different guitar
strings of
identical length
produces the
three different
deformations

shown as shaded segments. The string on the left is thin nylon, the one in the middle is thicker nylon, and the one on the right is steel.

Note:

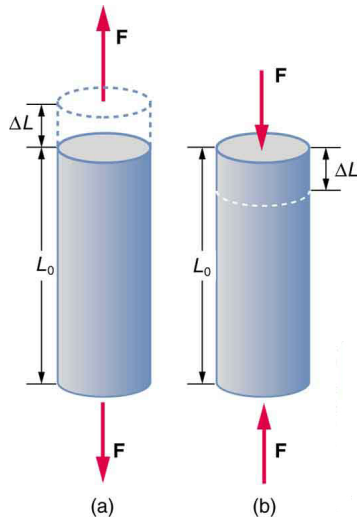
Stretch Yourself a Little

How would you go about measuring the proportionality constant k of a rubber band? If a rubber band stretched 3 cm when a 100-g mass was attached to it, then how much would it stretch if two similar rubber bands were attached to the same mass—even if put together in parallel or alternatively if tied together in series?

We now consider three specific types of deformations: changes in length (tension and compression), sideways shear (stress), and changes in volume. All deformations are assumed to be small unless otherwise stated.

Changes in Length—Tension and Compression: Elastic Modulus

A change in length ΔL is produced when a force is applied to a wire or rod parallel to its length L_0 , either stretching it (a tension) or compressing it. (See [\[link\]](#).)



(a) Tension. The rod is stretched a length ΔL when a force is applied parallel to its length. (b) Compression. The same rod is compressed by forces with the same magnitude in the opposite direction. For very small deformations and uniform materials, ΔL is approximately the same for the same magnitude of tension or compression. For larger deformations, the cross-

sectional area
changes as the
rod is
compressed or
stretched.

Experiments have shown that the change in length (ΔL) depends on only a few variables. As already noted, ΔL is proportional to the force F and depends on the substance from which the object is made. Additionally, the change in length is proportional to the original length L_0 and inversely proportional to the cross-sectional area of the wire or rod. For example, a long guitar string will stretch more than a short one, and a thick string will stretch less than a thin one. We can combine all these factors into one equation for ΔL :

Equation:

$$\Delta L = \frac{1}{Y} \frac{F}{A} L_0,$$

where ΔL is the change in length, F the applied force, Y is a factor, called the elastic modulus or Young's modulus, that depends on the substance, A is the cross-sectional area, and L_0 is the original length. [\[link\]](#) lists values of Y for several materials—those with a large Y are said to have a large **tensile strength** because they deform less for a given tension or compression.

Material	Young's modulus (tension–compression) Y (10^9 N/m^2)	Shear modulus S (10^9 N/m^2)	Bulk modulus B (10^9 N/m^2)
Aluminum	70	25	75
Bone – tension	16	80	8
Bone – compression	9		
Brass	90	35	75
Brick	15		
Concrete	20		
Glass	70	20	30
Granite	45	20	45
Hair (human)	10		
Hardwood	15	10	
Iron, cast	100	40	90
Lead	16	5	50
Marble	60	20	70
Nylon	5		

Polystyrene	3		
Silk	6		
Spider thread	3		
Steel	210	80	130
Tendon	1		
Acetone			0.7
Ethanol			0.9
Glycerin			4.5
Mercury			25
Water			2.2

Elastic Moduli[\[footnote\]](#)

Approximate and average values. Young's moduli Y for tension and compression sometimes differ but are averaged here. Bone has significantly different Young's moduli for tension and compression.

Young's moduli are not listed for liquids and gases in [\[link\]](#) because they cannot be stretched or compressed in only one direction. Note that there is an assumption that the object does not accelerate, so that there are actually two applied forces of magnitude F acting in opposite directions. For example, the strings in [\[link\]](#) are being pulled down by a force of magnitude w and held up by the ceiling, which also exerts a force of magnitude w .

Example: **The Stretch of a Long Cable**

Suspension cables are used to carry gondolas at ski resorts. (See [\[link\]](#)) Consider a suspension cable that includes an unsupported span of 3020 m. Calculate the amount of stretch in the steel cable. Assume that the cable has a diameter of 5.6 cm and the maximum tension it can withstand is $3.0 \times 10^6 \text{ N}$.



Gondolas travel along suspension cables at the Gala Yuzawa ski resort in Japan. (credit: Rudy Herman, Flickr)

Strategy

The force is equal to the maximum tension, or $F = 3.0 \times 10^6 \text{ N}$. The cross-sectional area is $\pi r^2 = 2.46 \times 10^{-3} \text{ m}^2$. The equation $\Delta L = \frac{1}{Y} \frac{F}{A} L_0$ can be used to find the change in length.

Solution

All quantities are known. Thus,

Equation:

$$\begin{aligned}\Delta L &= \left(\frac{1}{210 \times 10^9 \text{ N/m}^2} \right) \left(\frac{3.0 \times 10^6 \text{ N}}{2.46 \times 10^{-3} \text{ m}^2} \right) (3020 \text{ m}) \\ &= 18 \text{ m.}\end{aligned}$$

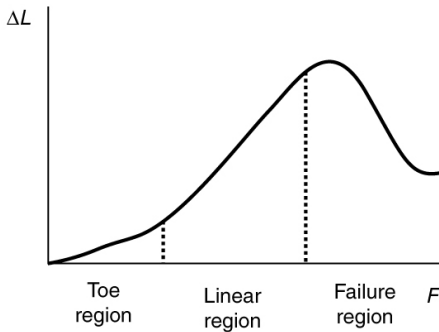
Discussion

This is quite a stretch, but only about 0.6% of the unsupported length. Effects of temperature upon length might be important in these

environments.

Bones, on the whole, do not fracture due to tension or compression. Rather they generally fracture due to sideways impact or bending, resulting in the bone shearing or snapping. The behavior of bones under tension and compression is important because it determines the load the bones can carry. Bones are classified as weight-bearing structures such as columns in buildings and trees. Weight-bearing structures have special features; columns in building have steel-reinforcing rods while trees and bones are fibrous. The bones in different parts of the body serve different structural functions and are prone to different stresses. Thus the bone in the top of the femur is arranged in thin sheets separated by marrow while in other places the bones can be cylindrical and filled with marrow or just solid. Overweight people have a tendency toward bone damage due to sustained compressions in bone joints and tendons.

Another biological example of Hooke's law occurs in tendons. Functionally, the tendon (the tissue connecting muscle to bone) must stretch easily at first when a force is applied, but offer a much greater restoring force for a greater strain. [\[link\]](#) shows a stress-strain relationship for a human tendon. Some tendons have a high collagen content so there is relatively little strain, or length change; others, like support tendons (as in the leg) can change length up to 10%. Note that this stress-strain curve is nonlinear, since the slope of the line changes in different regions. In the first part of the stretch called the toe region, the fibers in the tendon begin to align in the direction of the stress—this is called *uncrimping*. In the linear region, the fibrils will be stretched, and in the failure region individual fibers begin to break. A simple model of this relationship can be illustrated by springs in parallel: different springs are activated at different lengths of stretch. Examples of this are given in the problems at end of this chapter. Ligaments (tissue connecting bone to bone) behave in a similar way.



Typical stress-strain curve for mammalian tendon. Three regions are shown: (1) toe region (2) linear region, and (3) failure region.

Unlike bones and tendons, which need to be strong as well as elastic, the arteries and lungs need to be very stretchable. The elastic properties of the arteries are essential for blood flow. The pressure in the arteries increases and arterial walls stretch when the blood is pumped out of the heart. When the aortic valve shuts, the pressure in the arteries drops and the arterial walls relax to maintain the blood flow. When you feel your pulse, you are feeling exactly this—the elastic behavior of the arteries as the blood gushes through with each pump of the heart. If the arteries were rigid, you would not feel a pulse. The heart is also an organ with special elastic properties. The lungs expand with muscular effort when we breathe in but relax freely and elastically when we breathe out. Our skins are particularly elastic, especially for the young. A young person can go from 100 kg to 60 kg with no visible sag in their skins. The elasticity of all organs reduces with age. Gradual physiological aging through reduction in elasticity starts in the early 20s.

Example:

Calculating Deformation: How Much Does Your Leg Shorten When You Stand on It?

Calculate the change in length of the upper leg bone (the femur) when a 70.0 kg man supports 62.0 kg of his mass on it, assuming the bone to be equivalent to a uniform rod that is 40.0 cm long and 2.00 cm in radius.

Strategy

The force is equal to the weight supported, or

Equation:

$$F = mg = (62.0 \text{ kg})(9.80 \text{ m/s}^2) = 607.6 \text{ N},$$

and the cross-sectional area is $\pi r^2 = 1.257 \times 10^{-3} \text{ m}^2$. The equation $\Delta L = \frac{1}{Y} \frac{F}{A} L_0$ can be used to find the change in length.

Solution

All quantities except ΔL are known. Note that the compression value for Young's modulus for bone must be used here. Thus,

Equation:

$$\begin{aligned}\Delta L &= \left(\frac{1}{9 \times 10^9 \text{ N/m}^2} \right) \left(\frac{607.6 \text{ N}}{1.257 \times 10^{-3} \text{ m}^2} \right) (0.400 \text{ m}) \\ &= 2 \times 10^{-5} \text{ m}.\end{aligned}$$

Discussion

This small change in length seems reasonable, consistent with our experience that bones are rigid. In fact, even the rather large forces encountered during strenuous physical activity do not compress or bend bones by large amounts. Although bone is rigid compared with fat or muscle, several of the substances listed in [\[link\]](#) have larger values of Young's modulus Y . In other words, they are more rigid and have greater tensile strength.

The equation for change in length is traditionally rearranged and written in the following form:

Equation:

$$\frac{F}{A} = Y \frac{\Delta L}{L_0}.$$

The ratio of force to area, $\frac{F}{A}$, is defined as **stress** (measured in N/m^2), and the ratio of the change in length to length, $\frac{\Delta L}{L_0}$, is defined as **strain** (a unitless quantity). In other words,

Equation:

$$\text{stress} = Y \times \text{strain}.$$

In this form, the equation is analogous to Hooke's law, with stress analogous to force and strain analogous to deformation. If we again rearrange this equation to the form

Equation:

$$F = YA \frac{\Delta L}{L_0},$$

we see that it is the same as Hooke's law with a proportionality constant

Equation:

$$k = \frac{YA}{L_0}.$$

This general idea—that force and the deformation it causes are proportional for small deformations—applies to changes in length, sideways bending, and changes in volume.

Note:

Stress

The ratio of force to area, $\frac{F}{A}$, is defined as stress measured in N/m^2 .

Note:**Strain**

The ratio of the change in length to length, $\frac{\Delta L}{L_0}$, is defined as strain (a unitless quantity). In other words,

Equation:

$$\text{stress} = Y \times \text{strain}.$$

Sideways Stress: Shear Modulus

[\[link\]](#) illustrates what is meant by a sideways stress or a *shearing force*. Here the deformation is called Δx and it is perpendicular to L_0 , rather than parallel as with tension and compression. Shear deformation behaves similarly to tension and compression and can be described with similar equations. The expression for **shear deformation** is

Equation:

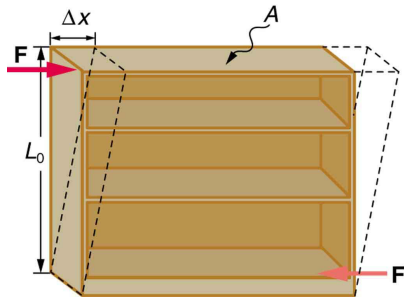
$$\Delta x = \frac{1}{S} \frac{F}{A} L_0,$$

where S is the shear modulus (see [\[link\]](#)) and F is the force applied perpendicular to L_0 and parallel to the cross-sectional area A . Again, to keep the object from accelerating, there are actually two equal and opposite forces F applied across opposite faces, as illustrated in [\[link\]](#). The equation is logical—for example, it is easier to bend a long thin pencil (small A) than a short thick one, and both are more easily bent than similar steel rods (large S).

Note:**Shear Deformation****Equation:**

$$\Delta x = \frac{1}{S} \frac{F}{A} L_0,$$

where S is the shear modulus and F is the force applied perpendicular to L_0 and parallel to the cross-sectional area A .



Shearing forces are applied perpendicular to the length L_0 and parallel to the area A , producing a deformation Δx . Vertical forces are not shown, but it should be kept in mind that in addition to the two shearing forces, \mathbf{F} , there must be supporting forces to keep the object from rotating. The distorting effects of these supporting forces are ignored in this treatment. The weight of the

object also is not shown, since it is usually negligible compared with forces large enough to cause significant deformations.

Examination of the shear moduli in [\[link\]](#) reveals some telling patterns. For example, shear moduli are less than Young's moduli for most materials. Bone is a remarkable exception. Its shear modulus is not only greater than its Young's modulus, but it is as large as that of steel. This is one reason that bones can be long and relatively thin. Bones can support loads comparable to that of concrete and steel. Most bone fractures are not caused by compression but by excessive twisting and bending.

The spinal column (consisting of 26 vertebral segments separated by discs) provides the main support for the head and upper part of the body. The spinal column has normal curvature for stability, but this curvature can be increased, leading to increased shearing forces on the lower vertebrae. Discs are better at withstanding compressional forces than shear forces. Because the spine is not vertical, the weight of the upper body exerts some of both. Pregnant women and people that are overweight (with large abdomens) need to move their shoulders back to maintain balance, thereby increasing the curvature in their spine and so increasing the shear component of the stress. An increased angle due to more curvature increases the shear forces along the plane. These higher shear forces increase the risk of back injury through ruptured discs. The lumbosacral disc (the wedge shaped disc below the last vertebrae) is particularly at risk because of its location.

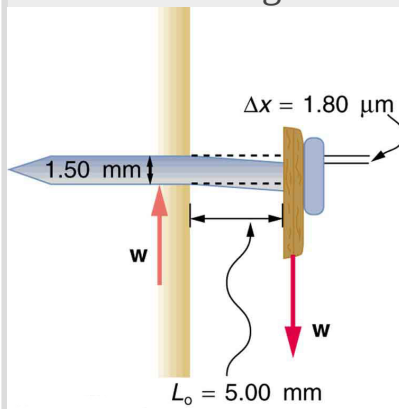
The shear moduli for concrete and brick are very small; they are too highly variable to be listed. Concrete used in buildings can withstand compression, as in pillars and arches, but is very poor against shear, as might be encountered in heavily loaded floors or during earthquakes. Modern structures were made possible by the use of steel and steel-reinforced

concrete. Almost by definition, liquids and gases have shear moduli near zero, because they flow in response to shearing forces.

Example:

Calculating Force Required to Deform: That Nail Does Not Bend Much Under a Load

Find the mass of the picture hanging from a steel nail as shown in [\[link\]](#), given that the nail bends only $1.80\ \mu\text{m}$. (Assume the shear modulus is known to two significant figures.)



Side view of a nail with a picture hung from it. The nail flexes very slightly (shown much larger than actual) because of the shearing effect of the supported weight. Also shown is the upward force of the wall on the nail, illustrating that there are equal and opposite forces applied across opposite cross

sections of the nail.

See [\[link\]](#) for a calculation of the mass of the picture.

Strategy

The force F on the nail (neglecting the nail's own weight) is the weight of the picture w . If we can find w , then the mass of the picture is just $\frac{w}{g}$. The equation $\Delta x = \frac{1}{S} \frac{F}{A} L_0$ can be solved for F .

Solution

Solving the equation $\Delta x = \frac{1}{S} \frac{F}{A} L_0$ for F , we see that all other quantities can be found:

Equation:

$$F = \frac{SA}{L_0} \Delta x.$$

S is found in [\[link\]](#) and is $S = 80 \times 10^9 \text{ N/m}^2$. The radius r is 0.750 mm (as seen in the figure), so the cross-sectional area is

Equation:

$$A = \pi r^2 = 1.77 \times 10^{-6} \text{ m}^2.$$

The value for L_0 is also shown in the figure. Thus,

Equation:

$$F = \frac{(80 \times 10^9 \text{ N/m}^2)(1.77 \times 10^{-6} \text{ m}^2)}{(5.00 \times 10^{-3} \text{ m})} (1.80 \times 10^{-6} \text{ m}) = 51 \text{ N}.$$

This 51 N force is the weight w of the picture, so the picture's mass is

Equation:

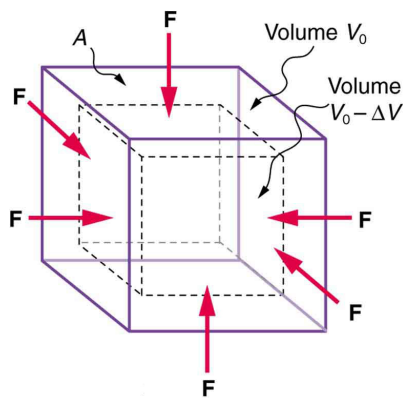
$$m = \frac{w}{g} = \frac{F}{g} = 5.2 \text{ kg}.$$

Discussion

This is a fairly massive picture, and it is impressive that the nail flexes only $1.80\text{ }\mu\text{m}$ —an amount undetectable to the unaided eye.

Changes in Volume: Bulk Modulus

An object will be compressed in all directions if inward forces are applied evenly on all its surfaces as in [\[link\]](#). It is relatively easy to compress gases and extremely difficult to compress liquids and solids. For example, air in a wine bottle is compressed when it is corked. But if you try corking a brim-full bottle, you cannot compress the wine—some must be removed if the cork is to be inserted. The reason for these different compressibilities is that atoms and molecules are separated by large empty spaces in gases but packed close together in liquids and solids. To compress a gas, you must force its atoms and molecules closer together. To compress liquids and solids, you must actually compress their atoms and molecules, and very strong electromagnetic forces in them oppose this compression.



An inward force on
all surfaces
compresses this
cube. Its change in
volume is
proportional to the
force per unit area

and its original volume, and is related to the compressibility of the substance.

We can describe the compression or volume deformation of an object with an equation. First, we note that a force “applied evenly” is defined to have the same stress, or ratio of force to area $\frac{F}{A}$ on all surfaces. The deformation produced is a change in volume ΔV , which is found to behave very similarly to the shear, tension, and compression previously discussed. (This is not surprising, since a compression of the entire object is equivalent to compressing each of its three dimensions.) The relationship of the change in volume to other physical quantities is given by

Equation:

$$\Delta V = -\frac{1}{B} \frac{F}{A} V_0,$$

where B is the bulk modulus (see [\[link\]](#)), V_0 is the original volume, and $\frac{F}{A}$ is the force per unit area applied uniformly inward on all surfaces. Note that no bulk moduli are given for gases.

What are some examples of bulk compression of solids and liquids? One practical example is the manufacture of industrial-grade diamonds by compressing carbon with an extremely large force per unit area. The carbon atoms rearrange their crystalline structure into the more tightly packed pattern of diamonds. In nature, a similar process occurs deep underground, where extremely large forces result from the weight of overlying material. Another natural source of large compressive forces is the pressure created by the weight of water, especially in deep parts of the oceans. Water exerts an inward force on all surfaces of a submerged object, and even on the water itself. At great depths, water is measurably compressed, as the following example illustrates.

Example:**Calculating Change in Volume with Deformation: How Much Is Water Compressed at Great Ocean Depths?**

Calculate the fractional decrease in volume ($\frac{\Delta V}{V_0}$) for seawater at 5.00 km depth, where the force per unit area is $5.00 \times 10^7 \text{ N/m}^2$.

Strategy

Equation $\Delta V = \frac{1}{B} \frac{F}{A} V_0$ is the correct physical relationship. All quantities in the equation except $\frac{\Delta V}{V_0}$ are known.

Solution

Solving for the unknown $\frac{\Delta V}{V_0}$ gives

Equation:

$$\frac{\Delta V}{V_0} = \frac{1}{B} \frac{F}{A}.$$

Substituting known values with the value for the bulk modulus B from [\[link\]](#),

Equation:

$$\begin{aligned} \frac{\Delta V}{V_0} &= \frac{5.00 \times 10^7 \text{ N/m}^2}{2.2 \times 10^9 \text{ N/m}^2} \\ &= 0.023 = 2.3\%. \end{aligned}$$

Discussion

Although measurable, this is not a significant decrease in volume considering that the force per unit area is about 500 atmospheres (1 million pounds per square foot). Liquids and solids are extraordinarily difficult to compress.

Conversely, very large forces are created by liquids and solids when they try to expand but are constrained from doing so—which is equivalent to compressing them to less than their normal volume. This often occurs when a contained material warms up, since most materials expand when their temperature increases. If the materials are tightly constrained, they deform

or break their container. Another very common example occurs when water freezes. Water, unlike most materials, expands when it freezes, and it can easily fracture a boulder, rupture a biological cell, or crack an engine block that gets in its way.

Other types of deformations, such as torsion or twisting, behave analogously to the tension, shear, and bulk deformations considered here.

Section Summary

- Hooke's law is given by
Equation:

$$F = k\Delta L,$$

where ΔL is the amount of deformation (the change in length), F is the applied force, and k is a proportionality constant that depends on the shape and composition of the object and the direction of the force. The relationship between the deformation and the applied force can also be written as

Equation:

$$\Delta L = \frac{1}{Y} \frac{F}{A} L_0,$$

where Y is *Young's modulus*, which depends on the substance, A is the cross-sectional area, and L_0 is the original length.

- The ratio of force to area, $\frac{F}{A}$, is defined as *stress*, measured in N/m².
- The ratio of the change in length to length, $\frac{\Delta L}{L_0}$, is defined as *strain* (a unitless quantity). In other words,

Equation:

$$\text{stress} = Y \times \text{strain}.$$

- The expression for shear deformation is
Equation:

$$\Delta x = \frac{1}{S} \frac{F}{A} L_0,$$

where S is the shear modulus and F is the force applied perpendicular to L_0 and parallel to the cross-sectional area A .

- The relationship of the change in volume to other physical quantities is given by

Equation:

$$\Delta V = \frac{1}{B} \frac{F}{A} V_0,$$

where B is the bulk modulus, V_0 is the original volume, and $\frac{F}{A}$ is the force per unit area applied uniformly inward on all surfaces.

Conceptual Questions

Exercise:

Problem:

The elastic properties of the arteries are essential for blood flow. Explain the importance of this in terms of the characteristics of the flow of blood (pulsating or continuous).

Exercise:

Problem:

What are you feeling when you feel your pulse? Measure your pulse rate for 10 s and for 1 min. Is there a factor of 6 difference?

Exercise:

Problem:

Examine different types of shoes, including sports shoes and thongs. In terms of physics, why are the bottom surfaces designed as they are? What differences will dry and wet conditions make for these surfaces?

Exercise:**Problem:**

Would you expect your height to be different depending upon the time of day? Why or why not?

Exercise:**Problem:**

Would you expect a large or small stress to be required to deform a spider web? Why is this elasticity an important feature for a spider web?

Exercise:**Problem:**

Explain why pregnant women often suffer from back strain late in their pregnancy.

Exercise:**Problem:**

An old carpenter's trick to keep nails from bending when they are pounded into hard materials is to grip the center of the nail firmly with pliers. Why does this help?

Exercise:**Problem:**

When a glass bottle full of vinegar warms up, both the vinegar and the glass expand, but vinegar expands significantly more with temperature than glass. The bottle will break if it was filled to its tightly capped lid. Explain why, and also explain how a pocket of air above the vinegar would prevent the break. (This is the function of the air above liquids in glass containers.)

Problems & Exercises

Exercise:**Problem:**

During a circus act, one performer swings upside down hanging from a trapeze holding another, also upside-down, performer by the legs. If the upward force on the lower performer is three times her weight, how much do the bones (the femurs) in her upper legs stretch? You may assume each is equivalent to a uniform rod 35.0 cm long and 1.80 cm in radius. Her mass is 60.0 kg.

Solution:**Equation:**

$$1.90 \times 10^{-3} \text{ cm}$$

Exercise:**Problem:**

During a wrestling match, a 150 kg wrestler briefly stands on one hand during a maneuver designed to perplex his already moribund adversary. By how much does the upper arm bone shorten in length? The bone can be represented by a uniform rod 38.0 cm in length and 2.10 cm in radius.

Exercise:**Problem:**

(a) The “lead” in pencils is a graphite composition with a Young's modulus of about $1 \times 10^9 \text{ N/m}^2$. Calculate the change in length of the lead in an automatic pencil if you tap it straight into the pencil with a force of 4.0 N. The lead is 0.50 mm in diameter and 60 mm long. (b) Is the answer reasonable? That is, does it seem to be consistent with what you have observed when using pencils?

Solution:

(a) 1 mm

(b) This does seem reasonable, since the lead does seem to shrink a little when you push on it.

Exercise:

Problem:

TV broadcast antennas are the tallest artificial structures on Earth. In 1987, a 72.0-kg physicist placed himself and 400 kg of equipment at the top of one 610-m high antenna to perform gravity experiments. By how much was the antenna compressed, if we consider it to be equivalent to a steel cylinder 0.150 m in radius?

Exercise:

Problem:

(a) By how much does a 65.0-kg mountain climber stretch her 0.800-cm diameter nylon rope when she hangs 35.0 m below a rock outcropping? (b) Does the answer seem to be consistent with what you have observed for nylon ropes? Would it make sense if the rope were actually a bungee cord?

Solution:

(a) 9 cm

(b) This seems reasonable for nylon climbing rope, since it is not supposed to stretch that much.

Exercise:

Problem:

A 20.0-m tall hollow aluminum flagpole is equivalent in strength to a solid cylinder 4.00 cm in diameter. A strong wind bends the pole much as a horizontal force of 900 N exerted at the top would. How far to the side does the top of the pole flex?

Exercise:

Problem:

As an oil well is drilled, each new section of drill pipe supports its own weight and that of the pipe and drill bit beneath it. Calculate the stretch in a new 6.00 m length of steel pipe that supports 3.00 km of pipe having a mass of 20.0 kg/m and a 100-kg drill bit. The pipe is equivalent in strength to a solid cylinder 5.00 cm in diameter.

Solution:

8.59 mm

Exercise:**Problem:**

Calculate the force a piano tuner applies to stretch a steel piano wire 8.00 mm, if the wire is originally 0.850 mm in diameter and 1.35 m long.

Exercise:**Problem:**

A vertebra is subjected to a shearing force of 500 N. Find the shear deformation, taking the vertebra to be a cylinder 3.00 cm high and 4.00 cm in diameter.

Solution:**Equation:**

$$1.49 \times 10^{-7} \text{ m}$$

Exercise:

Problem:

A disk between vertebrae in the spine is subjected to a shearing force of 600 N. Find its shear deformation, taking it to have the shear modulus of $1 \times 10^9 \text{ N/m}^2$. The disk is equivalent to a solid cylinder 0.700 cm high and 4.00 cm in diameter.

Exercise:**Problem:**

When using a pencil eraser, you exert a vertical force of 6.00 N at a distance of 2.00 cm from the hardwood-eraser joint. The pencil is 6.00 mm in diameter and is held at an angle of 20.0° to the horizontal. (a) By how much does the wood flex perpendicular to its length? (b) How much is it compressed lengthwise?

Solution:

(a) $3.99 \times 10^{-7} \text{ m}$

(b) $9.67 \times 10^{-8} \text{ m}$

Exercise:**Problem:**

To consider the effect of wires hung on poles, we take data from [\[link\]](#), in which tensions in wires supporting a traffic light were calculated. The left wire made an angle 30.0° below the horizontal with the top of its pole and carried a tension of 108 N. The 12.0 m tall hollow aluminum pole is equivalent in strength to a 4.50 cm diameter solid cylinder. (a) How far is it bent to the side? (b) By how much is it compressed?

Exercise:

Problem:

A farmer making grape juice fills a glass bottle to the brim and caps it tightly. The juice expands more than the glass when it warms up, in such a way that the volume increases by 0.2% (that is, $\Delta V/V_0 = 2 \times 10^{-3}$) relative to the space available. Calculate the magnitude of the normal force exerted by the juice per square centimeter if its bulk modulus is $1.8 \times 10^9 \text{ N/m}^2$, assuming the bottle does not break. In view of your answer, do you think the bottle will survive?

Solution:

$4 \times 10^6 \text{ N/m}^2$. This is about 36 atm, greater than a typical jar can withstand.

Exercise:**Problem:**

(a) When water freezes, its volume increases by 9.05% (that is, $\Delta V/V_0 = 9.05 \times 10^{-2}$). What force per unit area is water capable of exerting on a container when it freezes? (It is acceptable to use the bulk modulus of water in this problem.) (b) Is it surprising that such forces can fracture engine blocks, boulders, and the like?

Exercise:**Problem:**

This problem returns to the tightrope walker studied in [\[link\]](#), who created a tension of $3.94 \times 10^3 \text{ N}$ in a wire making an angle 5.0° below the horizontal with each supporting pole. Calculate how much this tension stretches the steel wire if it was originally 15 m long and 0.50 cm in diameter.

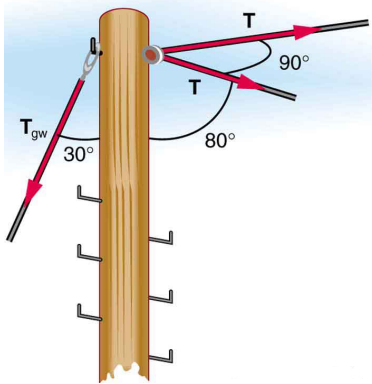
Solution:

1.4 cm

Exercise:

Problem:

The pole in [\[link\]](#) is at a 90.0° bend in a power line and is therefore subjected to more shear force than poles in straight parts of the line. The tension in each line is $4.00 \times 10^4 \text{ N}$, at the angles shown. The pole is 15.0 m tall, has an 18.0 cm diameter, and can be considered to have half the strength of hardwood. (a) Calculate the compression of the pole. (b) Find how much it bends and in what direction. (c) Find the tension in a guy wire used to keep the pole straight if it is attached to the top of the pole at an angle of 30.0° with the vertical. (Clearly, the guy wire must be in the opposite direction of the bend.)



This telephone pole is at a 90° bend in a power line. A guy wire is attached to the top of the pole at an angle of 30° with the vertical.

Glossary

deformation

change in shape due to the application of force

Hooke's law

proportional relationship between the force F on a material and the deformation ΔL it causes, $F = k\Delta L$

tensile strength

measure of deformation for a given tension or compression

stress

ratio of force to area

strain

ratio of change in length to original length

shear deformation

deformation perpendicular to the original length of an object

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class="introduction"

This
Australian
Grand Prix
Formula 1
race car
moves in a
circular
path as it
makes the
turn. Its
wheels also
spin rapidly
—the latter
completing
many
revolutions,
the former
only part of
one (a
circular
arc). The
same
physical
principles
are
involved in
each.
(credit:
Richard
Munckton)



Many motions, such as the arc of a bird's flight or Earth's path around the Sun, are curved. Recall that Newton's first law tells us that motion is along a straight line at constant speed unless there is a net external force. We will therefore study not only motion along curves, but also the forces that cause it, including gravitational forces. This chapter supports Big Idea 3 that interactions between objects are described by forces, and thus change in motion is a result of a net force exerted on an object. In this chapter, this idea is applied to uniform circular motion. In some ways, this chapter is a continuation of [Dynamics: Newton's Laws of Motion](#) as we study more applications of Newton's laws of motion.

This chapter deals with the simplest form of curved motion, **uniform circular motion**, which is motion in a circular path at constant speed. As an object moves on a circular path, the magnitude of its velocity remains constant, but the direction of the velocity is changing. This means there is an acceleration that we will refer to as a “centripetal” acceleration caused by a net external force, also called the “centripetal” force (Enduring

Understanding 3.B). The centripetal force is the net force totaling all external forces acting on the object (Essential Knowledge 3.B.1). In order to determine the net force, a free-body diagram may be useful (Essential Knowledge 3.B.2).

Studying this topic illustrates most of the concepts associated with rotational motion and leads to many new topics we group under the name *rotation*. This motion can be described using kinematics variables (Essential Knowledge 3.A.1), but in addition to linear variables, we will introduce angular variables. We use various ways to describe motion, namely, verbally, algebraically and graphically (Learning Objective 3.A.1.1). Pure *rotational motion* occurs when points in an object move in circular paths centered on one point. Pure *translational motion* is motion with no rotation. Some motion combines both types, such as a rotating hockey puck moving over ice. Some combinations of both types of motion are conveniently described with fictitious forces which appear as a result of using a non-inertial frame of reference (Enduring Understanding 3.A).

Furthermore, the properties of uniform circular motion can be applied to the motion of massive objects in a gravitational field. Thus, this chapter supports Big Idea 1 that gravitational mass is an important property of an object or a system.

We have experimental evidence that gravitational and inertial masses are equal (Enduring Understanding 1.C), and that gravitational mass is a measure of the strength of the gravitational interaction (Essential Knowledge 1.C.2). Therefore, this chapter will support Big Idea 2 that fields existing in space can be used to explain interactions, because any massive object creates a gravitational field in space (Enduring Understanding 2.B). Mathematically, we use Newton's universal law of gravitation to provide a model for the gravitational interaction between two massive objects (Essential Knowledge 2.B.2). We will discover that this model describes the interaction of one object with mass with another object with mass (Essential Knowledge 3.C.1), and also that gravitational force is a long-range force (Enduring Understanding 3.C).

The concepts in this chapter support:

Big Idea 1 Objects and systems have properties such as mass and charge. Systems may have internal structure.

Enduring Understanding 1.C Objects and systems have properties of inertial mass and gravitational mass that are experimentally verified to be the same and that satisfy conservation principles.

Essential Knowledge 1.C.2 Gravitational mass is the property of an object or a system that determines the strength of the gravitational interaction with other objects, systems, or gravitational fields.

Essential Knowledge 1.C.3 Objects and systems have properties of inertial mass and gravitational mass that are experimentally verified to be the same and that satisfy conservation principles.

Big Idea 2 Fields existing in space can be used to explain interactions.

Enduring Understanding 2.B A gravitational field is caused by an object with mass.

Essential Knowledge 2.B.2. The gravitational field caused by a spherically symmetric object with mass is radial and, outside the object, varies as the inverse square of the radial distance from the center of that object.

Big Idea 3 The interactions of an object with other objects can be described by forces.

Enduring Understanding 3.A All forces share certain common characteristics when considered by observers in inertial reference frames.

Essential Knowledge 3.A.1. An observer in a particular reference frame can describe the motion of an object using such quantities as position, displacement, distance, velocity, speed, and acceleration.

Essential Knowledge 3.A.3. A force exerted on an object is always due to the interaction of that object with another object.

Enduring Understanding 3.B Classically, the acceleration of an object interacting with other objects can be predicted by using $a = \sum F/m$.

Essential Knowledge 3.B.1 If an object of interest interacts with several other objects, the net force is the vector sum of the individual forces.

Essential Knowledge 3.B.2 Free-body diagrams are useful tools for visualizing forces being exerted on a single object and writing the equations that represent a physical situation.

Enduring Understanding 3.C At the macroscopic level, forces can be categorized as either long-range (action-at-a-distance) forces or contact forces.

Essential Knowledge 3.C.1. Gravitational force describes the interaction of one object that has mass with another object that has mass.

Glossary

uniform circular motion

the motion of an object in a circular path at constant speed

Rotation Angle and Angular Velocity

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define arc length, rotation angle, radius of curvature, and angular velocity.
- Calculate the angular velocity of a car wheel spin.

In [Kinematics](#), we studied motion along a straight line and introduced such concepts as displacement, velocity, and acceleration. [Two-Dimensional Kinematics](#) dealt with motion in two dimensions. Projectile motion is a special case of two-dimensional kinematics in which the object is projected into the air, while being subject to the gravitational force, and lands a distance away. In this chapter, we consider situations where the object does not land but moves in a curve. We begin the study of uniform circular motion by defining two angular quantities needed to describe rotational motion.

Rotation Angle

When objects rotate about some axis—for example, when the CD (compact disc) in [link](#) rotates about its center—each point in the object follows a circular arc. Consider a line from the center of the CD to its edge. Each **pit** used to record sound along this line moves through the same angle in the same amount of time. The rotation angle is the amount of rotation and is analogous to linear distance. We define the **rotation angle** $\Delta\theta$ to be the ratio of the arc length to the radius of curvature:

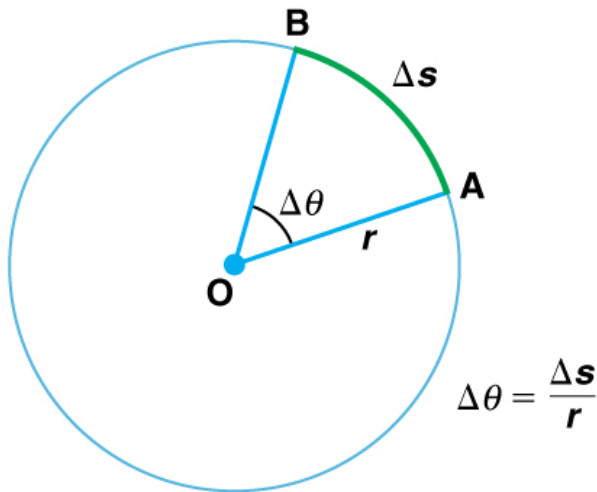
Equation:

$$\Delta\theta = \frac{\Delta s}{r}.$$



All points on a CD
travel in circular arcs.

The pits along a line
from the center to the
edge all move through
the same angle $\Delta\theta$ in
a time Δt .



The radius of a circle is rotated through an angle $\Delta\theta$. The arc length Δs is described on the circumference.

The **arc length** Δs is the distance traveled along a circular path as shown in [\[link\]](#) Note that r is the **radius of curvature** of the circular path.

We know that for one complete revolution, the arc length is the circumference of a circle of radius r . The circumference of a circle is $2\pi r$. Thus for one complete revolution the rotation angle is

Equation:

$$\Delta\theta = \frac{2\pi r}{r} = 2\pi.$$

This result is the basis for defining the units used to measure rotation angles, $\Delta\theta$ to be **radians** (rad), defined so that

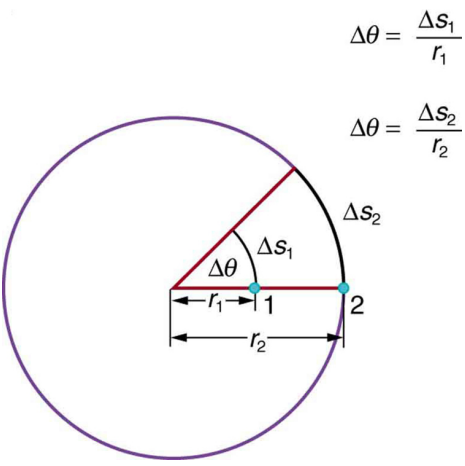
Equation:

$$2\pi \text{ rad} = 1 \text{ revolution.}$$

A comparison of some useful angles expressed in both degrees and radians is shown in [\[link\]](#).

Degree Measures	Radian Measure
30°	$\frac{\pi}{6}$
60°	$\frac{\pi}{3}$
90°	$\frac{\pi}{2}$
120°	$\frac{2\pi}{3}$
135°	$\frac{3\pi}{4}$
180°	π

Comparison of Angular Units



Points 1 and 2 rotate through the same angle ($\Delta\theta$), but point 2 moves through a greater arc length (Δs) because it is at a greater distance from the center of rotation (r).

If $\Delta\theta = 2\pi$ rad, then the CD has made one complete revolution, and every point on the CD is back at its original position. Because there are 360° in a circle or one revolution, the relationship between radians and degrees is thus

Equation:

$$2\pi \text{ rad} = 360^\circ$$

so that

Equation:

$$1 \text{ rad} = \frac{360^\circ}{2\pi} \approx 57.3^\circ.$$

Angular Velocity

How fast is an object rotating? We define **angular velocity** ω as the rate of change of an angle. In symbols, this is

Equation:

$$\omega = \frac{\Delta\theta}{\Delta t},$$

where an angular rotation $\Delta\theta$ takes place in a time Δt . The greater the rotation angle in a given amount of time, the greater the angular velocity. The units for angular velocity are radians per second (rad/s).

Angular velocity ω is analogous to linear velocity v . To get the precise relationship between angular and linear velocity, we again consider a pit on the rotating CD. This pit moves an arc length Δs in a time Δt , and so it has a linear velocity

Equation:

$$v = \frac{\Delta s}{\Delta t}.$$

From $\Delta\theta = \frac{\Delta s}{r}$ we see that $\Delta s = r\Delta\theta$. Substituting this into the expression for v gives

Equation:

$$v = \frac{r\Delta\theta}{\Delta t} = r\omega.$$

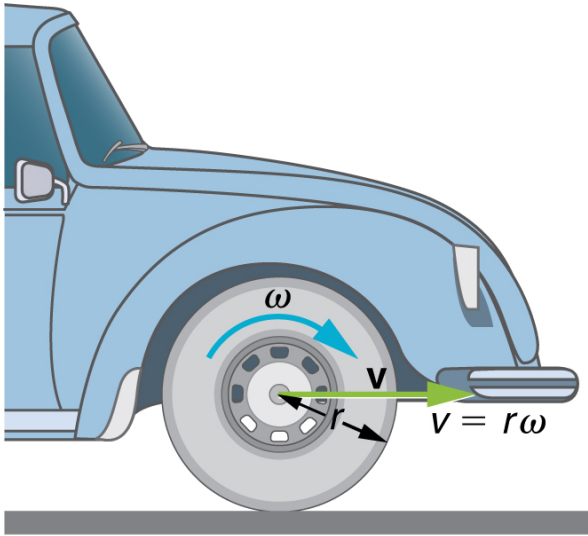
We write this relationship in two different ways and gain two different insights:

Equation:

$$v = r\omega \text{ or } \omega = \frac{v}{r}.$$

The first relationship in $v = r\omega$ or $\omega = \frac{v}{r}$ states that the linear velocity v is proportional to the distance from the center of rotation, thus, it is largest for a point on the rim (largest r), as you might expect. We can also call this linear speed v of a point on the rim the *tangential speed*. The second relationship in $v = r\omega$ or $\omega = \frac{v}{r}$ can be illustrated by considering the tire of a moving car. Note that the speed of a point on the rim of the tire is the same as the speed v of the car. See [\[link\]](#). So the faster the car moves, the faster the tire spins—large v means a large ω , because $v = r\omega$. Similarly, a

larger-radius tire rotating at the same angular velocity (ω) will produce a greater linear speed (v) for the car.



A car moving at a velocity v to the right has a tire rotating with an angular velocity ω . The speed of the tread of the tire relative to the axle is v , the same as if the car were jacked up. Thus the car moves forward at linear velocity $v = r\omega$, where r is the tire radius. A larger angular velocity for the tire means a greater velocity for the car.

Example:
How Fast Does a Car Tire Spin?

Calculate the angular velocity of a 0.300 m radius car tire when the car travels at 15.0 m/s (about 54 km/h). See [\[link\]](#).

Strategy

Because the linear speed of the tire rim is the same as the speed of the car, we have $v = 15.0 \text{ m/s}$. The radius of the tire is given to be $r = 0.300 \text{ m}$. Knowing v and r , we can use the second relationship in $v = r\omega$, $\omega = \frac{v}{r}$ to calculate the angular velocity.

Solution

To calculate the angular velocity, we will use the following relationship:

Equation:

$$\omega = \frac{v}{r}.$$

Substituting the knowns,

Equation:

$$\omega = \frac{15.0 \text{ m/s}}{0.300 \text{ m}} = 50.0 \text{ rad/s}.$$

Discussion

When we cancel units in the above calculation, we get 50.0/s. But the angular velocity must have units of rad/s. Because radians are actually unitless (radians are defined as a ratio of distance), we can simply insert them into the answer for the angular velocity. Also note that if an earth mover with much larger tires, say 1.20 m in radius, were moving at the same speed of 15.0 m/s, its tires would rotate more slowly. They would have an angular velocity

Equation:

$$\omega = (15.0 \text{ m/s})/(1.20 \text{ m}) = 12.5 \text{ rad/s}.$$

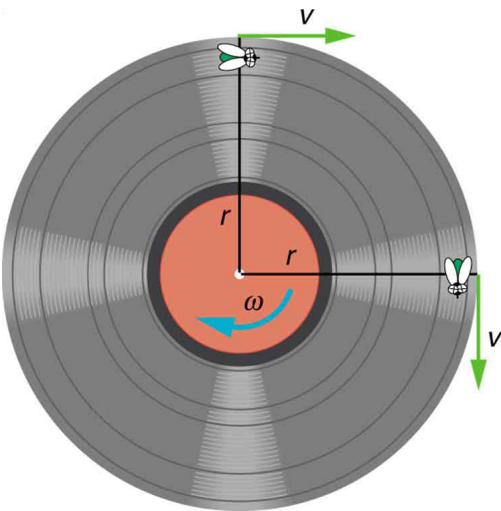
Both ω and v have directions (hence they are angular and linear *velocities*, respectively). Angular velocity has only two directions with respect to the

axis of rotation—it is either clockwise or counterclockwise. Linear velocity is tangent to the path, as illustrated in [\[link\]](#).

Note:

Take-Home Experiment

Tie an object to the end of a string and swing it around in a horizontal circle above your head (swing at your wrist). Maintain uniform speed as the object swings and measure the angular velocity of the motion. What is the approximate speed of the object? Identify a point close to your hand and take appropriate measurements to calculate the linear speed at this point. Identify other circular motions and measure their angular velocities.



As an object moves in a circle, here a fly on the edge of an old-fashioned vinyl record, its instantaneous velocity is always tangent to the circle. The direction of the angular velocity is clockwise in this case.

Note:

PhET Explorations: Ladybug Revolution

[Ladybug
Revolution](#)

Join the ladybug in an exploration of rotational motion. Rotate the merry-go-round to change its angle, or choose a constant angular velocity or angular acceleration. Explore how circular motion relates to the bug's x,y position, velocity, and acceleration using vectors or graphs.

Section Summary

- Uniform circular motion is motion in a circle at constant speed. The rotation angle $\Delta\theta$ is defined as the ratio of the arc length to the radius of curvature:

Equation:

$$\Delta\theta = \frac{\Delta s}{r},$$

where arc length Δs is distance traveled along a circular path and r is the radius of curvature of the circular path. The quantity $\Delta\theta$ is measured in units of radians (rad), for which

Equation:

$$2\pi \text{ rad} = 360^\circ = 1 \text{ revolution.}$$

- The conversion between radians and degrees is $1 \text{ rad} = 57.3^\circ$.
- Angular velocity ω is the rate of change of an angle,

Equation:

$$\omega = \frac{\Delta\theta}{\Delta t},$$

where a rotation $\Delta\theta$ takes place in a time Δt . The units of angular velocity are radians per second (rad/s). Linear velocity v and angular velocity ω are related by

Equation:

$$v = r\omega \text{ or } \omega = \frac{v}{r}.$$

Conceptual Questions

Exercise:

Problem:

There is an analogy between rotational and linear physical quantities. What rotational quantities are analogous to distance and velocity?

Problem Exercises

Exercise:

Problem:

Semi-trailer trucks have an odometer on one hub of a trailer wheel. The hub is weighted so that it does not rotate, but it contains gears to count the number of wheel revolutions—it then calculates the distance traveled. If the wheel has a 1.15 m diameter and goes through 200,000 rotations, how many kilometers should the odometer read?

Solution:

723 km

Exercise:**Problem:**

Microwave ovens rotate at a rate of about 6 rev/min. What is this in revolutions per second? What is the angular velocity in radians per second?

Exercise:**Problem:**

An automobile with 0.260 m radius tires travels 80,000 km before wearing them out. How many revolutions do the tires make, neglecting any backing up and any change in radius due to wear?

Solution:

5×10^7 rotations

Exercise:**Problem:**

(a) What is the period of rotation of Earth in seconds? (b) What is the angular velocity of Earth? (c) Given that Earth has a radius of 6.4×10^6 m at its equator, what is the linear velocity at Earth's surface?

Exercise:**Problem:**

A baseball pitcher brings his arm forward during a pitch, rotating the forearm about the elbow. If the velocity of the ball in the pitcher's hand is 35.0 m/s and the ball is 0.300 m from the elbow joint, what is the angular velocity of the forearm?

Solution:

117 rad/s

Exercise:**Problem:**

In lacrosse, a ball is thrown from a net on the end of a stick by rotating the stick and forearm about the elbow. If the angular velocity of the ball about the elbow joint is 30.0 rad/s and the ball is 1.30 m from the elbow joint, what is the velocity of the ball?

Exercise:**Problem:**

A truck with 0.420-m-radius tires travels at 32.0 m/s. What is the angular velocity of the rotating tires in radians per second? What is this in rev/min?

Solution:

76.2 rad/s

728 rpm

Exercise:**Problem:**

Integrated Concepts When kicking a football, the kicker rotates his leg about the hip joint.

(a) If the velocity of the tip of the kicker's shoe is 35.0 m/s and the hip joint is 1.05 m from the tip of the shoe, what is the shoe tip's angular velocity?

(b) The shoe is in contact with the initially stationary 0.500 kg football for 20.0 ms. What average force is exerted on the football to give it a velocity of 20.0 m/s?

(c) Find the maximum range of the football, neglecting air resistance.

Solution:

(a) 33.3 rad/s

(b) 500 N

(c) 40.8 m

Exercise:

Problem:Construct Your Own Problem

Consider an amusement park ride in which participants are rotated about a vertical axis in a cylinder with vertical walls. Once the angular velocity reaches its full value, the floor drops away and friction between the walls and the riders prevents them from sliding down. Construct a problem in which you calculate the necessary angular velocity that assures the riders will not slide down the wall. Include a free body diagram of a single rider. Among the variables to consider are the radius of the cylinder and the coefficients of friction between the riders' clothing and the wall.

Glossary

arc length

Δs , the distance traveled by an object along a circular path

pit

a tiny indentation on the spiral track moulded into the top of the polycarbonate layer of CD

rotation angle

the ratio of the arc length to the radius of curvature on a circular path:

$$\Delta\theta = \frac{\Delta s}{r}$$

radius of curvature
radius of a circular path

radians
a unit of angle measurement

angular velocity
 ω , the rate of change of the angle with which an object moves on a circular path

Centripetal Acceleration

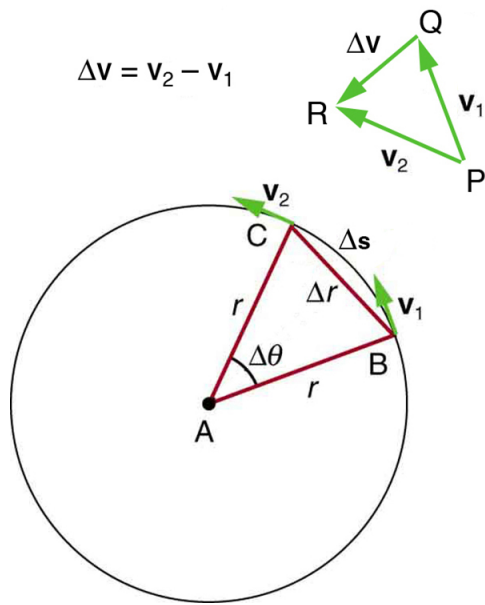
LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Establish the expression for centripetal acceleration.
- Explain the centrifuge.

We know from kinematics that acceleration is a change in velocity, either in its magnitude or in its direction, or both. In uniform circular motion, the direction of the velocity changes constantly, so there is always an associated acceleration, even though the magnitude of the velocity might be constant. You experience this acceleration yourself when you turn a corner in your car. (If you hold the wheel steady during a turn and move at constant speed, you are in uniform circular motion.) What you notice is a sideways acceleration because you and the car are changing direction. The sharper the curve and the greater your speed, the more noticeable this acceleration will become. In this section we examine the direction and magnitude of that acceleration.

[\[link\]](#) shows an object moving in a circular path at constant speed. The direction of the instantaneous velocity is shown at two points along the path. Acceleration is in the direction of the change in velocity, which points directly toward the center of rotation (the center of the circular path). This pointing is shown with the vector diagram in the figure. We call the acceleration of an object moving in uniform circular motion (resulting from a net external force) the **centripetal acceleration**(a_c); centripetal means “toward the center” or “center seeking.”



The directions of the velocity of an object at two different points are shown, and the change in velocity $\Delta\mathbf{v}$ is seen to point directly toward the center of curvature. (See small inset.) Because $\mathbf{a}_c = \Delta\mathbf{v}/\Delta t$, the acceleration is also toward the center; \mathbf{a}_c is called centripetal acceleration. (Because $\Delta\theta$ is very small, the arc length Δs is equal to the chord length Δr for small time differences.)

The direction of centripetal acceleration is toward the center of curvature, but what is its magnitude? Note that the triangle formed by the velocity vectors and the one formed by the radii r and Δs are similar. Both the

triangles ABC and PQR are isosceles triangles (two equal sides). The two equal sides of the velocity vector triangle are the speeds $v_1 = v_2 = v$. Using the properties of two similar triangles, we obtain

Equation:

$$\frac{\Delta v}{v} = \frac{\Delta s}{r}.$$

Acceleration is $\Delta v / \Delta t$, and so we first solve this expression for Δv :

Equation:

$$\Delta v = \frac{v}{r} \Delta s.$$

Then we divide this by Δt , yielding

Equation:

$$\frac{\Delta v}{\Delta t} = \frac{v}{r} \times \frac{\Delta s}{\Delta t}.$$

Finally, noting that $\Delta v / \Delta t = a_c$ and that $\Delta s / \Delta t = v$, the linear or tangential speed, we see that the magnitude of the centripetal acceleration is

Equation:

$$a_c = \frac{v^2}{r},$$

which is the acceleration of an object in a circle of radius r at a speed v . So, centripetal acceleration is greater at high speeds and in sharp curves (smaller radius), as you have noticed when driving a car. But it is a bit surprising that a_c is proportional to speed squared, implying, for example, that it is four times as hard to take a curve at 100 km/h than at 50 km/h. A sharp corner has a small radius, so that a_c is greater for tighter turns, as you have probably noticed.

It is also useful to express a_c in terms of angular velocity. Substituting $v = r\omega$ into the above expression, we find $a_c = (r\omega)^2/r = r\omega^2$. We can express the magnitude of centripetal acceleration using either of two equations:

Equation:

$$a_c = \frac{v^2}{r}; \quad a_c = r\omega^2.$$

Recall that the direction of a_c is toward the center. You may use whichever expression is more convenient, as illustrated in examples below.

A **centrifuge** (see [\[link\]](#)b) is a rotating device used to separate specimens of different densities. High centripetal acceleration significantly decreases the time it takes for separation to occur, and makes separation possible with small samples. Centrifuges are used in a variety of applications in science and medicine, including the separation of single cell suspensions such as bacteria, viruses, and blood cells from a liquid medium and the separation of macromolecules, such as DNA and protein, from a solution. Centrifuges are often rated in terms of their centripetal acceleration relative to acceleration due to gravity (g); maximum centripetal acceleration of several hundred thousand g is possible in a vacuum. Human centrifuges, extremely large centrifuges, have been used to test the tolerance of astronauts to the effects of accelerations larger than that of Earth's gravity.

Example:

How Does the Centripetal Acceleration of a Car Around a Curve Compare with That Due to Gravity?

What is the magnitude of the centripetal acceleration of a car following a curve of radius 500 m at a speed of 25.0 m/s (about 90 km/h)? Compare the acceleration with that due to gravity for this fairly gentle curve taken at highway speed. See [\[link\]](#)(a).

Strategy

Because v and r are given, the first expression in $a_c = \frac{v^2}{r}$; $a_c = r\omega^2$ is the most convenient to use.

Solution

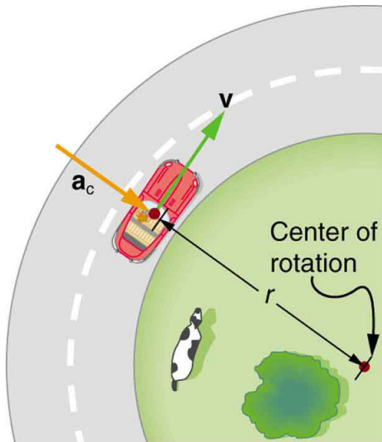
Entering the given values of $v = 25.0 \text{ m/s}$ and $r = 500 \text{ m}$ into the first expression for a_c gives

Equation:

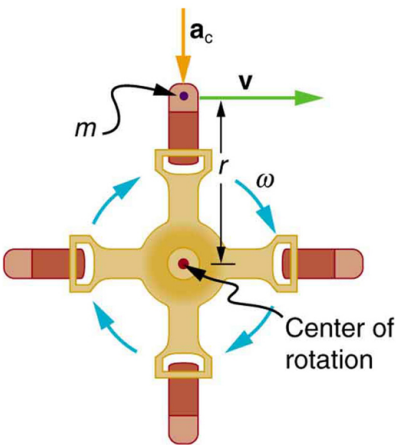
$$a_c = \frac{v^2}{r} = \frac{(25.0 \text{ m/s})^2}{500 \text{ m}} = 1.25 \text{ m/s}^2.$$

Discussion

To compare this with the acceleration due to gravity ($g = 9.80 \text{ m/s}^2$), we take the ratio of $a_c/g = (1.25 \text{ m/s}^2)/(9.80 \text{ m/s}^2) = 0.128$. Thus, $a_c = 0.128 g$ and is noticeable especially if you were not wearing a seat belt.



(a) Car around corner



(b) Centrifuge

(a) The car following a circular path at constant speed is accelerated perpendicular to its velocity, as shown. The magnitude of this centripetal acceleration is found in [\[link\]](#). (b) A particle of mass in a centrifuge is rotating at constant angular velocity . It

must be accelerated perpendicular to its velocity or it would continue in a straight line. The magnitude of the necessary acceleration is found in [\[link\]](#).

Example:

How Big Is the Centripetal Acceleration in an Ultracentrifuge?

Calculate the centripetal acceleration of a point 7.50 cm from the axis of an **ultracentrifuge** spinning at 7.5×10^4 rev/min. Determine the ratio of this acceleration to that due to gravity. See [\[link\]](#)(b).

Strategy

The term rev/min stands for revolutions per minute. By converting this to radians per second, we obtain the angular velocity ω . Because r is given, we can use the second expression in the equation $a_c = \frac{v^2}{r}$; $a_c = r\omega^2$ to calculate the centripetal acceleration.

Solution

To convert 7.50×10^4 rev/min to radians per second, we use the facts that one revolution is 2π rad and one minute is 60.0 s. Thus,

Equation:

$$\omega = 7.50 \times 10^4 \frac{\text{rev}}{\text{min}} \times \frac{2\pi \text{ rad}}{1 \text{ rev}} \times \frac{1 \text{ min}}{60.0 \text{ s}} = 7854 \text{ rad/s}.$$

Now the centripetal acceleration is given by the second expression in

$$a_c = \frac{v^2}{r}; a_c = r\omega^2 \text{ as}$$

Equation:

$$a_c = r\omega^2.$$

Converting 7.50 cm to meters and substituting known values gives

Equation:

$$a_c = (0.0750 \text{ m})(7854 \text{ rad/s})^2 = 4.63 \times 10^6 \text{ m/s}^2.$$

Note that the unitless radians are discarded in order to get the correct units for centripetal acceleration. Taking the ratio of a_c to g yields

Equation:

$$\frac{a_c}{g} = \frac{4.63 \times 10^6}{9.80} = 4.72 \times 10^5.$$

Discussion

This last result means that the centripetal acceleration is 472,000 times as strong as g . It is no wonder that such high ω centrifuges are called ultracentrifuges. The extremely large accelerations involved greatly decrease the time needed to cause the sedimentation of blood cells or other materials.

Of course, a net external force is needed to cause any acceleration, just as Newton proposed in his second law of motion. So a net external force is needed to cause a centripetal acceleration. In [Centripetal Force](#), we will consider the forces involved in circular motion.

Note:

PhET Explorations: Ladybug Motion 2D

Learn about position, velocity and acceleration vectors. Move the ladybug by setting the position, velocity or acceleration, and see how the vectors change. Choose linear, circular or elliptical motion, and record and playback the motion to analyze the behavior.

[Ladybu](#)

Section Summary

- Centripetal acceleration a_c is the acceleration experienced while in uniform circular motion. It always points toward the center of rotation. It is perpendicular to the linear velocity v and has the magnitude
Equation:

$$a_c = \frac{v^2}{r}; a_c = r\omega^2.$$

- The unit of centripetal acceleration is m/s^2 .

Conceptual Questions

Exercise:

Problem:

Can centripetal acceleration change the speed of circular motion? Explain.

Problem Exercises

Exercise:

Problem:

A fairground ride spins its occupants inside a flying saucer-shaped container. If the horizontal circular path the riders follow has an 8.00 m radius, at how many revolutions per minute will the riders be subjected to a centripetal acceleration whose magnitude is 1.50 times that due to gravity?

Solution:

12.9 rev/min

Exercise:**Problem:**

A runner taking part in the 200 m dash must run around the end of a track that has a circular arc with a radius of curvature of 30 m. If he completes the 200 m dash in 23.2 s and runs at constant speed throughout the race, what is the magnitude of his centripetal acceleration as he runs the curved portion of the track?

Exercise:**Problem:**

Taking the age of Earth to be about 4×10^9 years and assuming its orbital radius of 1.5×10^{11} m has not changed and is circular, calculate the approximate total distance Earth has traveled since its birth (in a frame of reference stationary with respect to the Sun).

Solution:

4×10^{21} m

Exercise:**Problem:**

The propeller of a World War II fighter plane is 2.30 m in diameter.

- (a) What is its angular velocity in radians per second if it spins at 1200 rev/min?
- (b) What is the linear speed of its tip at this angular velocity if the plane is stationary on the tarmac?
- (c) What is the centripetal acceleration of the propeller tip under these conditions? Calculate it in meters per second squared and convert to multiples of g .

Exercise:

Problem:

An ordinary workshop grindstone has a radius of 7.50 cm and rotates at 6500 rev/min.

- (a) Calculate the magnitude of the centripetal acceleration at its edge in meters per second squared and convert it to multiples of g .
- (b) What is the linear speed of a point on its edge?

Solution:

- a) $3.47 \times 10^4 \text{ m/s}^2$, $3.55 \times 10^3 g$
- b) 51.1 m/s

Exercise:

Problem:

Helicopter blades withstand tremendous stresses. In addition to supporting the weight of a helicopter, they are spun at rapid rates and experience large centripetal accelerations, especially at the tip.

- (a) Calculate the magnitude of the centripetal acceleration at the tip of a 4.00 m long helicopter blade that rotates at 300 rev/min.

(b) Compare the linear speed of the tip with the speed of sound (taken to be 340 m/s).

Exercise:

Problem: Olympic ice skaters are able to spin at about 5 rev/s.

- (a) What is their angular velocity in radians per second?
- (b) What is the centripetal acceleration of the skater's nose if it is 0.120 m from the axis of rotation?
- (c) An exceptional skater named Dick Button was able to spin much faster in the 1950s than anyone since—at about 9 rev/s. What was the centripetal acceleration of the tip of his nose, assuming it is at 0.120 m radius?
- (d) Comment on the magnitudes of the accelerations found. It is reputed that Button ruptured small blood vessels during his spins.

Solution:

a) 31.4 rad/s

b) 118 m/s

c) 384 m/s

d) The centripetal acceleration felt by Olympic skaters is 12 times larger than the acceleration due to gravity. That's quite a lot of acceleration in itself. The centripetal acceleration felt by Button's nose was 39.2 times larger than the acceleration due to gravity. It is no wonder that he ruptured small blood vessels in his spins.

Exercise:

Problem:

What percentage of the acceleration at Earth's surface is the acceleration due to gravity at the position of a satellite located 300 km above Earth?

Exercise:**Problem:**

Verify that the linear speed of an ultracentrifuge is about 0.50 km/s, and Earth in its orbit is about 30 km/s by calculating:

(a) The linear speed of a point on an ultracentrifuge 0.100 m from its center, rotating at 50,000 rev/min.

(b) The linear speed of Earth in its orbit about the Sun (use data from the text on the radius of Earth's orbit and approximate it as being circular).

Solution:

a) 0.524 km/s

b) 29.7 km/s

Exercise:**Problem:**

A rotating space station is said to create “artificial gravity”—a loosely-defined term used for an acceleration that would be crudely similar to gravity. The outer wall of the rotating space station would become a floor for the astronauts, and centripetal acceleration supplied by the floor would allow astronauts to exercise and maintain muscle and bone strength more naturally than in non-rotating space environments. If the space station is 200 m in diameter, what angular velocity would produce an “artificial gravity” of 9.80 m/s^2 at the rim?

Exercise:

Problem:

At takeoff, a commercial jet has a 60.0 m/s speed. Its tires have a diameter of 0.850 m.

- (a) At how many rev/min are the tires rotating?
 - (b) What is the centripetal acceleration at the edge of the tire?
 - (c) With what force must a determined 1.00×10^{-15} kg bacterium cling to the rim?
 - (d) Take the ratio of this force to the bacterium's weight.
-

Solution:

- (a) 1.35×10^3 rpm
- (b) 8.47×10^3 m/s²
- (c) 8.47×10^{-12} N
- (d) 865

Exercise:**Problem: Integrated Concepts**

Riders in an amusement park ride shaped like a Viking ship hung from a large pivot are rotated back and forth like a rigid pendulum. Sometime near the middle of the ride, the ship is momentarily motionless at the top of its circular arc. The ship then swings down under the influence of gravity.

- (a) Assuming negligible friction, find the speed of the riders at the bottom of its arc, given the system's center of mass travels in an arc having a radius of 14.0 m and the riders are near the center of mass.

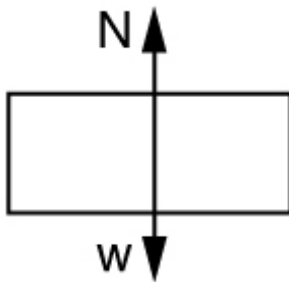
- (b) What is the centripetal acceleration at the bottom of the arc?
- (c) Draw a free body diagram of the forces acting on a rider at the bottom of the arc.
- (d) Find the force exerted by the ride on a 60.0 kg rider and compare it to her weight.
- (e) Discuss whether the answer seems reasonable.
-

Solution:

(a) 23.4 m/s

(b) 39.2 m/s²

(c)



(d) 2.9399×10^3 N or $5.00 w$, that is, the normal force (upward) is five times her weight.

(e) This answer seems reasonable, since she feels like she's being forced into the chair MUCH stronger than just by gravity.

Exercise:

Problem: Unreasonable Results

A mother pushes her child on a swing so that his speed is 9.00 m/s at the lowest point of his path. The swing is suspended 2.00 m above the child's center of mass.

- (a) What is the magnitude of the centripetal acceleration of the child at the low point?
- (b) What is the magnitude of the force the child exerts on the seat if his mass is 18.0 kg?
- (c) What is unreasonable about these results?
- (d) Which premises are unreasonable or inconsistent?
-

Solution:

- a) 40.5 m/s^2
- b) 905 N
- c) The force in part (b) is very large. The acceleration in part (a) is too much, about 4 g.
- d) The speed of the swing is too large. At the given velocity at the bottom of the swing, there is enough kinetic energy to send the child all the way over the top, ignoring friction.

Glossary

centripetal acceleration

the acceleration of an object moving in a circle, directed toward the center

ultracentrifuge

a centrifuge optimized for spinning a rotor at very high speeds

Centripetal Force

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Calculate coefficient of friction on a car tire.
- Calculate ideal speed and angle of a car on a turn.

Any force or combination of forces can cause a centripetal or radial acceleration. Just a few examples are the tension in the rope on a tether ball, the force of Earth's gravity on the Moon, friction between roller skates and a rink floor, a banked roadway's force on a car, and forces on the tube of a spinning centrifuge.

Any net force causing uniform circular motion is called a **centripetal force**. The direction of a centripetal force is toward the center of curvature, the same as the direction of centripetal acceleration. According to Newton's second law of motion, net force is mass times acceleration: net $F = ma$. For uniform circular motion, the acceleration is the centripetal acceleration — $a = a_c$. Thus, the magnitude of centripetal force F_c is

Equation:

$$F_c = ma_c.$$

By using the expressions for centripetal acceleration a_c from $a_c = \frac{v^2}{r}$; $a_c = r\omega^2$, we get two expressions for the centripetal force F_c in terms of mass, velocity, angular velocity, and radius of curvature:

Equation:

$$F_c = m \frac{v^2}{r}; F_c = mr\omega^2.$$

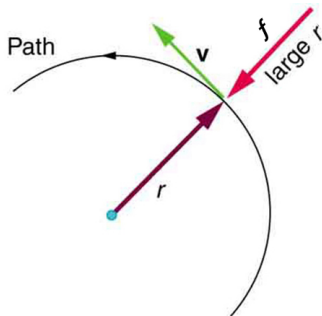
You may use whichever expression for centripetal force is more convenient. Centripetal force F_c is always perpendicular to the path and pointing to the center of curvature, because \mathbf{a}_c is perpendicular to the velocity and pointing to the center of curvature.

Note that if you solve the first expression for r , you get

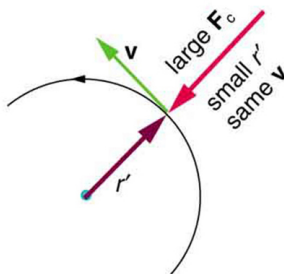
Equation:

$$r = \frac{mv^2}{F_c}.$$

This implies that for a given mass and velocity, a large centripetal force causes a small radius of curvature—that is, a tight curve.



$f = F_c$ is parallel to \mathbf{a}_c since $\mathbf{F}_c = m\mathbf{a}_c$



The frictional force
supplies the
centripetal force
and is numerically

equal to it.
Centripetal force is perpendicular to velocity and causes uniform circular motion. The larger the F_c , the smaller the radius of curvature r and the sharper the curve. The second curve has the same v , but a larger F_c produces a smaller r .

Example:

What Coefficient of Friction Do Car Tires Need on a Flat Curve?

- (a) Calculate the centripetal force exerted on a 900 kg car that negotiates a 500 m radius curve at 25.0 m/s.
(b) Assuming an unbanked curve, find the minimum static coefficient of friction, between the tires and the road, static friction being the reason that keeps the car from slipping (see [\[link\]](#)).

Strategy and Solution for (a)

We know that $F_c = \frac{mv^2}{r}$. Thus,

Equation:

$$F_c = \frac{mv^2}{r} = \frac{(900 \text{ kg})(25.0 \text{ m/s})^2}{(500 \text{ m})} = 1125 \text{ N}.$$

Strategy for (b)

[\[link\]](#) shows the forces acting on the car on an unbanked (level ground) curve. Friction is to the left, keeping the car from slipping, and because it

is the only horizontal force acting on the car, the friction is the centripetal force in this case. We know that the maximum static friction (at which the tires roll but do not slip) is $\mu_s N$, where μ_s is the static coefficient of friction and N is the normal force. The normal force equals the car's weight on level ground, so that $N = mg$. Thus the centripetal force in this situation is

Equation:

$$F_c = f = \mu_s N = \mu_s mg.$$

Now we have a relationship between centripetal force and the coefficient of friction. Using the first expression for F_c from the equation

Equation:

$$\left. \begin{aligned} F_c &= m \frac{v^2}{r} \\ F_c &= mr\omega^2 \end{aligned} \right\},$$

Equation:

$$m \frac{v^2}{r} = \mu_s mg.$$

We solve this for μ_s , noting that mass cancels, and obtain

Equation:

$$\mu_s = \frac{v^2}{rg}.$$

Solution for (b)

Substituting the knowns,

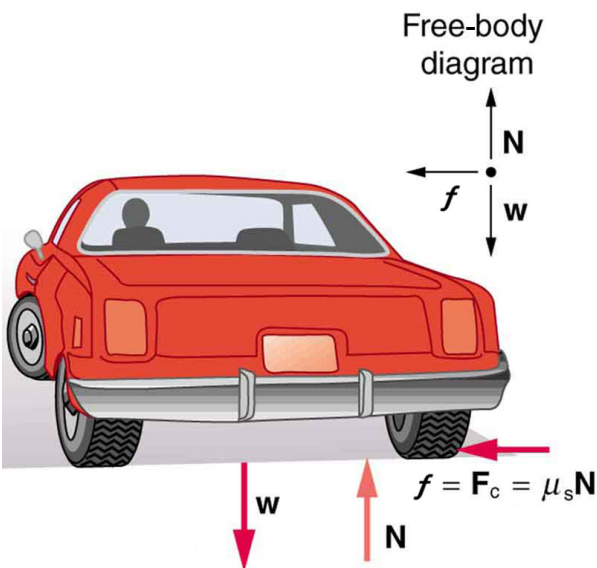
Equation:

$$\mu_s = \frac{(25.0 \text{ m/s})^2}{(500 \text{ m})(9.80 \text{ m/s}^2)} = 0.13.$$

(Because coefficients of friction are approximate, the answer is given to only two digits.)

Discussion

We could also solve part (a) using the first expression in $\left. \begin{aligned} F_c &= m\frac{v^2}{r} \\ F_c &= mr\omega^2 \end{aligned} \right\}$, because m, v , and r are given. The coefficient of friction found in part (b) is much smaller than is typically found between tires and roads. The car will still negotiate the curve if the coefficient is greater than 0.13, because static friction is a responsive force, being able to assume a value less than but no more than $\mu_s N$. A higher coefficient would also allow the car to negotiate the curve at a higher speed, but if the coefficient of friction is less, the safe speed would be less than 25 m/s. Note that mass cancels, implying that in this example, it does not matter how heavily loaded the car is to negotiate the turn. Mass cancels because friction is assumed proportional to the normal force, which in turn is proportional to mass. If the surface of the road were banked, the normal force would be less as will be discussed below.



This car on level ground is moving away and turning to the left. The centripetal force causing the car to turn in a circular path is due to friction

between the tires and the road.

A minimum coefficient of friction is needed, or the car will move in a larger-radius curve and leave the roadway.

Let us now consider **banked curves**, where the slope of the road helps you negotiate the curve. See [\[link\]](#). The greater the angle θ , the faster you can take the curve. Race tracks for bikes as well as cars, for example, often have steeply banked curves. In an “ideally banked curve,” the angle θ is such that you can negotiate the curve at a certain speed without the aid of friction between the tires and the road. We will derive an expression for θ for an ideally banked curve and consider an example related to it.

For **ideal banking**, the net external force equals the horizontal centripetal force in the absence of friction. The components of the normal force N in the horizontal and vertical directions must equal the centripetal force and the weight of the car, respectively. In cases in which forces are not parallel, it is most convenient to consider components along perpendicular axes—in this case, the vertical and horizontal directions.

[\[link\]](#) shows a free body diagram for a car on a frictionless banked curve. If the angle θ is ideal for the speed and radius, then the net external force will equal the necessary centripetal force. The only two external forces acting on the car are its weight \mathbf{w} and the normal force of the road \mathbf{N} . (A frictionless surface can only exert a force perpendicular to the surface—that is, a normal force.) These two forces must add to give a net external force that is horizontal toward the center of curvature and has magnitude mv^2/r .

Because this is the crucial force and it is horizontal, we use a coordinate system with vertical and horizontal axes. Only the normal force has a horizontal component, and so this must equal the centripetal force—that is,

Equation:

$$N \sin \theta = \frac{mv^2}{r}.$$

Because the car does not leave the surface of the road, the net vertical force must be zero, meaning that the vertical components of the two external forces must be equal in magnitude and opposite in direction. From the figure, we see that the vertical component of the normal force is $N \cos \theta$, and the only other vertical force is the car's weight. These must be equal in magnitude; thus,

Equation:

$$N \cos \theta = mg.$$

Now we can combine the last two equations to eliminate N and get an expression for θ , as desired. Solving the second equation for $N = mg/(\cos \theta)$, and substituting this into the first yields

Equation:

$$mg \frac{\sin \theta}{\cos \theta} = \frac{mv^2}{r}$$

Equation:

$$\begin{aligned} mg \tan(\theta) &= \frac{mv^2}{r} \\ \tan \theta &= \frac{v^2}{rg}. \end{aligned}$$

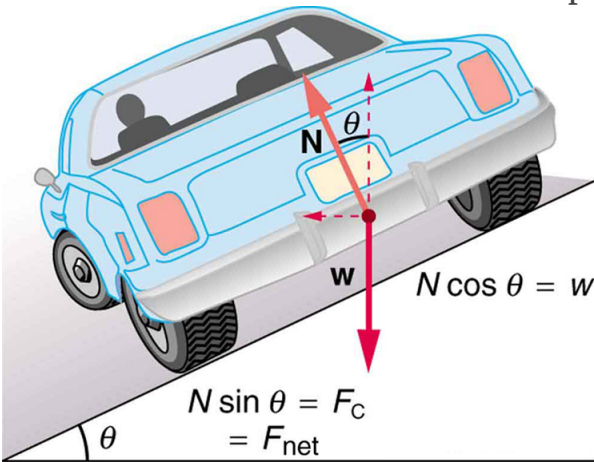
Taking the inverse tangent gives

Equation:

$$\theta = \tan^{-1} \left(\frac{v^2}{rg} \right) \text{ (ideally banked curve, no friction).}$$

This expression can be understood by considering how θ depends on v and r . A large θ will be obtained for a large v and a small r . That is, roads must be steeply banked for high speeds and sharp curves. Friction helps, because

it allows you to take the curve at greater or lower speed than if the curve is frictionless. Note that θ does not depend on the mass of the vehicle.



The car on this banked curve is moving away and turning to the left.

Example:

What Is the Ideal Speed to Take a Steeply Banked Tight Curve?

Curves on some test tracks and race courses, such as the Daytona International Speedway in Florida, are very steeply banked. This banking, with the aid of tire friction and very stable car configurations, allows the curves to be taken at very high speed. To illustrate, calculate the speed at which a 100 m radius curve banked at 65.0° should be driven if the road is frictionless.

Strategy

We first note that all terms in the expression for the ideal angle of a banked curve except for speed are known; thus, we need only rearrange it so that speed appears on the left-hand side and then substitute known quantities.

Solution

Starting with

Equation:

$$\tan \theta = \frac{v^2}{rg}$$

we get

Equation:

$$v = (rg \tan \theta)^{1/2}.$$

Noting that $\tan 65.0^\circ = 2.14$, we obtain

Equation:

$$\begin{aligned} v &= \left[(100 \text{ m})(9.80 \text{ m/s}^2)(2.14) \right]^{1/2} \\ &= 45.8 \text{ m/s.} \end{aligned}$$

Discussion

This is just about 165 km/h, consistent with a very steeply banked and rather sharp curve. Tire friction enables a vehicle to take the curve at significantly higher speeds.

Calculations similar to those in the preceding examples can be performed for a host of interesting situations in which centripetal force is involved—a number of these are presented in this chapter's Problems and Exercises.

Note:

Take-Home Experiment

Ask a friend or relative to swing a golf club or a tennis racquet. Take appropriate measurements to estimate the centripetal acceleration of the end of the club or racquet. You may choose to do this in slow motion.

Note:

PhET Explorations: Gravity and Orbits

Move the sun, earth, moon and space station to see how it affects their gravitational forces and orbital paths. Visualize the sizes and distances

between different heavenly bodies, and turn off gravity to see what would happen without it!

[Gravit](#)
[y and](#)
[Orbits](#)

Section Summary

- Centripetal force F_c is any force causing uniform circular motion. It is a “center-seeking” force that always points toward the center of rotation. It is perpendicular to linear velocity v and has magnitude

Equation:

$$F_c = ma_c,$$

which can also be expressed as

Equation:

$$\left. \begin{array}{l} F_c = m \frac{v^2}{r} \\ \text{or} \\ F_c = mr\omega^2 \end{array} \right\}$$

Conceptual Questions

Exercise:

Problem:

If you wish to reduce the stress (which is related to centripetal force) on high-speed tires, would you use large- or small-diameter tires? Explain.

Exercise:**Problem:**

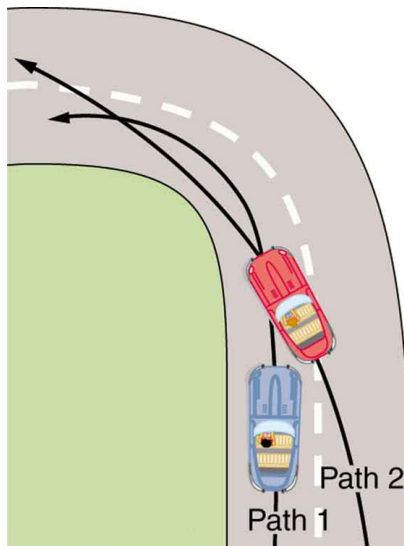
Define centripetal force. Can any type of force (for example, tension, gravitational force, friction, and so on) be a centripetal force? Can any combination of forces be a centripetal force?

Exercise:**Problem:**

If centripetal force is directed toward the center, why do you feel that you are 'thrown' away from the center as a car goes around a curve? Explain.

Exercise:**Problem:**

Race car drivers routinely cut corners as shown in [\[link\]](#). Explain how this allows the curve to be taken at the greatest speed.



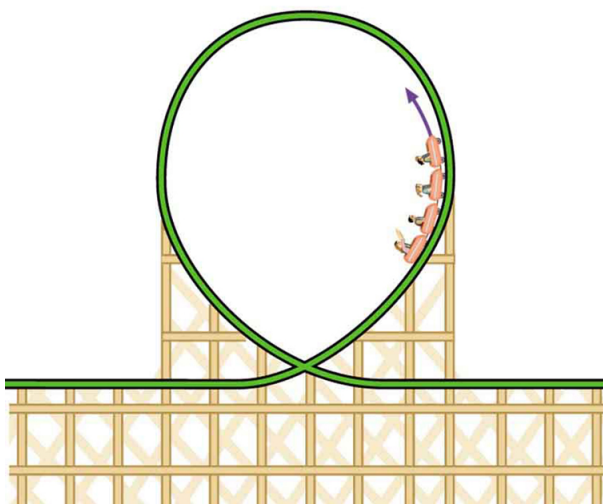
Two paths around a race track curve are shown. Race car drivers will take the inside path (called cutting the corner) whenever possible because it allows them to take the curve at the highest speed.

Exercise:

Problem:

A number of amusement parks have rides that make vertical loops like the one shown in [\[link\]](#). For safety, the cars are attached to the rails in such a way that they cannot fall off. If the car goes over the top at just the right speed, gravity alone will supply the centripetal force. What other force acts and what is its direction if:

- (a) The car goes over the top at faster than this speed?
- (b) The car goes over the top at slower than this speed?



Amusement rides with a vertical loop are an example of a form of curved motion.

Exercise:

Problem:

What is the direction of the force exerted by the car on the passenger as the car goes over the top of the amusement ride pictured in [\[link\]](#) under the following circumstances:

- (a) The car goes over the top at such a speed that the gravitational force is the only force acting?
- (b) The car goes over the top faster than this speed?
- (c) The car goes over the top slower than this speed?

Exercise:

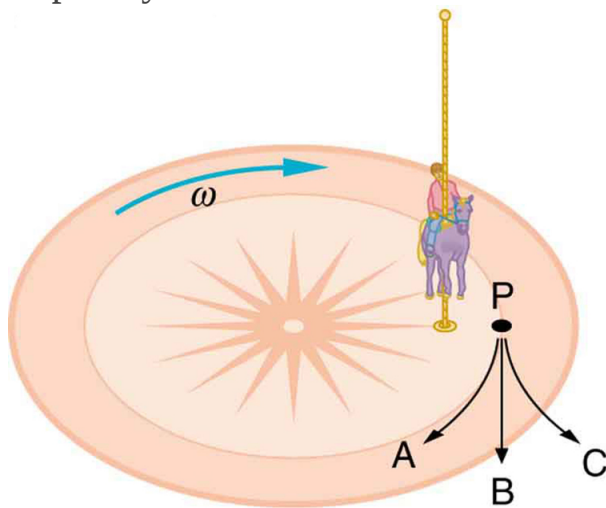
Problem:

As a skater forms a circle, what force is responsible for making her turn? Use a free body diagram in your answer.

Exercise:

Problem:

Suppose a child is riding on a merry-go-round at a distance about halfway between its center and edge. She has a lunch box resting on wax paper, so that there is very little friction between it and the merry-go-round. Which path shown in [\[link\]](#) will the lunch box take when she lets go? The lunch box leaves a trail in the dust on the merry-go-round. Is that trail straight, curved to the left, or curved to the right? Explain your answer.



Merry-go-round's rotating
frame of reference

A child riding on a merry-go-round releases her lunch box at point P. This is a view from above the clockwise rotation.

Assuming it slides with negligible friction, will it follow path A, B, or C, as viewed from Earth's frame of reference?

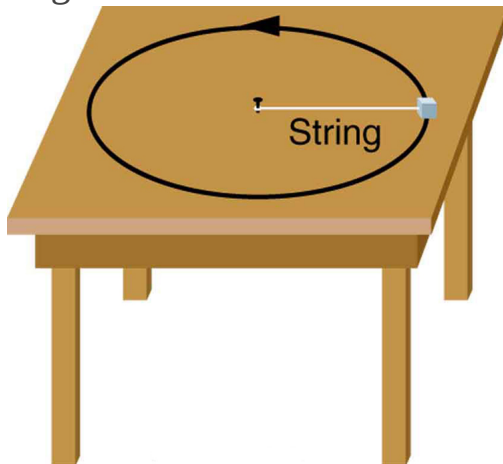
What will be the shape of the path it leaves in the dust on the merry-go-round?

Exercise:**Problem:**

Do you feel yourself thrown to either side when you negotiate a curve that is ideally banked for your car's speed? What is the direction of the force exerted on you by the car seat?

Exercise:**Problem:**

Suppose a mass is moving in a circular path on a frictionless table as shown in figure. In the Earth's frame of reference, there is no centrifugal force pulling the mass away from the centre of rotation, yet there is a very real force stretching the string attaching the mass to the nail. Using concepts related to centripetal force and Newton's third law, explain what force stretches the string, identifying its physical origin.



A mass attached to a nail on a frictionless table moves in a circular path. The force stretching the string is real and not fictional. What is the physical origin of the force on the string?

Problems Exercise

Exercise:

Problem:

- (a) A 22.0 kg child is riding a playground merry-go-round that is rotating at 40.0 rev/min. What centripetal force must she exert to stay on if she is 1.25 m from its center?
- (b) What centripetal force does she need to stay on an amusement park merry-go-round that rotates at 3.00 rev/min if she is 8.00 m from its center?
- (c) Compare each force with her weight.
-

Solution:

- a) 483 N
- b) 17.4 N
- c) 2.24 times her weight, 0.0807 times her weight

Exercise:

Problem:

Calculate the centripetal force on the end of a 100 m (radius) wind turbine blade that is rotating at 0.5 rev/s. Assume the mass is 4 kg.

Exercise:

Problem:

What is the ideal banking angle for a gentle turn of 1.20 km radius on a highway with a 105 km/h speed limit (about 65 mi/h), assuming everyone travels at the limit?

Solution:

4.14°

Exercise:**Problem:**

What is the ideal speed to take a 100 m radius curve banked at a 20.0° angle?

Exercise:**Problem:**

(a) What is the radius of a bobsled turn banked at 75.0° and taken at 30.0 m/s, assuming it is ideally banked?

(b) Calculate the centripetal acceleration.

(c) Does this acceleration seem large to you?

Solution:

a) 24.6 m

b) 36.6 m/s²

c) $a_c = 3.73 g$. This does not seem too large, but it is clear that bobsledders feel a lot of force on them going through sharply banked turns.

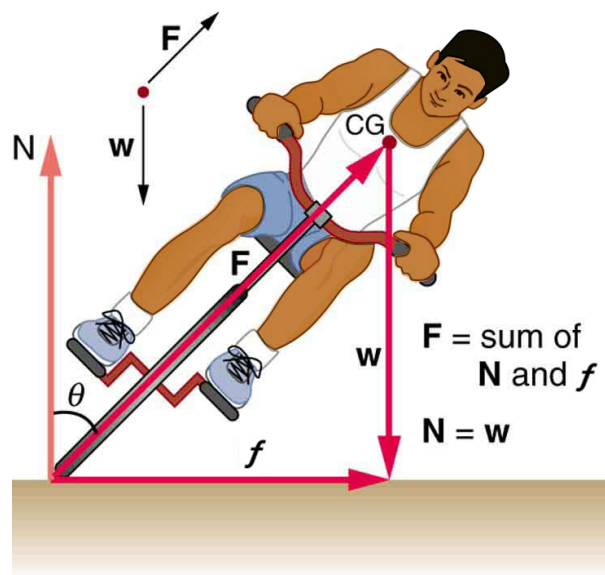
Exercise:

Problem:

Part of riding a bicycle involves leaning at the correct angle when making a turn, as seen in [\[link\]](#). To be stable, the force exerted by the ground must be on a line going through the center of gravity. The force on the bicycle wheel can be resolved into two perpendicular components—friction parallel to the road (this must supply the centripetal force), and the vertical normal force (which must equal the system's weight).

(a) Show that θ (as defined in the figure) is related to the speed v and radius of curvature r of the turn in the same way as for an ideally banked roadway—that is, $\theta = \tan^{-1} v^2 / rg$

(b) Calculate θ for a 12.0 m/s turn of radius 30.0 m (as in a race).
Free-body diagram



A bicyclist negotiating a turn on level ground must lean at the correct angle—the ability to do this becomes instinctive. The force of the ground on the wheel needs to be on a line through the center of gravity.

The net external force on the system is the centripetal force. The vertical component of the force on the wheel cancels the weight of the system while its horizontal component must supply the centripetal force.

This process produces a relationship among the angle θ , the speed v , and the radius of curvature r of the turn similar to that for the ideal banking of roadways.

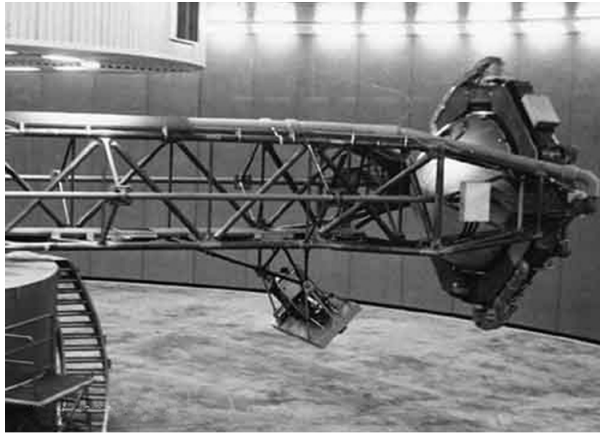
Exercise:

Problem:

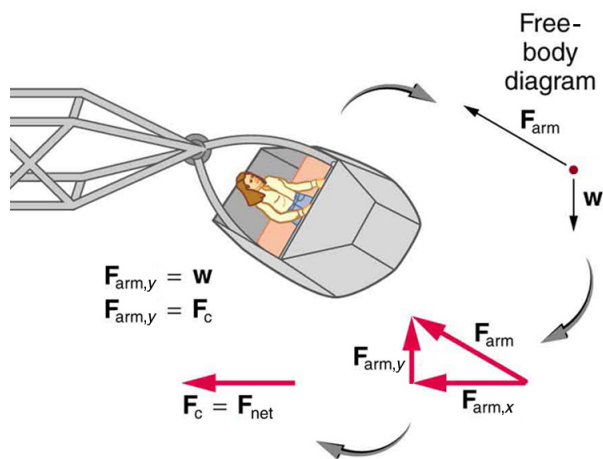
A large centrifuge, like the one shown in [\[link\]](#)(a), is used to expose aspiring astronauts to accelerations similar to those experienced in rocket launches and atmospheric reentries.

(a) At what angular velocity is the centripetal acceleration $10\ g$ if the rider is $15.0\ \text{m}$ from the center of rotation?

(b) The rider's cage hangs on a pivot at the end of the arm, allowing it to swing outward during rotation as shown in [\[link\]](#)(b). At what angle θ below the horizontal will the cage hang when the centripetal acceleration is $10\ g$? (Hint: The arm supplies centripetal force and supports the weight of the cage. Draw a free body diagram of the forces to see what the angle θ should be.)



(a) NASA centrifuge and ride



(b)

(a) NASA centrifuge used to subject trainees to accelerations similar to those experienced in rocket launches and reentries.

(credit: NASA) (b) Rider in cage showing how the cage pivots outward during rotation.

This allows the total force exerted on the rider by the cage to be along its axis at all times.

Solution:

a) 2.56 rad/s

b) 5.71°

Exercise:

Problem: Integrated Concepts

If a car takes a banked curve at less than the ideal speed, friction is needed to keep it from sliding toward the inside of the curve (a real problem on icy mountain roads). (a) Calculate the ideal speed to take a 100 m radius curve banked at 15.0° . (b) What is the minimum coefficient of friction needed for a frightened driver to take the same curve at 20.0 km/h?

Solution:

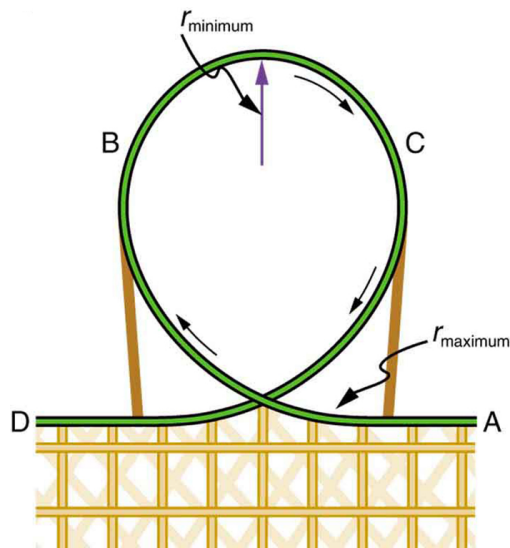
a) 16.2 m/s

b) 0.234

Exercise:

Problem:

Modern roller coasters have vertical loops like the one shown in [\[link\]](#). The radius of curvature is smaller at the top than on the sides so that the downward centripetal acceleration at the top will be greater than the acceleration due to gravity, keeping the passengers pressed firmly into their seats. What is the speed of the roller coaster at the top of the loop if the radius of curvature there is 15.0 m and the downward acceleration of the car is 1.50 g?



Teardrop-shaped loops are used in the latest roller coasters so that the radius of curvature gradually decreases to a minimum at the top. This means that the centripetal acceleration builds from zero to a maximum at the top and gradually decreases again. A circular loop would cause a jolting change in acceleration at entry, a disadvantage discovered long ago in railroad curve design. With a small radius of curvature at the top, the centripetal acceleration can more easily be kept greater than g so that the passengers do not lose contact with their seats nor do they

need seat belts to keep
them in place.

Exercise:

Problem: Unreasonable Results

- (a) Calculate the minimum coefficient of friction needed for a car to negotiate an unbanked 50.0 m radius curve at 30.0 m/s.
 - (b) What is unreasonable about the result?
 - (c) Which premises are unreasonable or inconsistent?
-

Solution:

- a) 1.84
- b) A coefficient of friction this much greater than 1 is unreasonable .
- c) The assumed speed is too great for the tight curve.

Glossary

centripetal force

any net force causing uniform circular motion

ideal banking

the sloping of a curve in a road, where the angle of the slope allows the vehicle to negotiate the curve at a certain speed without the aid of friction between the tires and the road; the net external force on the vehicle equals the horizontal centripetal force in the absence of friction

ideal speed

the maximum safe speed at which a vehicle can turn on a curve without the aid of friction between the tire and the road

ideal angle

the angle at which a car can turn safely on a steep curve, which is in proportion to the ideal speed

banked curve

the curve in a road that is sloping in a manner that helps a vehicle negotiate the curve

Fictitious Forces and Non-inertial Frames: The Coriolis Force

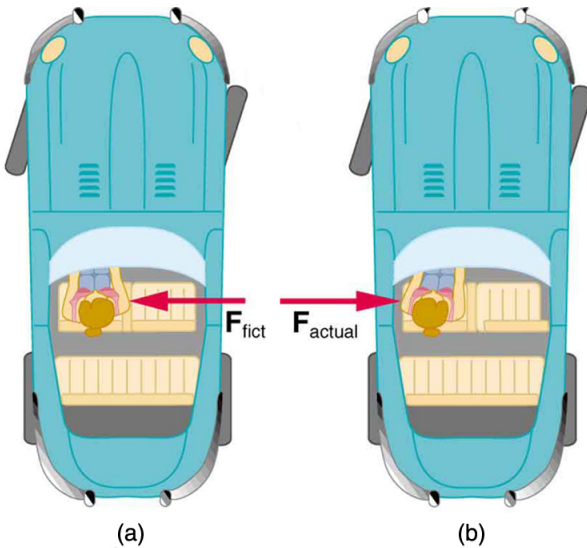
LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Discuss the inertial frame of reference.
- Discuss the non-inertial frame of reference.
- Describe the effects of the Coriolis force.

What do taking off in a jet airplane, turning a corner in a car, riding a merry-go-round, and the circular motion of a tropical cyclone have in common? Each exhibits fictitious forces—unreal forces that arise from motion and may *seem* real, because the observer's frame of reference is accelerating or rotating.

When taking off in a jet, most people would agree it feels as if you are being pushed back into the seat as the airplane accelerates down the runway. Yet a physicist would say that *you* tend to remain stationary while the *seat* pushes forward on you, and there is no real force backward on you. An even more common experience occurs when you make a tight curve in your car—say, to the right. You feel as if you are thrown (that is, *forced*) toward the left relative to the car. Again, a physicist would say that *you* are going in a straight line but the *car* moves to the right, and there is no real force on you to the left. Recall Newton's first law.

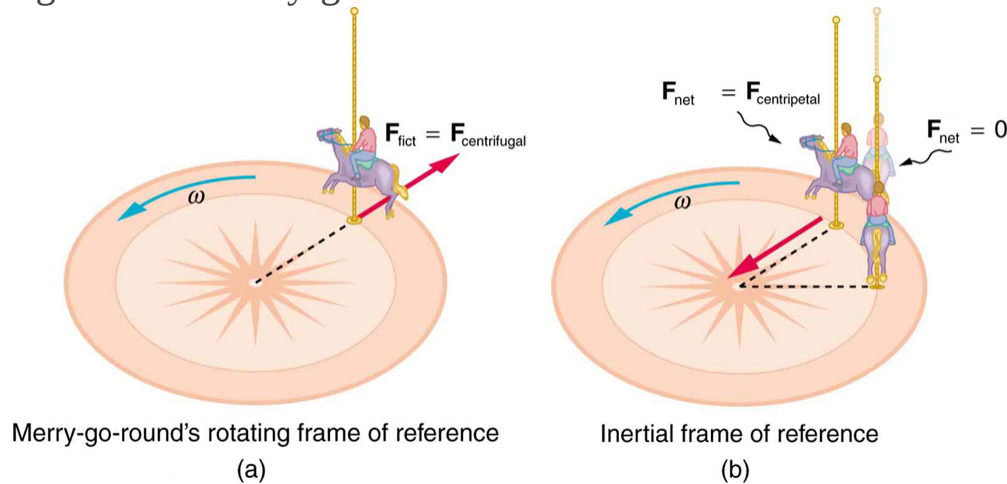


(a) The car driver feels herself forced to the left relative to the car when she makes a right turn. This is a fictitious force arising from the use of the car as a frame of reference. (b) In the Earth's frame of reference, the driver moves in a straight line, obeying Newton's first law, and the car moves to the right. There is no real force to the left on the driver relative to Earth. There is a real force to the right on the car to make it turn.

We can reconcile these points of view by examining the frames of reference used. Let us concentrate on people in a car. Passengers instinctively use the car as a frame of reference, while a physicist uses Earth. The physicist chooses Earth because it is very nearly an inertial frame of reference—one in which all forces are real (that is, in which all forces have an identifiable physical origin). In such a frame of reference, Newton's laws of motion take the form given in [Dynamics: Newton's Laws of Motion](#). The car is a

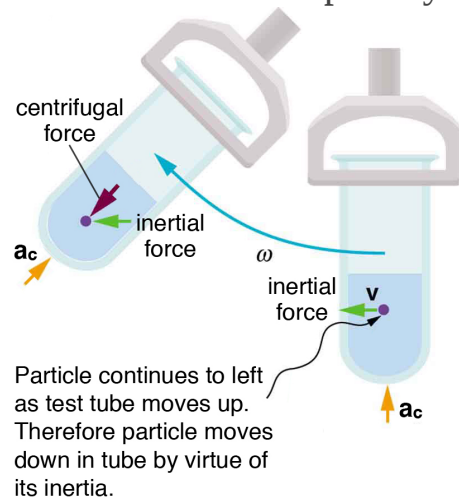
non-inertial frame of reference because it is accelerated to the side. The force to the left sensed by car passengers is a **fictitious force** having no physical origin. There is nothing real pushing them left—the car, as well as the driver, is actually accelerating to the right.

Let us now take a mental ride on a merry-go-round—specifically, a rapidly rotating playground merry-go-round. You take the merry-go-round to be your frame of reference because you rotate together. In that non-inertial frame, you feel a fictitious force, named **centrifugal force** (not to be confused with centripetal force), trying to throw you off. You must hang on tightly to counteract the centrifugal force. In Earth's frame of reference, there is no force trying to throw you off. Rather you must hang on to make yourself go in a circle because otherwise you would go in a straight line, right off the merry-go-round.



(a) A rider on a merry-go-round feels as if he is being thrown off. This fictitious force is called the centrifugal force—it explains the rider's motion in the rotating frame of reference. (b) In an inertial frame of reference and according to Newton's laws, it is his inertia that carries him off and not a real force (the unshaded rider has $F_{\text{net}} = 0$ and heads in a straight line). A real force, $F_{\text{centripetal}}$, is needed to cause a circular path.

This inertial effect, carrying you away from the center of rotation if there is no centripetal force to cause circular motion, is put to good use in centrifuges (see [\[link\]](#)). A centrifuge spins a sample very rapidly, as mentioned earlier in this chapter. Viewed from the rotating frame of reference, the fictitious centrifugal force throws particles outward, hastening their sedimentation. The greater the angular velocity, the greater the centrifugal force. But what really happens is that the inertia of the particles carries them along a line tangent to the circle while the test tube is forced in a circular path by a centripetal force.

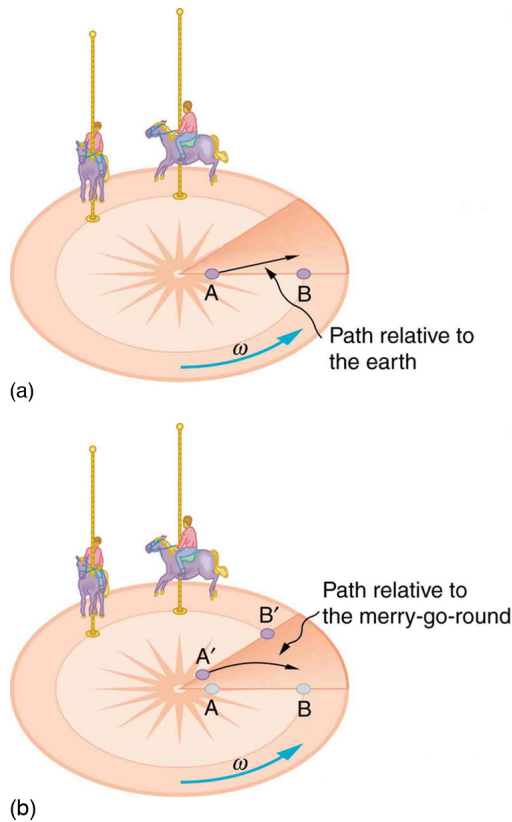


Centrifuges use inertia to perform their task. Particles in the fluid sediment come out because their inertia carries them away from the center of rotation. The large angular velocity of the centrifuge quickens the sedimentation.

Ultimately, the particles will come into contact with the test tube walls, which will then supply the

centripetal force
needed to make them
move in a circle of
constant radius.

Let us now consider what happens if something moves in a frame of reference that rotates. For example, what if you slide a ball directly away from the center of the merry-go-round, as shown in [\[link\]](#)? The ball follows a straight path relative to Earth (assuming negligible friction) and a path curved to the right on the merry-go-round's surface. A person standing next to the merry-go-round sees the ball moving straight and the merry-go-round rotating underneath it. In the merry-go-round's frame of reference, we explain the apparent curve to the right by using a fictitious force, called the **Coriolis force**, that causes the ball to curve to the right. The fictitious Coriolis force can be used by anyone in that frame of reference to explain why objects follow curved paths and allows us to apply Newton's Laws in non-inertial frames of reference.



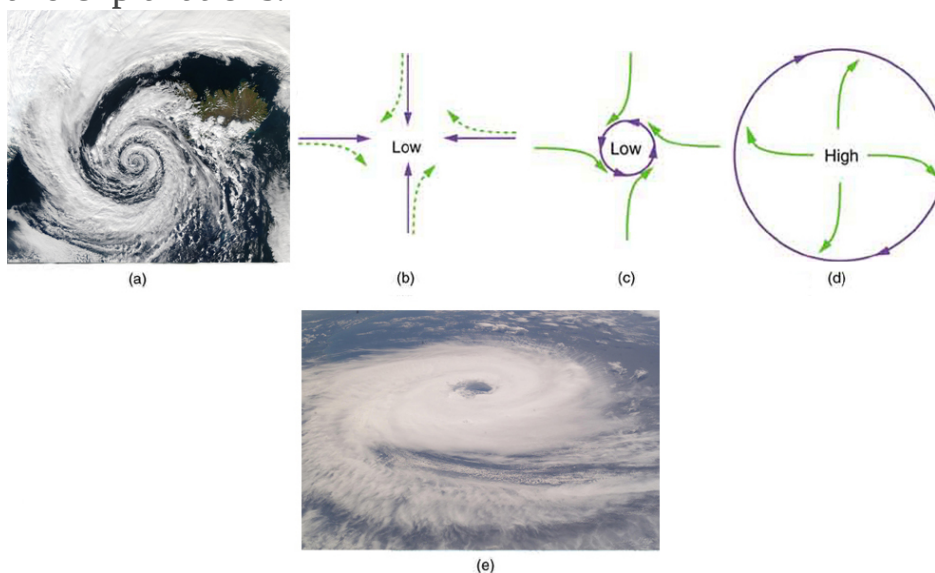
Looking down on the counterclockwise rotation of a merry-go-round, we see that a ball slid straight toward the edge follows a path curved to the right. The person slides the ball toward point B, starting at point A. Both points rotate to the shaded positions (A' and B') shown in the time that the ball follows the curved path in the rotating frame and a straight path in Earth's frame.

Up until now, we have considered Earth to be an inertial frame of reference with little or no worry about effects due to its rotation. Yet such effects *do* exist—in the rotation of weather systems, for example. Most consequences of Earth's rotation can be qualitatively understood by analogy with the merry-go-round. Viewed from above the North Pole, Earth rotates counterclockwise, as does the merry-go-round in [\[link\]](#). As on the merry-go-round, any motion in Earth's northern hemisphere experiences a Coriolis force to the right. Just the opposite occurs in the southern hemisphere; there, the force is to the left. Because Earth's angular velocity is small, the Coriolis force is usually negligible, but for large-scale motions, such as wind patterns, it has substantial effects.

The Coriolis force causes hurricanes in the northern hemisphere to rotate in the counterclockwise direction, while the tropical cyclones (what hurricanes are called below the equator) in the southern hemisphere rotate in the clockwise direction. The terms hurricane, typhoon, and tropical storm are regionally-specific names for tropical cyclones, storm systems characterized by low pressure centers, strong winds, and heavy rains. [\[link\]](#)

helps show how these rotations take place. Air flows toward any region of low pressure, and tropical cyclones contain particularly low pressures. Thus winds flow toward the center of a tropical cyclone or a low-pressure weather system at the surface. In the northern hemisphere, these inward winds are deflected to the right, as shown in the figure, producing a counterclockwise circulation at the surface for low-pressure zones of any type. Low pressure at the surface is associated with rising air, which also produces cooling and cloud formation, making low-pressure patterns quite visible from space. Conversely, wind circulation around high-pressure zones is clockwise in the northern hemisphere but is less visible because high pressure is associated with sinking air, producing clear skies.

The rotation of tropical cyclones and the path of a ball on a merry-go-round can just as well be explained by inertia and the rotation of the system underneath. When non-inertial frames are used, fictitious forces, such as the Coriolis force, must be invented to explain the curved path. There is no identifiable physical source for these fictitious forces. In an inertial frame, inertia explains the path, and no force is found to be without an identifiable source. Either view allows us to describe nature, but a view in an inertial frame is the simplest and truest, in the sense that all forces have real origins and explanations.



(a) The counterclockwise rotation of this northern hemisphere hurricane is a major consequence of the Coriolis force. (credit: NASA) (b) Without the

Coriolis force, air would flow straight into a low-pressure zone, such as that found in tropical cyclones. (c) The Coriolis force deflects the winds to the right, producing a counterclockwise rotation. (d) Wind flowing away from a high-pressure zone is also deflected to the right, producing a clockwise rotation. (e) The opposite direction of rotation is produced by the Coriolis force in the southern hemisphere, leading to tropical cyclones. (credit: NASA)

Section Summary

- Rotating and accelerated frames of reference are non-inertial.
- Fictitious forces, such as the Coriolis force, are needed to explain motion in such frames.

Conceptual Questions

Exercise:

Problem:

When a toilet is flushed or a sink is drained, the water (and other material) begins to rotate about the drain on the way down. Assuming no initial rotation and a flow initially directly straight toward the drain, explain what causes the rotation and which direction it has in the northern hemisphere. (Note that this is a small effect and in most toilets the rotation is caused by directional water jets.) Would the direction of rotation reverse if water were forced up the drain?

Exercise:

Problem:

Is there a real force that throws water from clothes during the spin cycle of a washing machine? Explain how the water is removed.

Exercise:**Problem:**

In one amusement park ride, riders enter a large vertical barrel and stand against the wall on its horizontal floor. The barrel is spun up and the floor drops away. Riders feel as if they are pinned to the wall by a force something like the gravitational force. This is a fictitious force sensed and used by the riders to explain events in the rotating frame of reference of the barrel. Explain in an inertial frame of reference (Earth is nearly one) what pins the riders to the wall, and identify all of the real forces acting on them.

Exercise:**Problem:**

Action at a distance, such as is the case for gravity, was once thought to be illogical and therefore untrue. What is the ultimate determinant of the truth in physics, and why was this action ultimately accepted?

Exercise:**Problem:**

Two friends are having a conversation. Anna says a satellite in orbit is in freefall because the satellite keeps falling toward Earth. Tom says a satellite in orbit is not in freefall because the acceleration due to gravity is not 9.80 m/s^2 . Who do you agree with and why?

Exercise:**Problem:**

A non-rotating frame of reference placed at the center of the Sun is very nearly an inertial one. Why is it not exactly an inertial frame?

Glossary

fictitious force

a force having no physical origin

centrifugal force

a fictitious force that tends to throw an object off when the object is rotating in a non-inertial frame of reference

Coriolis force

the fictitious force causing the apparent deflection of moving objects when viewed in a rotating frame of reference

non-inertial frame of reference

an accelerated frame of reference

Newton's Universal Law of Gravitation

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain Earth's gravitational force.
- Describe the gravitational effect of the Moon on Earth.
- Discuss weightlessness in space.
- Understand the Cavendish experiment.

The information presented in this section supports the following AP[®] learning objectives and science practices:

- **2.B.2.1** The student is able to apply $g = \frac{GM}{r^2}$ to calculate the gravitational field due to an object with mass M , where the field is a vector directed toward the center of the object of mass M . **(S.P. 2.2)**
- **2.B.2.2** The student is able to approximate a numerical value of the gravitational field (g) near the surface of an object from its radius and mass relative to those of the Earth or other reference objects. **(S.P. 2.2)**
- **3.A.3.4.** The student is able to make claims about the force on an object due to the presence of other objects with the same property: mass, electric charge. **(S.P. 6.1, 6.4)**

What do aching feet, a falling apple, and the orbit of the Moon have in common? Each is caused by the gravitational force. Our feet are strained by supporting our weight—the force of Earth's gravity on us. An apple falls

from a tree because of the same force acting a few meters above Earth's surface. And the Moon orbits Earth because gravity is able to supply the necessary centripetal force at a distance of hundreds of millions of meters. In fact, the same force causes planets to orbit the Sun, stars to orbit the center of the galaxy, and galaxies to cluster together. Gravity is another example of underlying simplicity in nature. It is the weakest of the four basic forces found in nature, and in some ways the least understood. It is a force that acts at a distance, without physical contact, and is expressed by a formula that is valid everywhere in the universe, for masses and distances that vary from the tiny to the immense.

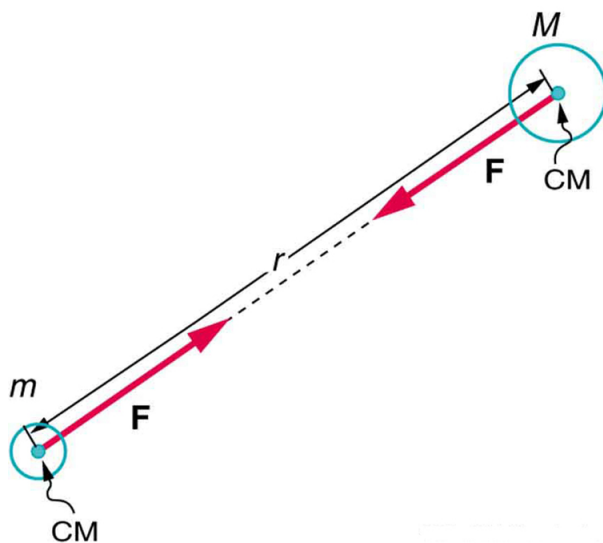
Sir Isaac Newton was the first scientist to precisely define the gravitational force, and to show that it could explain both falling bodies and astronomical motions. See [\[link\]](#). But Newton was not the first to suspect that the same force caused both our weight and the motion of planets. His forerunner Galileo Galilei had contended that falling bodies and planetary motions had the same cause. Some of Newton's contemporaries, such as Robert Hooke, Christopher Wren, and Edmund Halley, had also made some progress toward understanding gravitation. But Newton was the first to propose an exact mathematical form and to use that form to show that the motion of heavenly bodies should be conic sections—circles, ellipses, parabolas, and hyperbolas. This theoretical prediction was a major triumph—it had been known for some time that moons, planets, and comets follow such paths, but no one had been able to propose a mechanism that caused them to follow these paths and not others. This was one of the earliest examples of a theory derived from empirical evidence doing more than merely describing those empirical results; it made claims about the fundamental workings of the universe.



According to early accounts, Newton was inspired to make the connection between falling bodies and astronomical motions when he saw an apple fall from a tree and realized that if the gravitational force could extend above the ground to a tree, it might also reach the Sun. The inspiration of Newton's apple is a part of worldwide folklore and may even be based in fact. Great importance is attached to it because Newton's

universal law of
gravitation and his
laws of motion
answered very old
questions about nature
and gave tremendous
support to the notion
of underlying
simplicity and unity in
nature. Scientists still
expect underlying
simplicity to emerge
from their ongoing
inquiries into nature.

The gravitational force is relatively simple. It is always attractive, and it depends only on the masses involved and the distance between them. Stated in modern language, **Newton's universal law of gravitation** states that every particle in the universe attracts every other particle with a force along a line joining them. The force is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.



Gravitational attraction is along a line joining the centers of mass of these two bodies. The magnitude of the force is the same on each, consistent with Newton's third law.

Note:

Misconception Alert

The magnitude of the force on each object (one has larger mass than the other) is the same, consistent with Newton's third law.

The bodies we are dealing with tend to be large. To simplify the situation we assume that the body acts as if its entire mass is concentrated at one specific point called the **center of mass** (CM), which will be further explored in [Linear Momentum and Collisions](#). For two bodies having masses m and M with a distance r between their centers of mass, the equation for Newton's universal law of gravitation is

Equation:

$$F = G \frac{mM}{r^2},$$

where F is the magnitude of the gravitational force and G is a proportionality factor called the **gravitational constant**. G is a universal gravitational constant—that is, it is thought to be the same everywhere in the universe. It has been measured experimentally to be

Equation:

$$G = 6.673 \times 10^{-11} \frac{\text{N} \cdot \text{m}^2}{\text{kg}^2}$$

in SI units. Note that the units of G are such that a force in newtons is obtained from $F = G \frac{mM}{r^2}$, when considering masses in kilograms and distance in meters. For example, two 1.000 kg masses separated by 1.000 m will experience a gravitational attraction of 6.673×10^{-11} N. This is an extraordinarily small force. The small magnitude of the gravitational force is consistent with everyday experience. We are unaware that even large objects like mountains exert gravitational forces on us. In fact, our body weight is the force of attraction of the *entire Earth* on us with a mass of 6×10^{24} kg.

The experiment to measure G was first performed by Cavendish, and is explained in more detail later. The fundamental concept it is based on is having a known mass on a spring with a known force (or spring) constant. Then, a second known mass is placed at multiple known distances from the first, and the amount of stretch in the spring resulting from the gravitational attraction of the two masses is measured.

Recall that the acceleration due to gravity g is about 9.80 m/s^2 on Earth. We can now determine why this is so. The weight of an object mg is the gravitational force between it and Earth. Substituting mg for F in Newton's universal law of gravitation gives

Equation:

$$mg = G \frac{mM}{r^2},$$

where m is the mass of the object, M is the mass of Earth, and r is the distance to the center of Earth (the distance between the centers of mass of the object and Earth). See [\[link\]](#). The mass m of the object cancels, leaving an equation for g :

Equation:

$$g = G \frac{M}{r^2}.$$

Substituting known values for Earth's mass and radius (to three significant figures),

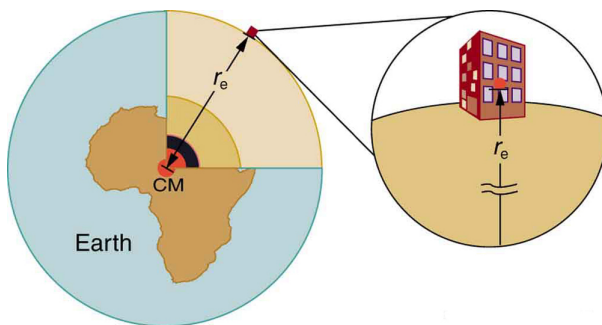
Equation:

$$g = \left(6.67 \times 10^{-11} \frac{\text{N} \cdot \text{m}^2}{\text{kg}^2} \right) \times \frac{5.98 \times 10^{24} \text{ kg}}{(6.38 \times 10^6 \text{ m})^2},$$

and we obtain a value for the acceleration of a falling body:

Equation:

$$g = 9.80 \text{ m/s}^2.$$



The distance between the centers of mass of Earth and an object on its surface is very nearly the same as the radius of Earth, because Earth is so much larger than the object.

This is the expected value *and is independent of the body's mass*. Newton's law of gravitation takes Galileo's observation that all masses fall with the same acceleration a step further, explaining the observation in terms of a force that causes objects to fall—in fact, in terms of a universally existing force of attraction between masses.

Note:**Gravitational Mass and Inertial Mass**

Notice that, in Equation 6.40, the mass of the objects under consideration is directly proportional to the gravitational force. More mass means greater forces, and vice versa. However, we have already seen the concept of mass before in a different context.

In Chapter 4, you read that mass is a measure of inertia. However, we normally measure the mass of an object by measuring the force of gravity (F) on it.

How do we know that inertial mass is identical to gravitational mass?

Assume that we compare the mass of two objects. The objects have inertial masses m_1 and m_2 . If the objects balance each other on a pan balance, we can conclude that they have the same gravitational mass, that is, that they experience the same force due to gravity, F . Using Newton's second law of motion, $F = ma$, we can write $m_1 a_1 = m_2 a_2$.

If we can show that the two objects experience the same acceleration due to gravity, we can conclude that $m_1 = m_2$, that is, that the objects' inertial masses are equal.

In fact, Galileo and others conducted experiments to show that, when factors such as wind resistance are kept constant, all objects, regardless of their mass, experience the same acceleration due to gravity. Galileo is famously said to have dropped two balls of different masses off the leaning tower of Pisa to demonstrate this. The balls accelerated at the same rate. Since acceleration due to gravity is constant for all objects on Earth, regardless of their mass or composition, i.e., $a_1 = a_2$, then $m_1 = m_2$. Thus, we can conclude that inertial mass is identical to gravitational mass. This allows us to calculate the acceleration of free fall due to gravity, such as in the orbits of planets.

Note:**Take-Home Experiment**

Take a marble, a ball, and a spoon and drop them from the same height. Do they hit the floor at the same time? If you drop a piece of paper as well, does it behave like the other objects? Explain your observations.

Note:**Making Connections: Gravitation, Other Forces, and General Relativity**

Attempts are still being made to understand the gravitational force. As we shall see in [Particle Physics](#), modern physics is exploring the connections of gravity to other forces, space, and time. General relativity alters our view of gravitation, leading us to think of gravitation as bending space and time.

Note:**Applying the Science Practices: All Objects Have Gravitational Fields**

We can use the formula developed above, $g = \frac{GM}{r^2}$, to calculate the gravitational fields of other objects.

For example, the Moon has a radius of 1.7×10^6 m and a mass of 7.3×10^{22} kg. The gravitational field on the surface of the Moon can be expressed as

$$\begin{aligned} g &= G \frac{M}{r^2} \\ &= \left(6.67 \times 10^{-11} \frac{\text{N} \cdot \text{m}^2}{\text{kg}^2} \right) \times \frac{7.3 \times 10^{22} \text{ kg}}{(1.7 \times 10^6 \text{ m})^2} \\ &= 1.685 \text{ m/s}^2 \end{aligned}$$

This is about 1/6 of the gravity on Earth, which seems reasonable, since the Moon has a much smaller mass than Earth does.

A person has a mass of 50 kg. The gravitational field 1.0 m from the person's center of mass can be expressed as

$$\begin{aligned} g &= G \frac{M}{r^2} \\ &= \left(6.67 \times 10^{-11} \frac{\text{N} \cdot \text{m}^2}{\text{kg}^2} \right) \times \frac{50 \text{ kg}}{(1 \text{ m})^2} \end{aligned}$$

$$= 3.34 \times 10^{-9} \text{ m/s}^2$$

This is less than one millionth of the gravitational field at the surface of Earth.

In the following example, we make a comparison similar to one made by Newton himself. He noted that if the gravitational force caused the Moon to orbit Earth, then the acceleration due to gravity should equal the centripetal acceleration of the Moon in its orbit. Newton found that the two accelerations agreed “pretty nearly.”

Example:

Earth's Gravitational Force Is the Centripetal Force Making the Moon Move in a Curved Path

- (a) Find the acceleration due to Earth's gravity at the distance of the Moon.
- (b) Calculate the centripetal acceleration needed to keep the Moon in its orbit (assuming a circular orbit about a fixed Earth), and compare it with the value of the acceleration due to Earth's gravity that you have just found.

Strategy for (a)

This calculation is the same as the one finding the acceleration due to gravity at Earth's surface, except that r is the distance from the center of Earth to the center of the Moon. The radius of the Moon's nearly circular orbit is $3.84 \times 10^8 \text{ m}$.

Solution for (a)

Substituting known values into the expression for g found above, remembering that M is the mass of Earth not the Moon, yields

Equation:

$$\begin{aligned} g &= G \frac{M}{r^2} = \left(6.67 \times 10^{-11} \frac{\text{N} \cdot \text{m}^2}{\text{kg}^2} \right) \times \frac{5.98 \times 10^{24} \text{ kg}}{(3.84 \times 10^8 \text{ m})^2} \\ &= 2.70 \times 10^{-3} \text{ m/s}^2 \end{aligned}$$

Strategy for (b)

Centripetal acceleration can be calculated using either form of

Equation:

$$\left. \begin{aligned} a_c &= \frac{v^2}{r} \\ a_c &= r\omega^2 \end{aligned} \right\}.$$

We choose to use the second form:

Equation:

$$a_c = r\omega^2,$$

where ω is the angular velocity of the Moon about Earth.

Solution for (b)

Given that the period (the time it takes to make one complete rotation) of the Moon's orbit is 27.3 days, (d) and using

Equation:

$$1 \text{ d} \times 24 \frac{\text{hr}}{\text{d}} \times 60 \frac{\text{min}}{\text{hr}} \times 60 \frac{\text{s}}{\text{min}} = 86,400 \text{ s}$$

we see that

Equation:

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi \text{ rad}}{(27.3 \text{ d})(86,400 \text{ s/d})} = 2.66 \times 10^{-6} \frac{\text{rad}}{\text{s}}.$$

The centripetal acceleration is

Equation:

$$\begin{aligned} a_c &= r\omega^2 = (3.84 \times 10^8 \text{ m})(2.66 \times 10^{-6} \text{ rad/s})^2 \\ &= 2.72 \times 10^{-3} \text{ m/s}^2 \end{aligned}$$

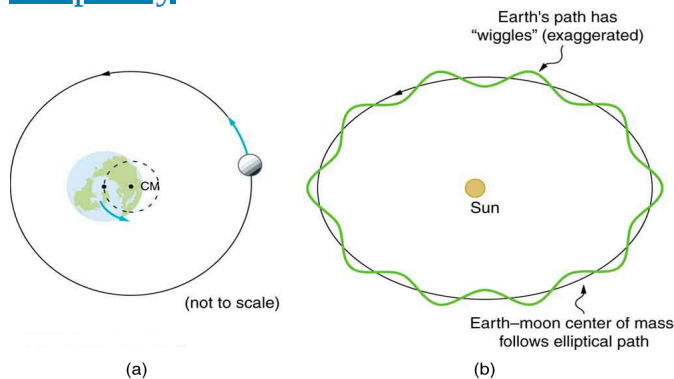
The direction of the acceleration is toward the center of the Earth.

Discussion

The centripetal acceleration of the Moon found in (b) differs by less than 1% from the acceleration due to Earth's gravity found in (a). This agreement is approximate because the Moon's orbit is slightly elliptical, and Earth is not stationary (rather the Earth-Moon system rotates about its

center of mass, which is located some 1700 km below Earth's surface). The clear implication is that Earth's gravitational force causes the Moon to orbit Earth.

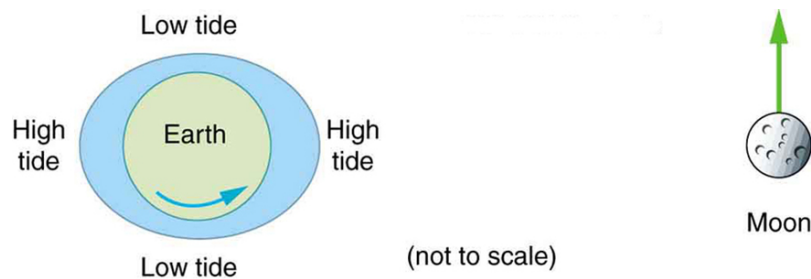
Why does Earth not remain stationary as the Moon orbits it? This is because, as expected from Newton's third law, if Earth exerts a force on the Moon, then the Moon should exert an equal and opposite force on Earth (see [\[link\]](#)). We do not sense the Moon's effect on Earth's motion, because the Moon's gravity moves our bodies right along with Earth but there are other signs on Earth that clearly show the effect of the Moon's gravitational force as discussed in [Satellites and Kepler's Laws: An Argument for Simplicity](#).



(a) Earth and the Moon rotate approximately once a month around their common center of mass. (b) Their center of mass orbits the Sun in an elliptical orbit, but Earth's path around the Sun has “wiggles” in it. Similar wiggles in the paths of stars have been observed and are considered direct evidence of planets orbiting those stars. This is important because the planets' reflected light is often too dim to be observed.

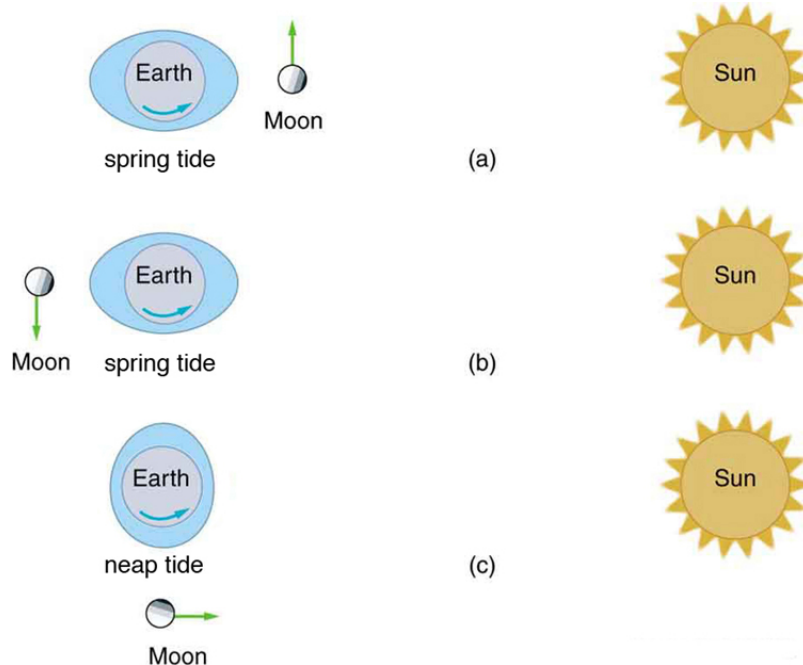
Tides

Ocean tides are one very observable result of the Moon's gravity acting on Earth. [\[link\]](#) is a simplified drawing of the Moon's position relative to the tides. Because water easily flows on Earth's surface, a high tide is created on the side of Earth nearest to the Moon, where the Moon's gravitational pull is strongest. Why is there also a high tide on the opposite side of Earth? The answer is that Earth is pulled toward the Moon more than the water on the far side, because Earth is closer to the Moon. So the water on the side of Earth closest to the Moon is pulled away from Earth, and Earth is pulled away from water on the far side. As Earth rotates, the tidal bulge (an effect of the tidal forces between an orbiting natural satellite and the primary planet that it orbits) keeps its orientation with the Moon. Thus there are two tides per day (the actual tidal period is about 12 hours and 25.2 minutes), because the Moon moves in its orbit each day as well).



The Moon causes ocean tides by attracting the water on the near side more than Earth, and by attracting Earth more than the water on the far side. The distances and sizes are not to scale. For this simplified representation of the Earth-Moon system, there are two high and two low tides per day at any location, because Earth rotates under the tidal bulge.

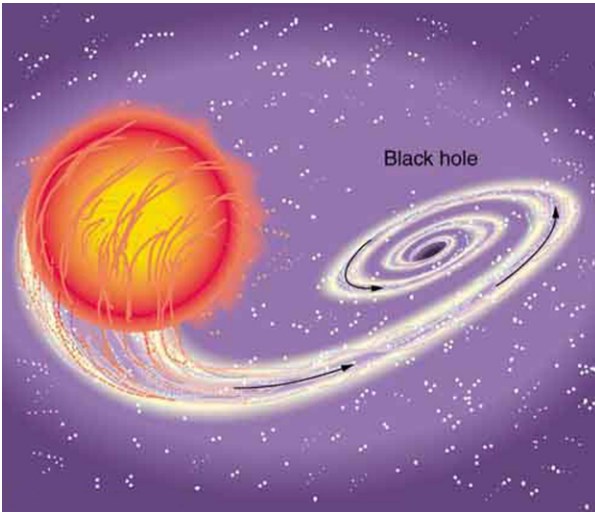
The Sun also affects tides, although it has about half the effect of the Moon. However, the largest tides, called spring tides, occur when Earth, the Moon, and the Sun are aligned. The smallest tides, called neap tides, occur when the Sun is at a 90° angle to the Earth-Moon alignment.



(a, b) Spring tides: The highest tides occur when Earth, the Moon, and the Sun are aligned. (c) Neap tide: The lowest tides occur when the Sun lies at 90° to the Earth-Moon alignment. Note that this figure is not drawn to scale.

Tides are not unique to Earth but occur in many astronomical systems. The most extreme tides occur where the gravitational force is the strongest and varies most rapidly, such as near black holes (see [\[link\]](#)). A few likely candidates for black holes have been observed in our galaxy. These have masses greater than the Sun but have diameters only a few kilometers

across. The tidal forces near them are so great that they can actually tear matter from a companion star.



A black hole is an object with such strong gravity that not even light can escape it. This black hole was created by the supernova of one star in a two-star system. The tidal forces created by the black hole are so great that it tears matter from the companion star. This matter is compressed and heated as it is sucked into the black hole, creating light and X-rays observable from Earth.

”Weightlessness” and Microgravity

In contrast to the tremendous gravitational force near black holes is the apparent gravitational field experienced by astronauts orbiting Earth. What

is the effect of “weightlessness” upon an astronaut who is in orbit for months? Or what about the effect of weightlessness upon plant growth? Weightlessness doesn't mean that an astronaut is not being acted upon by the gravitational force. There is no “zero gravity” in an astronaut's orbit. The term just means that the astronaut is in free-fall, accelerating with the acceleration due to gravity. If an elevator cable breaks, the passengers inside will be in free fall and will experience weightlessness. You can experience short periods of weightlessness in some rides in amusement parks.



Astronauts experiencing weightlessness on board the International Space Station. (credit: NASA)

Microgravity refers to an environment in which the apparent net acceleration of a body is small compared with that produced by Earth at its surface. Many interesting biology and physics topics have been studied over the past three decades in the presence of microgravity. Of immediate concern is the effect on astronauts of extended times in outer space, such as at the International Space Station. Researchers have observed that muscles will atrophy (waste away) in this environment. There is also a

corresponding loss of bone mass. Study continues on cardiovascular adaptation to space flight. On Earth, blood pressure is usually higher in the feet than in the head, because the higher column of blood exerts a downward force on it, due to gravity. When standing, 70% of your blood is below the level of the heart, while in a horizontal position, just the opposite occurs. What difference does the absence of this pressure differential have upon the heart?

Some findings in human physiology in space can be clinically important to the management of diseases back on Earth. On a somewhat negative note, spaceflight is known to affect the human immune system, possibly making the crew members more vulnerable to infectious diseases. Experiments flown in space also have shown that some bacteria grow faster in microgravity than they do on Earth. However, on a positive note, studies indicate that microbial antibiotic production can increase by a factor of two in space-grown cultures. One hopes to be able to understand these mechanisms so that similar successes can be achieved on the ground. In another area of physics space research, inorganic crystals and protein crystals have been grown in outer space that have much higher quality than any grown on Earth, so crystallography studies on their structure can yield much better results.

Plants have evolved with the stimulus of gravity and with gravity sensors. Roots grow downward and shoots grow upward. Plants might be able to provide a life support system for long duration space missions by regenerating the atmosphere, purifying water, and producing food. Some studies have indicated that plant growth and development are not affected by gravity, but there is still uncertainty about structural changes in plants grown in a microgravity environment.

The Cavendish Experiment: Then and Now

As previously noted, the universal gravitational constant G is determined experimentally. This definition was first done accurately by Henry Cavendish (1731–1810), an English scientist, in 1798, more than 100 years after Newton published his universal law of gravitation. The measurement of G is very basic and important because it determines the strength of one

of the four forces in nature. Cavendish's experiment was very difficult because he measured the tiny gravitational attraction between two ordinary-sized masses (tens of kilograms at most), using apparatus like that in [\[link\]](#). Remarkably, his value for G differs by less than 1% from the best modern value.

One important consequence of knowing G was that an accurate value for Earth's mass could finally be obtained. This was done by measuring the acceleration due to gravity as accurately as possible and then calculating the mass of Earth M from the relationship Newton's universal law of gravitation gives

Equation:

$$mg = G \frac{mM}{r^2},$$

where m is the mass of the object, M is the mass of Earth, and r is the distance to the center of Earth (the distance between the centers of mass of the object and Earth). See [\[link\]](#). The mass m of the object cancels, leaving an equation for g :

Equation:

$$g = G \frac{M}{r^2}.$$

Rearranging to solve for M yields

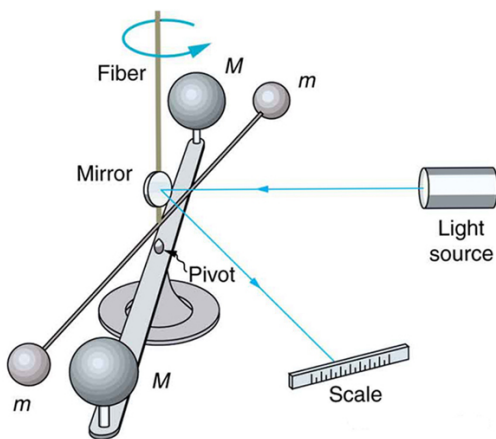
Equation:

$$M = \frac{gr^2}{G}.$$

So M can be calculated because all quantities on the right, including the radius of Earth r , are known from direct measurements. We shall see in [Satellites and Kepler's Laws: An Argument for Simplicity](#) that knowing G also allows for the determination of astronomical masses. Interestingly, of

all the fundamental constants in physics, G is by far the least well determined.

The Cavendish experiment is also used to explore other aspects of gravity. One of the most interesting questions is whether the gravitational force depends on substance as well as mass—for example, whether one kilogram of lead exerts the same gravitational pull as one kilogram of water. A Hungarian scientist named Roland von Eötvös pioneered this inquiry early in the 20th century. He found, with an accuracy of five parts per billion, that the gravitational force does not depend on the substance. Such experiments continue today, and have improved upon Eötvös' measurements. Cavendish-type experiments such as those of Eric Adelberger and others at the University of Washington, have also put severe limits on the possibility of a fifth force and have verified a major prediction of general relativity—that gravitational energy contributes to rest mass. Ongoing measurements there use a torsion balance and a parallel plate (not spheres, as Cavendish used) to examine how Newton's law of gravitation works over sub-millimeter distances. On this small-scale, do gravitational effects depart from the inverse square law? So far, no deviation has been observed.



Cavendish used an apparatus like this to measure the gravitational attraction between the two suspended spheres (

m) and the two on the stand (M) by observing the amount of torsion (twisting) created in the fiber. Distance between the masses can be varied to check the dependence of the force on distance. Modern experiments of this type continue to explore gravity.

Test Prep for AP Courses

Exercise:

Problem:

Jupiter has a mass approximately 300 times greater than Earth's and a radius about 11 times greater. How will the gravitational acceleration at the surface of Jupiter compare to that at the surface of the Earth?

- a. Greater
- b. Less
- c. About the same
- d. Not enough information

Solution:

(a)

Exercise:

Problem:

Given Newton's universal law of gravitation (Equation 6.40), under what circumstances is the force due to gravity maximized?

Exercise:

Problem: In the formula $g = \frac{GM}{r^2}$, what does G represent?

- a. The acceleration due to gravity
- b. A gravitational constant that is the same everywhere in the universe
- c. A gravitational constant that is inversely proportional to the radius
- d. The factor by which you multiply the inertial mass to obtain the gravitational mass

Solution:

(b)

Exercise:**Problem:**

Saturn's moon Titan has a radius of 2.58×10^6 m and a measured gravitational field of 1.35 m/s^2 . What is its mass?

Exercise:**Problem:**

A recently discovered planet has a mass twice as great as Earth's and a radius twice as large as Earth's. What will be the approximate size of its gravitational field?

- a. 19 m/s^2
- b. 4.9 m/s^2
- c. 2.5 m/s^2

d. 9.8 m/s^2

Solution:

(b)

Exercise:

Problem:

4. Earth is $1.5 \times 10^{11} \text{ m}$ from the Sun. Mercury is $5.7 \times 10^{10} \text{ m}$ from the Sun. How does the gravitational field of the Sun on Mercury (g_{SM}) compare to the gravitational field of the Sun on Earth (g_{SE})?

Section Summary

- Newton's universal law of gravitation: Every particle in the universe attracts every other particle with a force along a line joining them. The force is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. In equation form, this is

Equation:

$$F = G \frac{mM}{r^2},$$

where F is the magnitude of the gravitational force. G is the gravitational constant, given by $G = 6.673 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2$.

- Newton's law of gravitation applies universally.

Conceptual Questions

Exercise:

Problem:

Action at a distance, such as is the case for gravity, was once thought to be illogical and therefore untrue. What is the ultimate determinant of the truth in physics, and why was this action ultimately accepted?

Exercise:**Problem:**

Two friends are having a conversation. Anna says a satellite in orbit is in freefall because the satellite keeps falling toward Earth. Tom says a satellite in orbit is not in freefall because the acceleration due to gravity is not 9.80 m/s^2 . Who do you agree with and why?

Exercise:**Problem:**

Draw a free body diagram for a satellite in an elliptical orbit showing why its speed increases as it approaches its parent body and decreases as it moves away.

Exercise:**Problem:**

Newton's laws of motion and gravity were among the first to convincingly demonstrate the underlying simplicity and unity in nature. Many other examples have since been discovered, and we now expect to find such underlying order in complex situations. Is there proof that such order will always be found in new explorations?

Problem Exercises**Exercise:**

Problem:

- (a) Calculate Earth's mass given the acceleration due to gravity at the North Pole is 9.830 m/s^2 and the radius of the Earth is 6371 km from center to pole.
- (b) Compare this with the accepted value of $5.979 \times 10^{24} \text{ kg}$.
-

Solution:

- a) $5.979 \times 10^{24} \text{ kg}$
- b) This is identical to the best value to three significant figures.

Exercise:**Problem:**

- (a) Calculate the magnitude of the acceleration due to gravity on the surface of Earth due to the Moon.
- (b) Calculate the magnitude of the acceleration due to gravity at Earth due to the Sun.
- (c) Take the ratio of the Moon's acceleration to the Sun's and comment on why the tides are predominantly due to the Moon in spite of this number.

Exercise:**Problem:**

- (a) What is the acceleration due to gravity on the surface of the Moon?
- (b) On the surface of Mars? The mass of Mars is $6.418 \times 10^{23} \text{ kg}$ and its radius is $3.38 \times 10^6 \text{ m}$.
-

Solution:

- a) 1.62 m/s^2

b) 3.75 m/s^2

Exercise:

Problem:

(a) Calculate the acceleration due to gravity on the surface of the Sun.

(b) By what factor would your weight increase if you could stand on the Sun? (Never mind that you cannot.)

Exercise:

Problem:

The Moon and Earth rotate about their common center of mass, which is located about 4700 km from the center of Earth. (This is 1690 km below the surface.)

(a) Calculate the magnitude of the acceleration due to the Moon's gravity at that point.

(b) Calculate the magnitude of the centripetal acceleration of the center of Earth as it rotates about that point once each lunar month (about 27.3 d) and compare it with the acceleration found in part (a).

Comment on whether or not they are equal and why they should or should not be.

Solution:

a) $3.42 \times 10^{-5} \text{ m/s}^2$

b) $3.34 \times 10^{-5} \text{ m/s}^2$

The values are nearly identical. One would expect the gravitational force to be the same as the centripetal force at the core of the system.

Exercise:

Problem: Solve part (b) of [\[link\]](#) using $a_c = v^2/r$.

Exercise:**Problem:**

Astrology, that unlikely and vague pseudoscience, makes much of the position of the planets at the moment of one's birth. The only known force a planet exerts on Earth is gravitational.

(a) Calculate the magnitude of the gravitational force exerted on a 4.20 kg baby by a 100 kg father 0.200 m away at birth (he is assisting, so he is close to the child).

(b) Calculate the magnitude of the force on the baby due to Jupiter if it is at its closest distance to Earth, some 6.29×10^{11} m away. How does the force of Jupiter on the baby compare to the force of the father on the baby? Other objects in the room and the hospital building also exert similar gravitational forces. (Of course, there could be an unknown force acting, but scientists first need to be convinced that there is even an effect, much less that an unknown force causes it.)

Solution:

a) 7.01×10^{-7} N

b) 1.35×10^{-6} N, 0.521

Exercise:**Problem:**

The existence of the dwarf planet Pluto was proposed based on irregularities in Neptune's orbit. Pluto was subsequently discovered near its predicted position. But it now appears that the discovery was fortuitous, because Pluto is small and the irregularities in Neptune's orbit were not well known. To illustrate that Pluto has a minor effect on the orbit of Neptune compared with the closest planet to Neptune:

(a) Calculate the acceleration due to gravity at Neptune due to Pluto when they are 4.50×10^{12} m apart, as they are at present. The mass of

Pluto is 1.4×10^{22} kg.

(b) Calculate the acceleration due to gravity at Neptune due to Uranus, presently about 2.50×10^{12} m apart, and compare it with that due to Pluto. The mass of Uranus is 8.62×10^{25} kg.

Exercise:

Problem:

(a) The Sun orbits the Milky Way galaxy once each 2.60×10^8 y, with a roughly circular orbit averaging 3.00×10^4 light years in radius. (A light year is the distance traveled by light in 1 y.) Calculate the centripetal acceleration of the Sun in its galactic orbit. Does your result support the contention that a nearly inertial frame of reference can be located at the Sun?

(b) Calculate the average speed of the Sun in its galactic orbit. Does the answer surprise you?

Solution:

a) $1.66 \times 10^{-10} \text{ m/s}^2$

b) $2.17 \times 10^5 \text{ m/s}$

Exercise:

Problem: Unreasonable Result

A mountain 10.0 km from a person exerts a gravitational force on him equal to 2.00% of his weight.

(a) Calculate the mass of the mountain.

(b) Compare the mountain's mass with that of Earth.

(c) What is unreasonable about these results?

(d) Which premises are unreasonable or inconsistent? (Note that accurate gravitational measurements can easily detect the effect of nearby mountains and variations in local geology.)

Solution:

a) 2.94×10^{17} kg

b) 4.92×10^{-8}

of the Earth's mass.

c) The mass of the mountain and its fraction of the Earth's mass are too great.

d) The gravitational force assumed to be exerted by the mountain is too great.

Glossary

gravitational constant, G

a proportionality factor used in the equation for Newton's universal law of gravitation; it is a universal constant—that is, it is thought to be the same everywhere in the universe

center of mass

the point where the entire mass of an object can be thought to be concentrated

microgravity

an environment in which the apparent net acceleration of a body is small compared with that produced by Earth at its surface

Newton's universal law of gravitation

every particle in the universe attracts every other particle with a force along a line joining them; the force is directly proportional to the

product of their masses and inversely proportional to the square of the distance between them

Satellites and Kepler's Laws: An Argument for Simplicity

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- State Kepler's laws of planetary motion.
- Derive Kepler's third law for circular orbits.
- Discuss the Ptolemaic model of the universe.

Examples of gravitational orbits abound. Hundreds of artificial satellites orbit Earth together with thousands of pieces of debris. The Moon's orbit about Earth has intrigued humans from time immemorial. The orbits of planets, asteroids, meteors, and comets about the Sun are no less interesting. If we look further, we see almost unimaginable numbers of stars, galaxies, and other celestial objects orbiting one another and interacting through gravity.

All these motions are governed by gravitational force, and it is possible to describe them to various degrees of precision. Precise descriptions of complex systems must be made with large computers. However, we can describe an important class of orbits without the use of computers, and we shall find it instructive to study them. These orbits have the following characteristics:

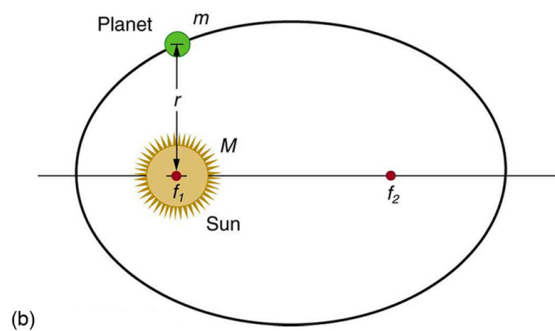
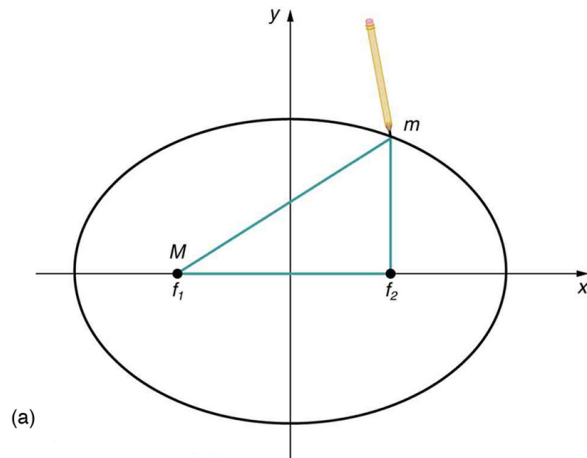
1. *A small mass m orbits a much larger mass M .* This allows us to view the motion as if M were stationary—in fact, as if from an inertial frame of reference placed on M —without significant error. Mass m is the satellite of M , if the orbit is gravitationally bound.
2. *The system is isolated from other masses.* This allows us to neglect any small effects due to outside masses.

The conditions are satisfied, to good approximation, by Earth's satellites (including the Moon), by objects orbiting the Sun, and by the satellites of other planets. Historically, planets were studied first, and there is a classical set of three laws, called Kepler's laws of planetary motion, that describe the orbits of all bodies satisfying the two previous conditions (not just planets in our solar system). These descriptive laws are named for the German astronomer Johannes Kepler (1571–1630), who devised them after careful study (over some 20 years) of a large amount of meticulously recorded observations of planetary motion done by Tycho Brahe (1546–1601). Such careful collection and detailed recording of methods and data are hallmarks of good science. Data constitute the evidence from which new interpretations and meanings can be constructed.

Kepler's Laws of Planetary Motion

Kepler's First Law

The orbit of each planet about the Sun is an ellipse with the Sun at one focus.



(a) An ellipse is a closed curve such that the sum of the distances from a point on the curve to the two foci (f_1 and f_2) is a constant. You can draw an ellipse as shown by putting a pin at each focus, and then placing a string around a pencil and the pins and tracing a line on paper. A circle is a special case of an ellipse in which the two foci coincide (thus any point on the circle is the same distance from the center). (b) For any closed gravitational orbit, m follows an elliptical path with M at one focus. Kepler's first law states this fact for planets orbiting the Sun.

Kepler's Second Law

Each planet moves so that an imaginary line drawn from the Sun to the planet sweeps out equal areas in equal times (see [link](#)).

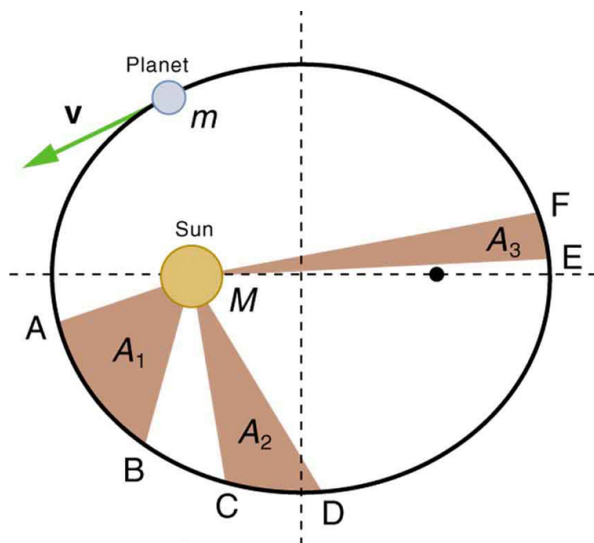
Kepler's Third Law

The ratio of the squares of the periods of any two planets about the Sun is equal to the ratio of the cubes of their average distances from the Sun. In equation form, this is

Equation:

$$\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3},$$

where T is the period (time for one orbit) and r is the average radius. This equation is valid only for comparing two small masses orbiting the same large one. Most importantly, this is a descriptive equation only, giving no information as to the cause of the equality.



The shaded regions have equal areas. It takes equal times for m to go from A to B, from C to D, and from E to F. The mass m moves fastest when it is closest to M . Kepler's second law was originally devised for planets orbiting the Sun, but it has broader validity.

Note again that while, for historical reasons, Kepler's laws are stated for planets orbiting the Sun, they are actually valid for all bodies satisfying the two previously stated conditions.

Example:**Find the Time for One Orbit of an Earth Satellite**

Given that the Moon orbits Earth each 27.3 d and that it is an average distance of 3.84×10^8 m from the center of Earth, calculate the period of an artificial satellite orbiting at an average altitude of 1500 km above Earth's surface.

Strategy

The period, or time for one orbit, is related to the radius of the orbit by

Kepler's third law, given in mathematical form in $\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3}$. Let us use the

subscript 1 for the Moon and the subscript 2 for the satellite. We are asked to find T_2 . The given information tells us that the orbital radius of the Moon is $r_1 = 3.84 \times 10^8$ m, and that the period of the Moon is $T_1 = 27.3$ d. The height of the artificial satellite above Earth's surface is given, and so we must add the radius of Earth (6380 km) to get

$r_2 = (1500 + 6380)$ km = 7880 km. Now all quantities are known, and so T_2 can be found.

Solution

Kepler's third law is

Equation:

$$\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3}.$$

To solve for T_2 , we cross-multiply and take the square root, yielding
Equation:

$$T_2^2 = T_1^2 \left(\frac{r_2}{r_1} \right)^3$$

Equation:

$$T_2 = T_1 \left(\frac{r_2}{r_1} \right)^{3/2}.$$

Substituting known values yields
Equation:

$$\begin{aligned} T_2 &= 27.3 \text{ d} \times \frac{24.0 \text{ h}}{\text{d}} \times \left(\frac{7880 \text{ km}}{3.84 \times 10^5 \text{ km}} \right)^{3/2} \\ &= 1.93 \text{ h.} \end{aligned}$$

Discussion This is a reasonable period for a satellite in a fairly low orbit. It is interesting that any satellite at this altitude will orbit in the same amount of time. This fact is related to the condition that the satellite's mass is small compared with that of Earth.

People immediately search for deeper meaning when broadly applicable laws, like Kepler's, are discovered. It was Newton who took the next giant step when he proposed the law of universal gravitation. While Kepler was able to discover *what* was happening, Newton discovered that gravitational force was the cause.

Derivation of Kepler's Third Law for Circular Orbits

We shall derive Kepler's third law, starting with Newton's laws of motion and his universal law of gravitation. The point is to demonstrate that the force of gravity is the cause for Kepler's laws (although we will only derive the third one).

Let us consider a circular orbit of a small mass m around a large mass M , satisfying the two conditions stated at the beginning of this section. Gravity supplies the centripetal force to mass m . Starting with Newton's second law applied to circular motion,

Equation:

$$F_{\text{net}} = ma_c = m \frac{v^2}{r}.$$

The net external force on mass m is gravity, and so we substitute the force of gravity for F_{net} :

Equation:

$$G \frac{mM}{r^2} = m \frac{v^2}{r}.$$

The mass m cancels, yielding

Equation:

$$G \frac{M}{r} = v^2.$$

The fact that m cancels out is another aspect of the oft-noted fact that at a given location all masses fall with the same acceleration. Here we see that at a given orbital radius r , all masses orbit at the same speed. (This was implied by the result of the preceding worked example.) Now, to get at Kepler's third law, we must get the period T into the equation. By definition, period T is the time for one complete orbit. Now the average speed v is the circumference divided by the period—that is,

Equation:

$$v = \frac{2\pi r}{T}.$$

Substituting this into the previous equation gives

Equation:

$$G \frac{M}{r} = \frac{4\pi^2 r^2}{T^2}.$$

Solving for T^2 yields

Equation:

$$T^2 = \frac{4\pi^2}{GM} r^3.$$

Using subscripts 1 and 2 to denote two different satellites, and taking the ratio of the last equation for satellite 1 to satellite 2 yields

Equation:

$$\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3}.$$

This is Kepler's third law. Note that Kepler's third law is valid only for comparing satellites of the same parent body, because only then does the mass of the parent body M cancel.

Now consider what we get if we solve $T^2 = \frac{4\pi^2}{GM} r^3$ for the ratio r^3/T^2 . We obtain a relationship that can be used to determine the mass M of a parent body from the orbits of its satellites:

Equation:

$$\frac{r^3}{T^2} = \frac{G}{4\pi^2} M.$$

If r and T are known for a satellite, then the mass M of the parent can be calculated. This principle has been used extensively to find the masses of heavenly bodies that have satellites. Furthermore, the ratio r^3/T^2 should be a constant for all satellites of the same parent body (because $r^3/T^2 = GM/4\pi^2$). (See [\[link\]](#)).

It is clear from [\[link\]](#) that the ratio of r^3/T^2 is constant, at least to the third digit, for all listed satellites of the Sun, and for those of Jupiter. Small variations in that ratio have two causes—uncertainties in the r and T data, and perturbations of the orbits due to other bodies. Interestingly, those perturbations can be—and have been—used to predict the location of new planets and moons. This is another verification of Newton's universal law of gravitation.

Note:**Making Connections: General Relativity and Mercury**

Newton's universal law of gravitation is modified by Einstein's general theory of relativity, as we shall see in [Particle Physics](#). Newton's gravity is not seriously in error—it was and still is an extremely good approximation for most situations. Einstein's modification is most noticeable in extremely large gravitational fields, such as near black holes. However, general relativity also explains such phenomena as small but long-known deviations of the orbit of the planet Mercury from classical predictions.

The Case for Simplicity

The development of the universal law of gravitation by Newton played a pivotal role in the history of ideas. While it is beyond the scope of this text to cover that history in any detail, we note some important points. The definition of planet set in 2006 by the International Astronomical Union (IAU) states that in the solar system, a planet is a celestial body that:

1. is in orbit around the Sun,
2. has sufficient mass to assume hydrostatic equilibrium and

3. has cleared the neighborhood around its orbit.

A non-satellite body fulfilling only the first two of the above criteria is classified as “dwarf planet.”

In 2006, Pluto was demoted to a ‘dwarf planet’ after scientists revised their definition of what constitutes a “true” planet.

Parent	Satellite	Average orbital radius $r(\text{km})$	Period $T(\text{y})$	$r^3 / T^2 (\text{km}^3 / \text{y}^2)$
Earth	Moon	3.84×10^5	0.07481	1.01×10^{19}
Sun	Mercury	5.79×10^7	0.2409	3.34×10^{24}
	Venus	1.082×10^8	0.6150	3.35×10^{24}
	Earth	1.496×10^8	1.000	3.35×10^{24}
	Mars	2.279×10^8	1.881	3.35×10^{24}

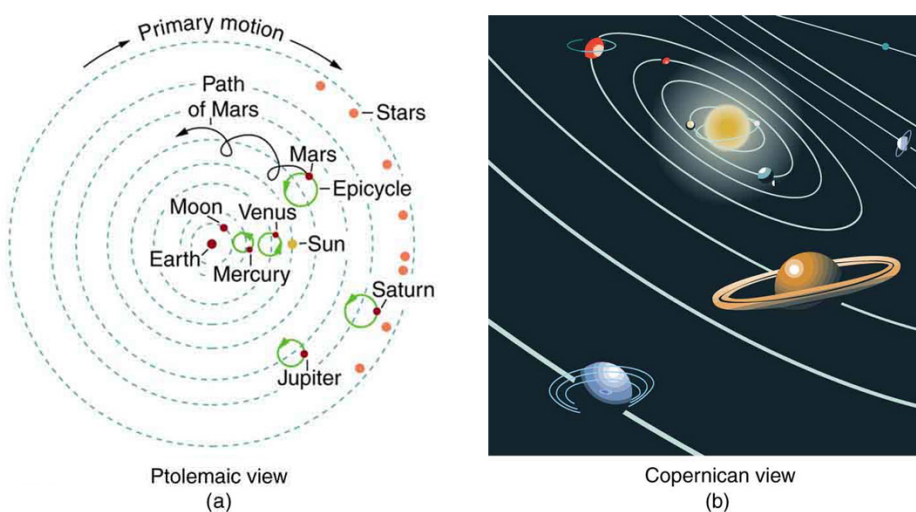
Parent	Satellite	Average orbital radius $r(\text{km})$	Period $T(\text{y})$	$r^3 / T^2 (\text{km}^3 / \text{y}^2)$
	Jupiter	7.783×10^8	11.86	3.35×10^{24}
	Saturn	1.427×10^9	29.46	3.35×10^{24}
	Neptune	4.497×10^9	164.8	3.35×10^{24}
	Pluto	5.90×10^9	248.3	3.33×10^{24}
Jupiter	Io	4.22×10^5	0.00485 (1.77 d)	3.19×10^{21}
	Europa	6.71×10^5	0.00972 (3.55 d)	3.20×10^{21}
	Ganymede	1.07×10^6	0.0196 (7.16 d)	3.19×10^{21}
	Callisto	1.88×10^6	0.0457 (16.19 d)	3.20×10^{21}

Orbital Data and Kepler's Third Law

The universal law of gravitation is a good example of a physical principle that is very broadly applicable. That single equation for the gravitational force describes all situations in which gravity acts. It gives a cause for a vast number of effects, such as the orbits of the planets and moons in the solar system. It epitomizes the underlying unity and simplicity of physics.

Before the discoveries of Kepler, Copernicus, Galileo, Newton, and others, the solar system was thought to revolve around Earth as shown in [\[link\]](#)(a). This is called the Ptolemaic view, for the Greek philosopher who lived in the second century AD. This model is characterized by a list of facts for the motions of planets with no cause and effect explanation. There tended to be a different rule for each heavenly body and a general lack of simplicity.

[\[link\]](#)(b) represents the modern or Copernican model. In this model, a small set of rules and a single underlying force explain not only all motions in the solar system, but all other situations involving gravity. The breadth and simplicity of the laws of physics are compelling. As our knowledge of nature has grown, the basic simplicity of its laws has become ever more evident.



(a) The Ptolemaic model of the universe has Earth at the center with the Moon, the planets, the Sun, and the stars revolving about it in complex superpositions of circular paths. This geocentric model, which can be made progressively more accurate by adding more

circles, is purely descriptive, containing no hints as to what are the causes of these motions. (b) The Copernican model has the Sun at the center of the solar system. It is fully explained by a small number of laws of physics, including Newton's universal law of gravitation.

Section Summary

- Kepler's laws are stated for a small mass m orbiting a larger mass M in near-isolation. Kepler's laws of planetary motion are then as follows:

Kepler's first law

The orbit of each planet about the Sun is an ellipse with the Sun at one focus.

Kepler's second law

Each planet moves so that an imaginary line drawn from the Sun to the planet sweeps out equal areas in equal times.

Kepler's third law

The ratio of the squares of the periods of any two planets about the Sun is equal to the ratio of the cubes of their average distances from the Sun:

Equation:

$$\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3},$$

where T is the period (time for one orbit) and r is the average radius of the orbit.

- The period and radius of a satellite's orbit about a larger body M are related by

Equation:

$$T^2 = \frac{4\pi^2}{GM} r^3$$

or

Equation:

$$\frac{r^3}{T^2} = \frac{G}{4\pi^2} M.$$

Conceptual Questions

Exercise:

Problem:

In what frame(s) of reference are Kepler's laws valid? Are Kepler's laws purely descriptive, or do they contain causal information?

Problem Exercises

Exercise:

Problem:

A geosynchronous Earth satellite is one that has an orbital period of precisely 1 day. Such orbits are useful for communication and weather observation because the satellite remains above the same point on Earth (provided it orbits in the equatorial plane in the same direction as Earth's rotation). Calculate the radius of such an orbit based on the data for the moon in [\[link\]](#).

Exercise:

Problem:

Calculate the mass of the Sun based on data for Earth's orbit and compare the value obtained with the Sun's actual mass.

Solution:

$$1.98 \times 10^{30} \text{ kg}$$

Exercise:**Problem:**

Find the mass of Jupiter based on data for the orbit of one of its moons, and compare your result with its actual mass.

Exercise:**Problem:**

Find the ratio of the mass of Jupiter to that of Earth based on data in [\[link\]](#).

Solution:

$$\frac{M_J}{M_E} = 316$$

Exercise:**Problem:**

Astronomical observations of our Milky Way galaxy indicate that it has a mass of about 8.0×10^{11} solar masses. A star orbiting on the galaxy's periphery is about 6.0×10^4 light years from its center. (a) What should the orbital period of that star be? (b) If its period is 6.0×10^7 instead, what is the mass of the galaxy? Such calculations are used to imply the existence of “dark matter” in the universe and have indicated, for example, the existence of very massive black holes at the centers of some galaxies.

Exercise:

Problem: Integrated Concepts

Space debris left from old satellites and their launchers is becoming a hazard to other satellites. (a) Calculate the speed of a satellite in an orbit 900 km above Earth's surface. (b) Suppose a loose rivet is in an orbit of the same radius that intersects the satellite's orbit at an angle of 90° relative to Earth. What is the velocity of the rivet relative to the satellite just before striking it? (c) Given the rivet is 3.00 mm in size, how long will its collision with the satellite last? (d) If its mass is 0.500 g, what is the average force it exerts on the satellite? (e) How much energy in joules is generated by the collision? (The satellite's velocity does not change appreciably, because its mass is much greater than the rivet's.)

Solution:

a) $7.4 \times 10^3 \text{ m/s}$

b) $1.05 \times 10^3 \text{ m/s}$

c) $2.86 \times 10^{-7} \text{ s}$

d) $1.84 \times 10^7 \text{ N}$

e) $2.76 \times 10^4 \text{ J}$

Exercise:**Problem: Unreasonable Results**

(a) Based on Kepler's laws and information on the orbital characteristics of the Moon, calculate the orbital radius for an Earth satellite having a period of 1.00 h. (b) What is unreasonable about this result? (c) What is unreasonable or inconsistent about the premise of a 1.00 h orbit?

Solution:

a) $5.08 \times 10^3 \text{ km}$

- b) This radius is unreasonable because it is less than the radius of earth.
- c) The premise of a one-hour orbit is inconsistent with the known radius of the earth.

Exercise:

Problem: Construct Your Own Problem

On February 14, 2000, the NEAR spacecraft was successfully inserted into orbit around Eros, becoming the first artificial satellite of an asteroid. Construct a problem in which you determine the orbital speed for a satellite near Eros. You will need to find the mass of the asteroid and consider such things as a safe distance for the orbit. Although Eros is not spherical, calculate the acceleration due to gravity on its surface at a point an average distance from its center of mass. Your instructor may also wish to have you calculate the escape velocity from this point on Eros.

Connection for AP® Courses
class="introduction"

How many
forms of
energy can
you identify
in this
photograph
of a wind
farm in
Iowa?
(credit:
Jürgen from
Sandesneben
, Germany,
Wikimedia
Commons)



Energy plays an essential role both in everyday events and in scientific phenomena. You can no doubt name many forms of energy, from that provided by our foods to the energy we use to run our cars and the sunlight that warms us on the beach. You can also cite examples of what people call “energy” that may not be scientific, such as someone having an energetic personality. Not only does energy have many interesting forms, it is

involved in almost all phenomena, and is one of the most important concepts of physics.

There is no simple and accurate scientific definition for energy. Energy is characterized by its many forms and the fact that it is conserved. We can loosely define energy as the ability to do **work**, admitting that in some circumstances not all energy is available to do work. Because of the association of energy with work, we begin the chapter with a discussion of work. Work is intimately related to energy and how energy moves from one system to another or changes form. The work-energy theorem supports Big Idea 3, that interactions between objects are described by forces. In particular, exerting a force on an object may do work on it, changing its energy (Enduring Understanding 3.E). The work-energy theorem, introduced in this chapter, establishes the relationship between work done on an object by an external force and changes in the object's kinetic energy (Essential Knowledge 3.E.1).

Similarly, systems can do work on each other, supporting Big Idea 4, that interactions between systems can result in changes in those systems—in this case, changes in the total energy of the system (Enduring Understanding 4.C). The total energy of the system is the sum of its kinetic energy, potential energy, and microscopic internal energy (Essential Knowledge 4.C.1). In this chapter students learn how to calculate kinetic, gravitational, and elastic potential energy in order to determine the total mechanical energy of a system. The transfer of mechanical energy into or out of a system is equal to the work done on the system by an external force with a nonzero component parallel to the displacement (Essential Knowledge 4.C.2).

An important aspect of energy is that the total amount of energy in the universe is constant. Energy can change forms, but it cannot appear from nothing or disappear without a trace. Energy is thus one of a handful of physical quantities that we say is “conserved.” **Conservation of energy** (as physicists call the principle that energy can neither be created nor destroyed) is based on experiment. Even as scientists discovered new forms of energy, conservation of energy has always been found to apply. Perhaps the most dramatic example of this was supplied by Einstein when he

suggested that mass is equivalent to energy (his famous equation $E = mc^2$). This is one of the most important applications of Big Idea 5, that changes that occur as a result of interactions are constrained by conservation laws. Specifically, there are many situations where conservation of energy (Enduring Understanding 5.B) is both a useful concept and starting point for calculations related to the system. Note, however, that conservation doesn't necessarily mean that energy in a system doesn't change. Energy may be transferred into or out of the system, and the change must be equal to the amount transferred (Enduring Understanding 5.A). This may occur if there is an external force or a transfer between external objects and the system (Essential Knowledge 5.A.3). Energy is one of the fundamental quantities that are conserved for all systems (Essential Knowledge 5.A.2). The chapter introduces concepts of kinetic energy and potential energy. Kinetic energy is introduced as an energy of motion that can be changed by the amount of work done by an external force. Potential energy can only exist when objects interact with each other via conservative forces according to classical physics (Essential Knowledge 5.B.3). Because of this, a single object can only have kinetic energy and no potential energy (Essential Knowledge 5.B.1). The chapter also introduces the idea that the energy transfer is equal to the work done on the system by external forces and the rate of energy transfer is defined as power (Essential Knowledge 5.B.5).

From a societal viewpoint, energy is one of the major building blocks of modern civilization. Energy resources are key limiting factors to economic growth. The world use of energy resources, especially oil, continues to grow, with ominous consequences economically, socially, politically, and environmentally. We will briefly examine the world's energy use patterns at the end of this chapter.

The concepts in this chapter support:

Big Idea 3 The interactions of an object with other objects can be described by forces.

Enduring Understanding 3.E A force exerted on an object can change the kinetic energy of the object.

Essential Knowledge 3.E.1 The change in the kinetic energy of an object depends on the force exerted on the object and on the displacement of the object during the interval that the force is exerted.

Big Idea 4 Interactions between systems can result in changes in those systems.

Enduring Understanding 4.C Interactions with other objects or systems can change the total energy of a system.

Essential Knowledge 4.C.1 The energy of a system includes its kinetic energy, potential energy, and microscopic internal energy. Examples should include gravitational potential energy, elastic potential energy, and kinetic energy.

Essential Knowledge 4.C.2 Mechanical energy (the sum of kinetic and potential energy) is transferred into or out of a system when an external force is exerted on a system such that a component of the force is parallel to its displacement. The process through which the energy is transferred is called work.

Big Idea 5 Changes that occur as a result of interactions are constrained by conservation laws.

Enduring Understanding 5.A Certain quantities are conserved, in the sense that the changes of those quantities in a given system are always equal to the transfer of that quantity to or from the system by all possible interactions with other systems.

Essential Knowledge 5.A.2 For all systems under all circumstances, energy, charge, linear momentum, and angular momentum are conserved.

Essential Knowledge 5.A.3 An interaction can be either a force exerted by objects outside the system or the transfer of some quantity with objects outside the system.

Enduring Understanding 5.B The energy of a system is conserved.

Essential Knowledge 5.B.1 Classically, an object can only have kinetic energy since potential energy requires an interaction between two or more objects.

Essential Knowledge 5.B.3 A system with internal structure can have potential energy. Potential energy exists within a system if the objects within that system interact with conservative forces.

Essential Knowledge 5.B.5 Energy can be transferred by an external force exerted on an object or system that moves the object or system through a distance; this energy transfer is called work. Energy transfer in mechanical or electrical systems may occur at different rates. Power is defined as the rate of energy transfer into, out of, or within a system.

Work: The Scientific Definition

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain how an object must be displaced for a force on it to do work.
- Explain how relative directions of force and displacement of an object determine whether the work done on the object is positive, negative, or zero.

The information presented in this section supports the following AP® learning objectives and science practices:

- **5.B.5.1** The student is able to design an experiment and analyze data to examine how a force exerted on an object or system does work on the object or system as it moves through a distance. (S.P. 4.2, 5.1)
- **5.B.5.2** The student is able to design an experiment and analyze graphical data in which interpretations of the area under a force-distance curve are needed to determine the work done on or by the object or system. (S.P. 4.5, 5.1)
- **5.B.5.3** The student is able to predict and calculate from graphical data the energy transfer to or work done on an object or system from information about a force exerted on the object or system through a distance. (S.P. 1.5, 2.2, 6.4)

What It Means to Do Work

The scientific definition of work differs in some ways from its everyday meaning. Certain things we think of as hard work, such as writing an exam or carrying a heavy load on level ground, are not work as defined by a scientist. The scientific definition of work reveals its relationship to energy—whenever work is done, energy is transferred.

For work, in the scientific sense, to be done on an object, a force must be exerted on that object and there must be displacement of that object in the direction of the force.

Formally, the **work** done on a system by a constant force is defined to be *the product of the component of the force in the direction of motion and the distance through which the force acts*. For a constant force, this is expressed in equation form as

Equation:

$$W = |\mathbf{F}| (\cos \theta) |\mathbf{d}|,$$

where W is work, \mathbf{d} is the displacement of the system, and θ is the angle between the force vector \mathbf{F} and the displacement vector \mathbf{d} , as in [\[link\]](#). We can also write this as

Equation:

$$W = Fd \cos \theta.$$

To find the work done on a system that undergoes motion that is not one-way or that is in two or three dimensions, we divide the motion into one-way one-dimensional segments and add up the work done over each segment.

Note:

What is Work?

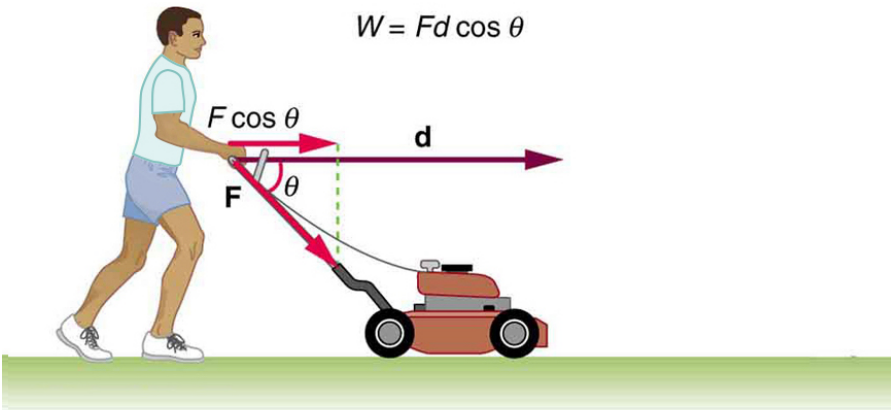
The work done on a system by a constant force is *the product of the component of the force in the direction of motion times the distance*

through which the force acts. For one-way motion in one dimension, this is expressed in equation form as

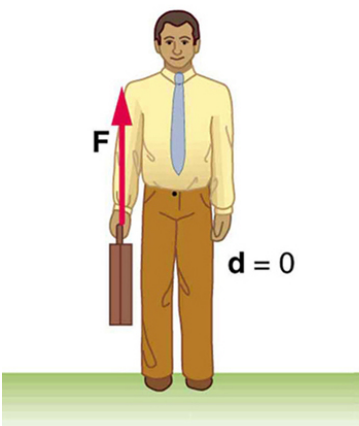
Equation:

$$W = Fd \cos \theta,$$

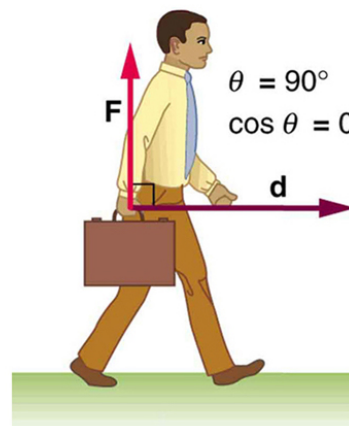
where W is work, F is the magnitude of the force on the system, d is the magnitude of the displacement of the system, and θ is the angle between the force vector \mathbf{F} and the displacement vector \mathbf{d} .



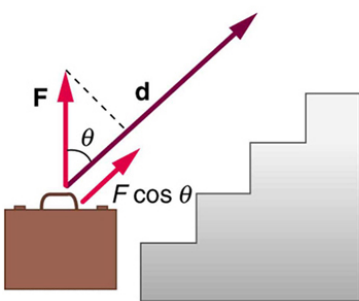
(a)



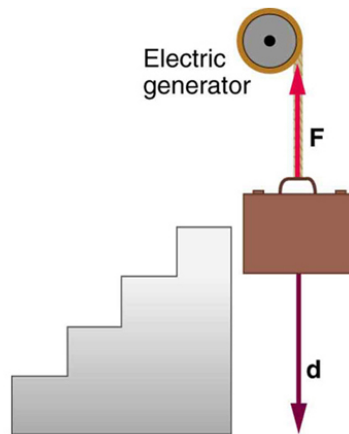
(b)



(c)



(d)



(e)

Examples of work. (a) The work done by the force \mathbf{F} on this lawn mower is $Fd \cos \theta$. Note that $F \cos \theta$ is the component of the force in the direction of motion. (b) A person holding a briefcase does no work on it, because there is no

displacement. No energy is transferred to or from the briefcase. (c) The person moving the briefcase horizontally at a constant speed does no work on it, and transfers no energy to it. (d) Work is done on the briefcase by carrying it up stairs at constant speed, because there is necessarily a component of force \mathbf{F} in the direction of the motion. Energy is transferred to the briefcase and could in turn be used to do work. (e) When the briefcase is lowered, energy is transferred out of the briefcase and into an electric generator. Here the work done on the briefcase by the generator is negative, removing energy from the briefcase, because \mathbf{F} and \mathbf{d} are in opposite directions.

To examine what the definition of work means, let us consider the other situations shown in [\[link\]](#). The person holding the briefcase in [\[link\]\(b\)](#) does no work, for example. Here $d = 0$, so $W = 0$. Why is it you get tired just holding a load? The answer is that your muscles are doing work against one another, *but they are doing no work on the system of interest* (the “briefcase-Earth system”—see [Gravitational Potential Energy](#) for more details). There must be displacement for work to be done, and there must be a component of the force in the direction of the motion. For example, the person carrying the briefcase on level ground in [\[link\]\(c\)](#) does no work on it, because the force is perpendicular to the motion. That is, $\cos 90^\circ = 0$, and so $W = 0$.

In contrast, when a force exerted on the system has a component in the direction of motion, such as in [\[link\]\(d\)](#), work is done—energy is transferred to the briefcase. Finally, in [\[link\]\(e\)](#), energy is transferred from the briefcase to a generator. There are two good ways to interpret this energy transfer. One interpretation is that the briefcase’s weight does work on the generator, giving it energy. The other interpretation is that the generator does negative work on the briefcase, thus removing energy from it. The drawing shows the latter, with the force from the generator upward

on the briefcase, and the displacement downward. This makes $\theta = 180^\circ$, and $\cos 180^\circ = -1$; therefore, W is negative.

Note:

Real World Connections: When Work Happens

Note that work as we define it is not the same as effort. You can push against a concrete wall all you want, but you won't move it. While the pushing represents effort on your part, the fact that you have not changed the wall's state in any way indicates that you haven't done work. If you did somehow push the wall over, this would indicate a change in the wall's state, and therefore you would have done work.

This can also be shown with [\[link\]](#)(a): as you push a lawnmower against friction, both you and friction are changing the lawnmower's state.

However, only the component of the force parallel to the movement is changing the lawnmower's state. The component perpendicular to the motion is trying to push the lawnmower straight into Earth; the lawnmower does not move into Earth, and therefore the lawnmower's state is not changing in the direction of Earth.

Similarly, in [\[link\]](#)(c), both your hand and gravity are exerting force on the briefcase. However, they are both acting perpendicular to the direction of motion, hence they are not changing the condition of the briefcase and do no work. However, if the briefcase were dropped, then its displacement would be parallel to the force of gravity, which would do work on it, changing its state (it would fall to the ground).

Calculating Work

Work and energy have the same units. From the definition of work, we see that those units are force times distance. Thus, in SI units, work and energy are measured in **newton-meters**. A newton-meter is given the special name **joule** (J), and $1 \text{ J} = 1 \text{ N} \cdot \text{m} = 1 \text{ kg} \cdot \text{m}^2/\text{s}^2$. One joule is not a large amount of energy; it would lift a small 100-gram apple a distance of about 1 meter.

Example:**Calculating the Work You Do to Push a Lawn Mower Across a Large Lawn**

How much work is done on the lawn mower by the person in [\[link\]](#)(a) if he exerts a constant force of 75.0 N at an angle 35° below the horizontal and pushes the mower 25.0 m on level ground? Convert the amount of work from joules to kilocalories and compare it with this person's average daily intake of 10,000 kJ (about 2400 kcal) of food energy. One *calorie* (1 cal) of heat is the amount required to warm 1 g of water by 1°C, and is equivalent to 4.184 J, while one *food calorie* (1 kcal) is equivalent to 4184 J.

Strategy

We can solve this problem by substituting the given values into the definition of work done on a system, stated in the equation $W = Fd \cos \theta$. The force, angle, and displacement are given, so that only the work W is unknown.

Solution

The equation for the work is

Equation:

$$W = Fd \cos \theta.$$

Substituting the known values gives

Equation:

$$\begin{aligned} W &= (75.0 \text{ N})(25.0 \text{ m}) \cos (35.0^\circ) \\ &= 1536 \text{ J} = 1.54 \times 10^3 \text{ J}. \end{aligned}$$

Converting the work in joules to kilocalories yields

$W = (1536 \text{ J})(1 \text{ kcal}/4184 \text{ J}) = 0.367 \text{ kcal}$. The ratio of the work done to the daily consumption is

Equation:

$$\frac{W}{2400 \text{ kcal}} = 1.53 \times 10^{-4}.$$

Discussion

This ratio is a tiny fraction of what the person consumes, but it is typical. Very little of the energy released in the consumption of food is used to do work. Even when we “work” all day long, less than 10% of our food energy intake is used to do work and more than 90% is converted to thermal energy or stored as chemical energy in fat.

Note:

Applying the Science Practices: Boxes on Floors

Plan and design an experiment to determine how much work you do on a box when you are pushing it over different floor surfaces. Make sure your experiment can help you answer the following questions: What happens on different surfaces? What happens if you take different routes across the same surface? Do you get different results with two people pushing on perpendicular surfaces of the box? What if you vary the mass in the box? Remember to think about both your effort in any given instant (a proxy for force exerted) and the total work you do. Also, when planning your experiments, remember that in any given set of trials you should only change one variable.

You should find that you have to exert more effort on surfaces that will create more friction with the box, though you might be surprised by which surfaces the box slides across easily. Longer routes result in your doing more work, even though the box ends up in the same place. Two people pushing on perpendicular sides do less work for their total effort, due to the forces and displacement not being parallel. A more massive box will take more effort to move.

Note:

Applying the Science Practices: Force-Displacement Diagrams

Suppose you are given two carts and a track to run them on, a motion detector, a force sensor, and a computer that can record the data from the two sensors. Plan and design an experiment to measure the work done on one of the carts, and compare your results to the work-energy theorem. Note that the motion detector can measure both displacement and velocity

versus time, while the force sensor measures force over time, and the carts have known masses. Recall that the work-energy theorem states that the work done on a system (force over displacement) should equal the change in kinetic energy. In your experimental design, describe and compare two possible ways to calculate the work done.

Sample Response: One possible technique is to set up the motion detector at one end of the track, and have the computer record both displacement and velocity over time. Then attach the force sensor to one of the carts, and use this cart, through the force sensor, to push the second cart toward the motion detector. Calculate the difference between the final and initial kinetic energies (the kinetic energies after and before the push), and compare this to the area of a graph of force versus displacement for the duration of the push. They should be the same.

Test Prep for AP Courses

Exercise:

Problem:

Given [\[link\]](#) about how much force does the rocket engine exert on the 3.0-kg payload?

Distance traveled with rocket engine firing (m)	Payload final velocity (m/s)
500	310
490	300
1020	450

Distance traveled with rocket engine firing (m)	Payload final velocity (m/s)
505	312

- a. 150 N
- b. 300 N
- c. 450 N
- d. 600 N

Solution:

(b)

Exercise:

Problem:

You have a cart track, a cart, several masses, and a position-sensing pulley. Design an experiment to examine how the force exerted on the cart does work as it moves through a distance.

Exercise:

Problem:

Look at [\[link\]](#)(c). You compress a spring by x , and then release it. Next you compress the spring by $2x$. How much more work did you do the second time than the first?

- a. Half as much
- b. The same
- c. Twice as much
- d. Four times as much

Solution:

(d)

Exercise:

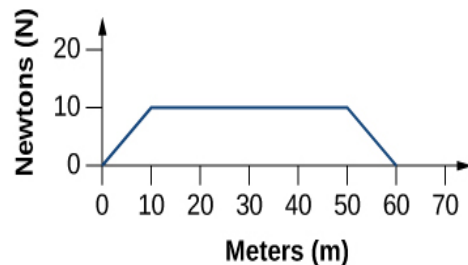
Problem:

You have a cart track, two carts, several masses, a position-sensing pulley, and a piece of carpet (a rough surface) that will fit over the track. Design an experiment to examine how the force exerted on the cart does work as the cart moves through a distance.

Exercise:

Problem:

A crane is lifting construction materials from the ground to an elevation of 60 m. Over the first 10 m, the motor linearly increases the force it exerts from 0 to 10 kN. It exerts that constant force for the next 40 m, and then winds down to 0 N again over the last 10 m, as shown in the figure. What is the total work done on the construction materials?



- a. 500 kJ
- b. 600 kJ
- c. 300 kJ
- d. 18 MJ

Solution:

(a)

Section Summary

- Work is the transfer of energy by a force acting on an object as it is displaced.
- The work W that a force \mathbf{F} does on an object is the product of the magnitude F of the force, times the magnitude d of the displacement, times the cosine of the angle θ between them. In symbols,

Equation:

$$W = Fd \cos \theta.$$

- The SI unit for work and energy is the joule (J), where $1 \text{ J} = 1 \text{ N} \cdot \text{m} = 1 \text{ kg} \cdot \text{m}^2/\text{s}^2$.
- The work done by a force is zero if the displacement is either zero or perpendicular to the force.
- The work done is positive if the force and displacement have the same direction, and negative if they have opposite direction.

Conceptual Questions

Exercise:

Problem:

Give an example of something we think of as work in everyday circumstances that is not work in the scientific sense. Is energy transferred or changed in form in your example? If so, explain how this is accomplished without doing work.

Exercise:

Problem:

Give an example of a situation in which there is a force and a displacement, but the force does no work. Explain why it does no work.

Exercise:

Problem:

Describe a situation in which a force is exerted for a long time but does no work. Explain.

Problems & Exercises**Exercise:****Problem:**

How much work does a supermarket checkout attendant do on a can of soup he pushes 0.600 m horizontally with a force of 5.00 N? Express your answer in joules and kilocalories.

Solution:

$$3.00 \text{ J} = 7.17 \times 10^{-4} \text{ kcal}$$

Exercise:**Problem:**

A 75.0-kg person climbs stairs, gaining 2.50 meters in height. Find the work done to accomplish this task.

Exercise:**Problem:**

(a) Calculate the work done on a 1500-kg elevator car by its cable to lift it 40.0 m at constant speed, assuming friction averages 100 N. (b) What is the work done on the lift by the gravitational force in this process? (c) What is the total work done on the lift?

Solution:

$$(a) 5.92 \times 10^5 \text{ J}$$

(b) $-5.88 \times 10^5 \text{ J}$

(c) The net force is zero.

Exercise:

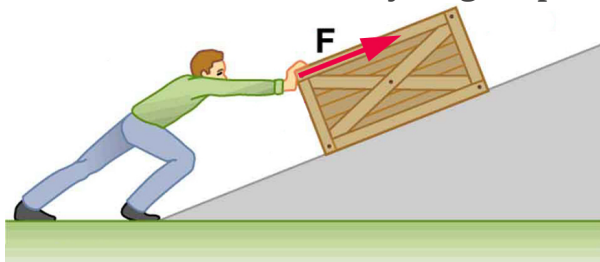
Problem:

Suppose a car travels 108 km at a speed of 30.0 m/s, and uses 2.0 gal of gasoline. Only 30% of the gasoline goes into useful work by the force that keeps the car moving at constant speed despite friction. (See [\[link\]](#) for the energy content of gasoline.) (a) What is the magnitude of the force exerted to keep the car moving at constant speed? (b) If the required force is directly proportional to speed, how many gallons will be used to drive 108 km at a speed of 28.0 m/s?

Exercise:

Problem:

Calculate the work done by an 85.0-kg man who pushes a crate 4.00 m up along a ramp that makes an angle of 20.0° with the horizontal. (See [\[link\]](#).) He exerts a force of 500 N on the crate parallel to the ramp and moves at a constant speed. Be certain to include the work he does on the crate *and* on his body to get up the ramp.



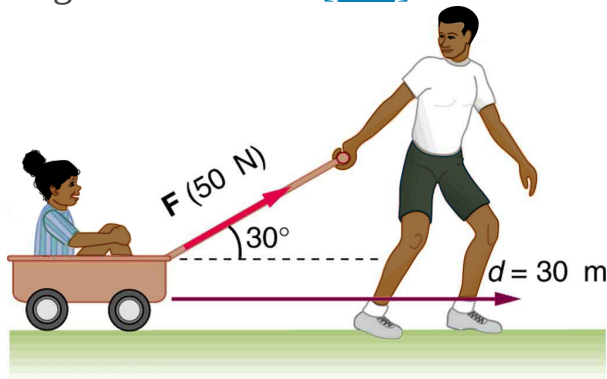
A man pushes a crate up a ramp.

Solution:

$3.14 \times 10^3 \text{ J}$

Exercise:**Problem:**

How much work is done by the boy pulling his sister 30.0 m in a wagon as shown in [\[link\]](#)? Assume no friction acts on the wagon.



The boy does work on the system of the wagon and the child when he pulls them as shown.

Exercise:**Problem:**

A shopper pushes a grocery cart 20.0 m at constant speed on level ground, against a 35.0 N frictional force. He pushes in a direction 25.0° below the horizontal. (a) What is the work done on the cart by friction? (b) What is the work done on the cart by the gravitational force? (c) What is the work done on the cart by the shopper? (d) Find the force the shopper exerts, using energy considerations. (e) What is the total work done on the cart?

Solution:

(a) -700 J

(b) 0

(c) 700 J

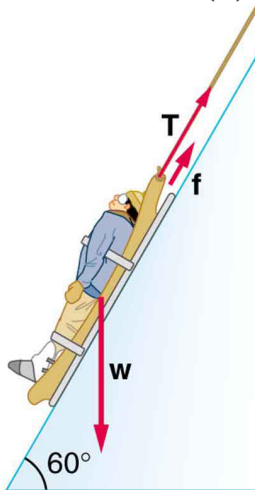
(d) 38.6 N

(e) 0

Exercise:

Problem:

Suppose the ski patrol lowers a rescue sled and victim, having a total mass of 90.0 kg, down a 60.0° slope at constant speed, as shown in [\[link\]](#). The coefficient of friction between the sled and the snow is 0.100. (a) How much work is done by friction as the sled moves 30.0 m along the hill? (b) How much work is done by the rope on the sled in this distance? (c) What is the work done by the gravitational force on the sled? (d) What is the total work done?



A rescue sled and victim are lowered down a steep slope.

Glossary

energy

the ability to do work

work

the transfer of energy by a force that causes an object to be displaced;
the product of the component of the force in the direction of the
displacement and the magnitude of the displacement

joule

SI unit of work and energy, equal to one newton-meter

Kinetic Energy and the Work-Energy Theorem

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain work as a transfer of energy and net work as the work done by the net force.
- Explain and apply the work-energy theorem.

The information presented in this section supports the following AP® learning objectives and science practices:

- **3.E.1.1** The student is able to make predictions about the changes in kinetic energy of an object based on considerations of the direction of the net force on the object as the object moves. **(S.P. 6.4, 7.2)**
- **3.E.1.2** The student is able to use net force and velocity vectors to determine qualitatively whether kinetic energy of an object would increase, decrease, or remain unchanged. **(S.P. 1.4)**
- **3.E.1.3** The student is able to use force and velocity vectors to determine qualitatively or quantitatively the net force exerted on an object and qualitatively whether kinetic energy of that object would increase, decrease, or remain unchanged. **(S.P. 1.4, 2.2)**
- **3.E.1.4** The student is able to apply mathematical routines to determine the change in kinetic energy of an object given the forces on the object and the displacement of the object. **(S.P. 2.2)**
- **4.C.1.1** The student is able to calculate the total energy of a system and justify the mathematical routines used in the

calculation of component types of energy within the system whose sum is the total energy. (S.P. 1.4, 2.1, 2.2)

- **4.C.2.1** The student is able to make predictions about the changes in the mechanical energy of a system when a component of an external force acts parallel or antiparallel to the direction of the displacement of the center of mass. (S.P. 6.4)
- **4.C.2.2** The student is able to apply the concepts of conservation of energy and the work-energy theorem to determine qualitatively and/or quantitatively that work done on a two-object system in linear motion will change the kinetic energy of the center of mass of the system, the potential energy of the systems, and/or the internal energy of the system. (S.P. 1.4, 2.2, 7.2)
- **5.B.5.3** The student is able to predict and calculate from graphical data the energy transfer to or work done on an object or system from information about a force exerted on the object or system through a distance. (S.P. 1.5, 2.2, 6.4)

Work Transfers Energy

What happens to the work done on a system? Energy is transferred into the system, but in what form? Does it remain in the system or move on? The answers depend on the situation. For example, if the lawn mower in [\[link\]](#) (a) is pushed just hard enough to keep it going at a constant speed, then energy put into the mower by the person is removed continuously by friction, and eventually leaves the system in the form of heat transfer. In contrast, work done on the briefcase by the person carrying it up stairs in [\[link\]](#) (d) is stored in the briefcase-Earth system and can be recovered at any time, as shown in [\[link\]](#) (e). In fact, the building of the pyramids in ancient Egypt is an example of storing energy in a system by doing work on the system. Some of the energy imparted to the stone blocks in lifting them

during construction of the pyramids remains in the stone-Earth system and has the potential to do work.

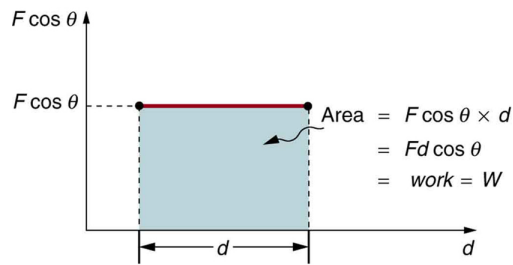
In this section we begin the study of various types of work and forms of energy. We will find that some types of work leave the energy of a system constant, for example, whereas others change the system in some way, such as making it move. We will also develop definitions of important forms of energy, such as the energy of motion.

Net Work and the Work-Energy Theorem

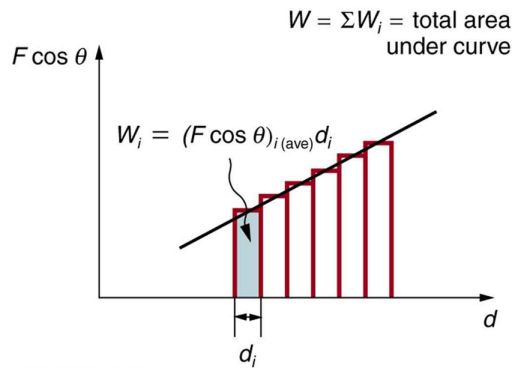
We know from the study of Newton's laws in [Dynamics: Force and Newton's Laws of Motion](#) that net force causes acceleration. We will see in this section that work done by the net force gives a system energy of motion, and in the process we will also find an expression for the energy of motion.

Let us start by considering the total, or net, work done on a system. Net work is defined to be the sum of work done by all external forces—that is, **net work** is the work done by the net external force \mathbf{F}_{net} . In equation form, this is $W_{\text{net}} = F_{\text{net}}d \cos \theta$ where θ is the angle between the force vector and the displacement vector.

[\[link\]](#)(a) shows a graph of force versus displacement for the component of the force in the direction of the displacement—that is, an $F \cos \theta$ vs. d graph. In this case, $F \cos \theta$ is constant. You can see that the area under the graph is $Fd \cos \theta$, or the work done. [\[link\]](#)(b) shows a more general process where the force varies. The area under the curve is divided into strips, each having an average force $(F \cos \theta)_{i(\text{ave})}$. The work done is $(F \cos \theta)_{i(\text{ave})}d_i$ for each strip, and the total work done is the sum of the W_i . Thus the total work done is the total area under the curve, a useful property to which we shall refer later.



(a)



(b)

(a) A graph of $F \cos \theta$ vs. d , when $F \cos \theta$ is constant. The area under the curve represents the work done by the force.

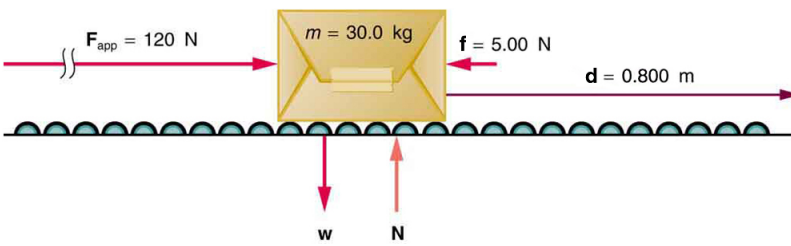
(b) A graph of $F \cos \theta$ vs. d in which the force varies. The work done for each interval is the area of each strip; thus, the total area under the curve equals the total work done.

Note:

Real World Connections: Work and Direction

Consider driving in a car. While moving, you have forward velocity and therefore kinetic energy. When you hit the brakes, they exert a force opposite to your direction of motion (acting through the wheels). The brakes do work on your car and reduce the kinetic energy. Similarly, when you accelerate, the engine (acting through the wheels) exerts a force in the direction of motion. The engine does work on your car, and increases the kinetic energy. Finally, if you go around a corner at a constant speed, you have the same kinetic energy both before and after the corner. The force exerted by the engine was perpendicular to the direction of motion, and therefore did no work and did not change the kinetic energy.

Net work will be simpler to examine if we consider a one-dimensional situation where a force is used to accelerate an object in a direction parallel to its initial velocity. Such a situation occurs for the package on the roller belt conveyor system shown in [\[link\]](#).



A package on a roller belt is pushed horizontally through a distance d .

The force of gravity and the normal force acting on the package are perpendicular to the displacement and do no work. Moreover, they are also equal in magnitude and opposite in direction so they cancel in calculating the net force. The net force arises solely from the horizontal applied force F_{app} and the horizontal friction force f . Thus, as expected, the net force is

parallel to the displacement, so that $\theta = 0^\circ$ and $\cos \theta = 1$, and the net work is given by

Equation:

$$W_{\text{net}} = F_{\text{net}}d.$$

The effect of the net force \mathbf{F}_{net} is to accelerate the package from v_0 to v . The kinetic energy of the package increases, indicating that the net work done on the system is positive. (See [\[link\]](#).) By using Newton's second law, and doing some algebra, we can reach an interesting conclusion. Substituting $F_{\text{net}} = ma$ from Newton's second law gives

Equation:

$$W_{\text{net}} = mad.$$

To get a relationship between net work and the speed given to a system by the net force acting on it, we take $d = x - x_0$ and use the equation studied in [Motion Equations for Constant Acceleration in One Dimension](#) for the change in speed over a distance d if the acceleration has the constant value a ; namely, $v^2 = v_0^2 + 2ad$ (note that a appears in the expression for the net work). Solving for acceleration gives $a = \frac{v^2 - v_0^2}{2d}$. When a is substituted into the preceding expression for W_{net} , we obtain

Equation:

$$W_{\text{net}} = m \left(\frac{v^2 - v_0^2}{2d} \right) d.$$

The d cancels, and we rearrange this to obtain

Equation:

$$W_{\text{net}} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2.$$

This expression is called the **work-energy theorem**, and it actually applies *in general* (even for forces that vary in direction and magnitude), although we have derived it for the special case of a constant force parallel to the displacement. The theorem implies that the net work on a system equals the change in the quantity $\frac{1}{2}mv^2$. This quantity is our first example of a form of energy.

Note:

The Work-Energy Theorem

The net work on a system equals the change in the quantity $\frac{1}{2}mv^2$.

Equation:

$$W_{\text{net}} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$$

The quantity $\frac{1}{2}mv^2$ in the work-energy theorem is defined to be the translational **kinetic energy** (KE) of a mass m moving at a speed v . (*Translational* kinetic energy is distinct from *rotational* kinetic energy, which is considered later.) In equation form, the translational kinetic energy,

Equation:

$$\text{KE} = \frac{1}{2}mv^2,$$

is the energy associated with translational motion. Kinetic energy is a form of energy associated with the motion of a particle, single body, or system of objects moving together.

We are aware that it takes energy to get an object, like a car or the package in [\[link\]](#), up to speed, but it may be a bit surprising that kinetic energy is proportional to speed squared. This proportionality means, for example, that a car traveling at 100 km/h has four times the kinetic energy it has at 50

km/h, helping to explain why high-speed collisions are so devastating. We will now consider a series of examples to illustrate various aspects of work and energy.

Note:

Applying the Science Practices: Cars on a Hill

Assemble a ramp suitable for rolling some toy cars up or down. Then plan a series of experiments to determine how the direction of a force relative to the velocity of an object alters the kinetic energy of the object. Note that gravity will be pointing down in all cases. What happens if you start the car at the top? How about at the bottom, with an initial velocity that is increasing? If your ramp is wide enough, what happens if you send the toy car straight across? Does varying the surface of the ramp change your results?

Sample Response: When the toy car is going down the ramp, with a component of gravity in the same direction, the kinetic energy increases. Sending the car up the ramp decreases the kinetic energy, as gravity is opposing the motion. Sending the car sideways should result in little to no change. If you have a surface that generates more friction than a smooth surface (carpet), note that the friction always opposed the motion, and hence decreases the kinetic energy.

Example:

Calculating the Kinetic Energy of a Package

Suppose a 30.0-kg package on the roller belt conveyor system in [\[link\]](#) is moving at 0.500 m/s. What is its kinetic energy?

Strategy

Because the mass m and speed v are given, the kinetic energy can be calculated from its definition as given in the equation $KE = \frac{1}{2}mv^2$.

Solution

The kinetic energy is given by

Equation:

$$\text{KE} = \frac{1}{2}mv^2.$$

Entering known values gives

Equation:

$$\text{KE} = 0.5(30.0 \text{ kg})(0.500 \text{ m/s})^2,$$

which yields

Equation:

$$\text{KE} = 3.75 \text{ kg} \cdot \text{m}^2/\text{s}^2 = 3.75 \text{ J}.$$

Discussion

Note that the unit of kinetic energy is the joule, the same as the unit of work, as mentioned when work was first defined. It is also interesting that, although this is a fairly massive package, its kinetic energy is not large at this relatively low speed. This fact is consistent with the observation that people can move packages like this without exhausting themselves.

Note:

Real World Connections: Center of Mass

Suppose we have two experimental carts, of equal mass, latched together on a track with a compressed spring between them. When the latch is released, the spring does 10 J of work on the carts (we'll see how in a couple of sections). The carts move relative to the spring, which is the center of mass of the system. However, the center of mass stays fixed. How can we consider the kinetic energy of this system?

By the work-energy theorem, the work done by the spring on the carts must turn into kinetic energy. So this system has 10 J of kinetic energy. The total kinetic energy of the system is the kinetic energy of the center of mass of the system relative to the fixed origin plus the kinetic energy of each cart relative to the center of mass. We know that the center of mass relative to the fixed origin does not move, and therefore all of the kinetic energy must be distributed among the carts relative to the center of mass. Since the

carts have equal mass, they each receive an equal amount of kinetic energy, so each cart has 5.0 J of kinetic energy.

In our example, the forces between the spring and each cart are internal to the system. According to Newton's third law, these internal forces will cancel since they are equal and opposite in direction. However, this does not imply that these internal forces will not do work. Thus, the change in kinetic energy of the system is caused by work done by the force of the spring, and results in the motion of the two carts relative to the center of mass.

Example:

Determining the Work to Accelerate a Package

Suppose that you push on the 30.0-kg package in [\[link\]](#) with a constant force of 120 N through a distance of 0.800 m, and that the opposing friction force averages 5.00 N.

(a) Calculate the net work done on the package. (b) Solve the same problem as in part (a), this time by finding the work done by each force that contributes to the net force.

Strategy and Concept for (a)

This is a motion in one dimension problem, because the downward force (from the weight of the package) and the normal force have equal magnitude and opposite direction, so that they cancel in calculating the net force, while the applied force, friction, and the displacement are all horizontal. (See [\[link\]](#).) As expected, the net work is the net force times distance.

Solution for (a)

The net force is the push force minus friction, or

$F_{\text{net}} = 120 \text{ N} - 5.00 \text{ N} = 115 \text{ N}$. Thus the net work is

Equation:

$$\begin{aligned} W_{\text{net}} &= F_{\text{net}}d = (115 \text{ N})(0.800 \text{ m}) \\ &= 92.0 \text{ N} \cdot \text{m} = 92.0 \text{ J}. \end{aligned}$$

Discussion for (a)

This value is the net work done on the package. The person actually does more work than this, because friction opposes the motion. Friction does negative work and removes some of the energy the person expends and converts it to thermal energy. The net work equals the sum of the work done by each individual force.

Strategy and Concept for (b)

The forces acting on the package are gravity, the normal force, the force of friction, and the applied force. The normal force and force of gravity are each perpendicular to the displacement, and therefore do no work.

Solution for (b)

The applied force does work.

Equation:

$$\begin{aligned}W_{\text{app}} &= F_{\text{app}}d \cos(0^\circ) = F_{\text{app}}d \\&= (120 \text{ N})(0.800 \text{ m}) \\&= 96.0 \text{ J}\end{aligned}$$

The friction force and displacement are in opposite directions, so that $\theta = 180^\circ$, and the work done by friction is

Equation:

$$\begin{aligned}W_{\text{fr}} &= F_{\text{fr}}d \cos(180^\circ) = -F_{\text{fr}}d \\&= -(5.00 \text{ N})(0.800 \text{ m}) \\&= -4.00 \text{ J}.\end{aligned}$$

So the amounts of work done by gravity, by the normal force, by the applied force, and by friction are, respectively,

Equation:

$$\begin{aligned}W_{\text{gr}} &= 0, \\W_{\text{N}} &= 0, \\W_{\text{app}} &= 96.0 \text{ J}, \\W_{\text{fr}} &= -4.00 \text{ J}.\end{aligned}$$

The total work done as the sum of the work done by each force is then seen to be

Equation:

$$W_{\text{total}} = W_{\text{gr}} + W_{\text{N}} + W_{\text{app}} + W_{\text{fr}} = 92.0 \text{ J}.$$

Discussion for (b)

The calculated total work W_{total} as the sum of the work by each force agrees, as expected, with the work W_{net} done by the net force. The work done by a collection of forces acting on an object can be calculated by either approach.

Example:**Determining Speed from Work and Energy**

Find the speed of the package in [\[link\]](#) at the end of the push, using work and energy concepts.

Strategy

Here the work-energy theorem can be used, because we have just calculated the net work, W_{net} , and the initial kinetic energy, $\frac{1}{2}mv_0^2$. These calculations allow us to find the final kinetic energy, $\frac{1}{2}mv^2$, and thus the final speed v .

Solution

The work-energy theorem in equation form is

Equation:

$$W_{\text{net}} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2.$$

Solving for $\frac{1}{2}mv^2$ gives

Equation:

$$\frac{1}{2}mv^2 = W_{\text{net}} + \frac{1}{2}mv_0^2.$$

Thus,

Equation:

$$\frac{1}{2}mv^2 = 92.0 \text{ J} + 3.75 \text{ J} = 95.75 \text{ J}.$$

Solving for the final speed as requested and entering known values gives
Equation:

$$\begin{aligned} v &= \sqrt{\frac{2(95.75 \text{ J})}{m}} = \sqrt{\frac{191.5 \text{ kg}\cdot\text{m}^2/\text{s}^2}{30.0 \text{ kg}}} \\ &= 2.53 \text{ m/s.} \end{aligned}$$

Discussion

Using work and energy, we not only arrive at an answer, we see that the final kinetic energy is the sum of the initial kinetic energy and the net work done on the package. This means that the work indeed adds to the energy of the package.

Example:

Work and Energy Can Reveal Distance, Too

How far does the package in [\[link\]](#) coast after the push, assuming friction remains constant? Use work and energy considerations.

Strategy

We know that once the person stops pushing, friction will bring the package to rest. In terms of energy, friction does negative work until it has removed all of the package's kinetic energy. The work done by friction is the force of friction times the distance traveled times the cosine of the angle between the friction force and displacement; hence, this gives us a way of finding the distance traveled after the person stops pushing.

Solution

The normal force and force of gravity cancel in calculating the net force. The horizontal friction force is then the net force, and it acts opposite to the displacement, so $\theta = 180^\circ$. To reduce the kinetic energy of the package to zero, the work W_{fr} by friction must be minus the kinetic energy that the package started with plus what the package accumulated due to the pushing. Thus $W_{\text{fr}} = -95.75 \text{ J}$. Furthermore, $W_{\text{fr}} = f d' \cos \theta = -f d'$, where d' is the distance it takes to stop. Thus,

Equation:

$$d = -\frac{W_{\text{fr}}}{f} = -\frac{-95.75 \text{ J}}{5.00 \text{ N}},$$

and so

Equation:

$$d = 19.2 \text{ m.}$$

Discussion

This is a reasonable distance for a package to coast on a relatively friction-free conveyor system. Note that the work done by friction is negative (the force is in the opposite direction of motion), so it removes the kinetic energy.

Some of the examples in this section can be solved without considering energy, but at the expense of missing out on gaining insights about what work and energy are doing in this situation. On the whole, solutions involving energy are generally shorter and easier than those using kinematics and dynamics alone.

Test Prep for AP Courses

Exercise:

Problem:

A toy car is going around a loop-the-loop. Gravity ____ the kinetic energy on the upward side of the loop, ____ the kinetic energy at the top, and ____ the kinetic energy on the downward side of the loop.

- a. increases, decreases, has no effect on
- b. decreases, has no effect on, increases
- c. increases, has no effect on, decreases
- d. decreases, increases, has no effect on

Exercise:**Problem:**

A roller coaster is set up with a track in the form of a perfect cosine. Describe and graph what happens to the kinetic energy of a cart as it goes through the first full period of the track.

Solution:

The kinetic energy should change in the form of $-\cos$, with an initial value of 0 or slightly above, and ending at the same level.

Exercise:**Problem:**

If wind is blowing horizontally toward a car with an angle of 30 degrees from the direction of travel, the kinetic energy will _____. If the wind is blowing at a car at 135 degrees from the direction of travel, the kinetic energy will _____.

- a. increase, increase
- b. increase, decrease
- c. decrease, increase
- d. decrease, decrease

Exercise:**Problem:**

In what direction relative to the direction of travel can a force act on a car (traveling on level ground), and not change the kinetic energy? Can you give examples of such forces?

Solution:

Any force acting perpendicular will have no effect on kinetic energy. Obvious examples are gravity and the normal force, but others include

wind directly from the side and rain or other precipitation falling straight down.

Exercise:

Problem:

A 2000-kg airplane is coming in for a landing, with a velocity 5 degrees below the horizontal and a drag force of 40 kN acting directly rearward. Kinetic energy will _____ due to the net force of _____.

- a. increase, 20 kN
- b. decrease, 40 kN
- c. increase, 45 kN
- d. decrease, 45 kN

Exercise:

Problem:

You are participating in the Iditarod, and your sled dogs are pulling you across a frozen lake with a force of 1200 N while a 300 N wind is blowing at you at 135 degrees from your direction of travel. What is the net force, and will your kinetic energy increase or decrease?

Solution:

Note that the wind is pushing from behind and one side, so your KE will increase. The net force has components of 1400 N in the direction of travel and 212 N perpendicular to the direction of travel. So the net force is 1420 N at 8.5 degrees from the direction of travel.

Exercise:

Problem:

A model drag car is being accelerated along its track from rest by a motor with a force of 75 N, but there is a drag force of 30 N due to the track. What is the kinetic energy after 2 m of travel?

- a. 90 J
- b. 150 J
- c. 210 J
- d. 60 J

Exercise:

Problem:

You are launching a 2-kg potato out of a potato cannon. The cannon is 1.5 m long and is aimed 30 degrees above the horizontal. It exerts a 50 N force on the potato. What is the kinetic energy of the potato as it leaves the muzzle of the potato cannon?

Solution:

Gravity has a component perpendicular to the cannon (and to displacement, so it is irrelevant) and has a component parallel to the cannon. The latter is equal to 9.8 N. Thus the net force in the direction of the displacement is 50 N – 9.8 N, and the kinetic energy is 60 J.

Exercise:

Problem:

When the force acting on an object is parallel to the direction of the motion of the center of mass, the mechanical energy _____. When the force acting on an object is antiparallel to the direction of the center of mass, the mechanical energy _____.

- a. increases, increases
- b. increases, decreases
- c. decreases, increases
- d. decreases, decreases

Exercise:

Problem:

Describe a system in which the main forces acting are parallel or antiparallel to the center of mass, and justify your answer.

Solution:

The potato cannon (and many other projectile launchers) above is an option, with a force launching the projectile, friction, potentially gravity depending on the direction it is pointed, etc. A drag (or other) car accelerating is another possibility.

Exercise:**Problem:**

A child is pulling two red wagons, with the second one tied to the first by a (non-stretching) rope. Each wagon has a mass of 10 kg. If the child exerts a force of 30 N for 5.0 m, how much has the kinetic energy of the two-wagon system changed?

- a. 300 J
- b. 150 J
- c. 75 J
- d. 60 J

Exercise:**Problem:**

A child has two red wagons, with the rear one tied to the front by a (non-stretching) rope. If the child pushes on the rear wagon, what happens to the kinetic energy of each of the wagons, and the two-wagon system?

Solution:

The kinetic energy of the rear wagon increases. The front wagon does not, until the rear wagon collides with it. The total system may be

treated by its center of mass, halfway between the wagons, and its energy increases by the same amount as the sum of the two individual wagons.

Exercise:

Problem:

Draw a graph of the force parallel to displacement exerted on a stunt motorcycle going through a loop-the-loop versus the distance traveled around the loop. Explain the net change in energy.

Section Summary

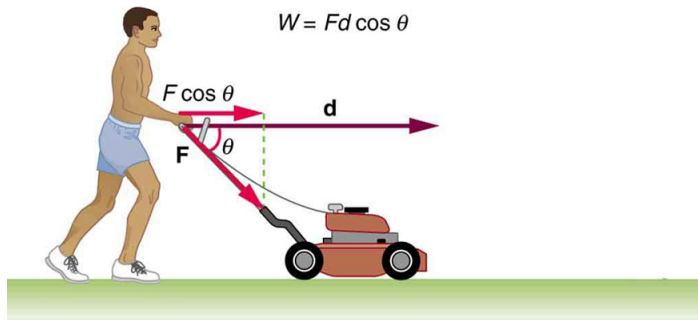
- The net work W_{net} is the work done by the net force acting on an object.
- Work done on an object transfers energy to the object.
- The translational kinetic energy of an object of mass m moving at speed v is $\text{KE} = \frac{1}{2}mv^2$.
- The work-energy theorem states that the net work W_{net} on a system changes its kinetic energy, $W_{\text{net}} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$.

Conceptual Questions

Exercise:

Problem:

The person in [\[link\]](#) does work on the lawn mower. Under what conditions would the mower gain energy? Under what conditions would it lose energy?



Exercise:

Problem:

Work done on a system puts energy into it. Work done by a system removes energy from it. Give an example for each statement.

Exercise:

Problem:

When solving for speed in [\[link\]](#), we kept only the positive root. Why?

Problems & Exercises

Exercise:

Problem:

Compare the kinetic energy of a 20,000-kg truck moving at 110 km/h with that of an 80.0-kg astronaut in orbit moving at 27,500 km/h.

Solution:

1/250

Exercise:

Problem:

(a) How fast must a 3000-kg elephant move to have the same kinetic energy as a 65.0-kg sprinter running at 10.0 m/s? (b) Discuss how the larger energies needed for the movement of larger animals would relate to metabolic rates.

Exercise:**Problem:**

Confirm the value given for the kinetic energy of an aircraft carrier in [\[link\]](#). You will need to look up the definition of a nautical mile (1 knot = 1 nautical mile/h).

Solution:

$$1.1 \times 10^{10} \text{ J}$$

Exercise:**Problem:**

(a) Calculate the force needed to bring a 950-kg car to rest from a speed of 90.0 km/h in a distance of 120 m (a fairly typical distance for a non-panic stop). (b) Suppose instead the car hits a concrete abutment at full speed and is brought to a stop in 2.00 m. Calculate the force exerted on the car and compare it with the force found in part (a).

Exercise:**Problem:**

A car's bumper is designed to withstand a 4.0-km/h (1.1-m/s) collision with an immovable object without damage to the body of the car. The bumper cushions the shock by absorbing the force over a distance. Calculate the magnitude of the average force on a bumper that collapses 0.200 m while bringing a 900-kg car to rest from an initial speed of 1.1 m/s.

Solution:

$$2.8 \times 10^3 \text{ N}$$

Exercise:**Problem:**

Boxing gloves are padded to lessen the force of a blow. (a) Calculate the force exerted by a boxing glove on an opponent's face, if the glove and face compress 7.50 cm during a blow in which the 7.00-kg arm and glove are brought to rest from an initial speed of 10.0 m/s. (b) Calculate the force exerted by an identical blow in the gory old days when no gloves were used and the knuckles and face would compress only 2.00 cm. (c) Discuss the magnitude of the force with glove on. Does it seem high enough to cause damage even though it is lower than the force with no glove?

Exercise:**Problem:**

Using energy considerations, calculate the average force a 60.0-kg sprinter exerts backward on the track to accelerate from 2.00 to 8.00 m/s in a distance of 25.0 m, if he encounters a headwind that exerts an average force of 30.0 N against him.

Solution:

102 N

Glossary**net work**

work done by the net force, or vector sum of all the forces, acting on an object

work-energy theorem

the result, based on Newton's laws, that the net work done on an object is equal to its change in kinetic energy

kinetic energy

the energy an object has by reason of its motion, equal to $\frac{1}{2}mv^2$ for the translational (i.e., non-rotational) motion of an object of mass m moving at speed v

Gravitational Potential Energy

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain gravitational potential energy in terms of work done against gravity.
- Show that the gravitational potential energy of an object of mass m at height h on Earth is given by $PE_g = mgh$.
- Show how knowledge of potential energy as a function of position can be used to simplify calculations and explain physical phenomena.

The information presented in this section supports the following AP® learning objectives and science practices:

- **4.C.1.1** The student is able to calculate the total energy of a system and justify the mathematical routines used in the calculation of component types of energy within the system whose sum is the total energy. **(S.P. 1.4, 2.1, 2.2)**
- **5.B.1.1** The student is able to set up a representation or model showing that a single object can only have kinetic energy and use information about that object to calculate its kinetic energy. **(S.P. 1.4, 2.2)**
- **5.B.1.2** The student is able to translate between a representation of a single object, which can only have kinetic energy, and a system that includes the object, which may have both kinetic and potential energies. **(S.P. 1.5)**

Work Done Against Gravity

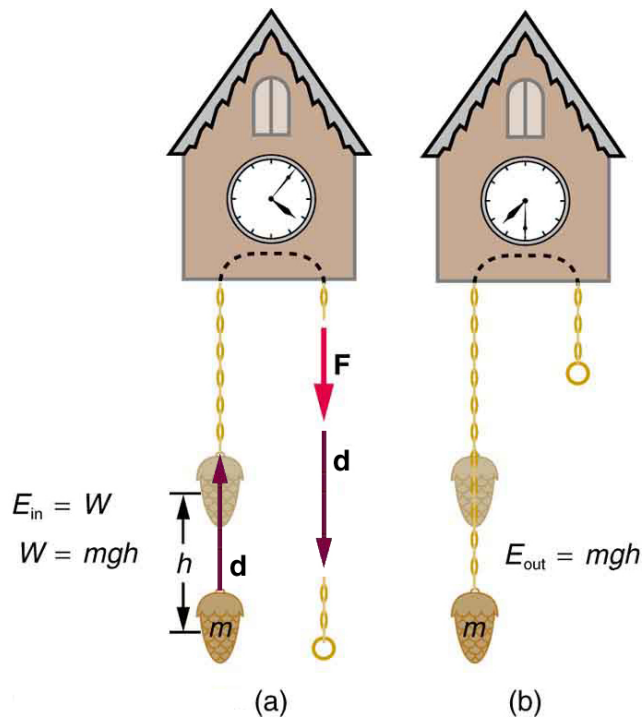
Climbing stairs and lifting objects is work in both the scientific and everyday sense—it is work done against the gravitational force. When there is work, there is a

transformation of energy. The work done against the gravitational force goes into an important form of stored energy that we will explore in this section.

Let us calculate the work done in lifting an object of mass m through a height h , such as in [\[link\]](#). If the object is lifted straight up at constant speed, then the force needed to lift it is equal to its weight mg . The work done on the mass is then $W = Fd = mgh$. We define this to be the **gravitational potential energy** (PE_g) put into (or gained by) the object-Earth system. This energy is associated with the state of separation between two objects that attract each other by the gravitational force. For convenience, we refer to this as the PE_g gained by the object, recognizing that this is energy stored in the gravitational field of Earth. Why do we use the word “system”? Potential energy is a property of a system rather than of a single object—due to its physical position. An object’s gravitational potential is due to its position relative to the surroundings within the Earth-object system. The force applied to the object is an external force, from outside the system. When it does positive work it increases the gravitational potential energy of the system. Because gravitational potential energy depends on relative position, we need a reference level at which to set the potential energy equal to 0. We usually choose this point to be Earth’s surface, but this point is arbitrary; what is important is the *difference* in gravitational potential energy, because this difference is what relates to the work done. The difference in gravitational potential energy of an object (in the Earth-object system) between two rungs of a ladder will be the same for the first two rungs as for the last two rungs.

Converting Between Potential Energy and Kinetic Energy

Gravitational potential energy may be converted to other forms of energy, such as kinetic energy. If we release the mass, gravitational force will do an amount of work equal to mgh on it, thereby increasing its kinetic energy by that same amount (by the work-energy theorem). We will find it more useful to consider just the conversion of PE_g to KE without explicitly considering the intermediate step of work. (See [\[link\]](#).) This shortcut makes it easier to solve problems using energy (if possible) rather than explicitly using forces.



(a) The work done to lift the weight is stored in the mass-Earth system as gravitational potential energy. (b) As the weight moves downward, this gravitational potential energy is transferred to the cuckoo clock.

More precisely, we define the *change* in gravitational potential energy ΔPE_g to be
Equation:

$$\Delta\text{PE}_g = mgh,$$

where, for simplicity, we denote the change in height by h rather than the usual Δh . Note that h is positive when the final height is greater than the initial height, and vice versa. For example, if a 0.500-kg mass hung from a cuckoo clock is raised 1.00 m, then its change in gravitational potential energy is

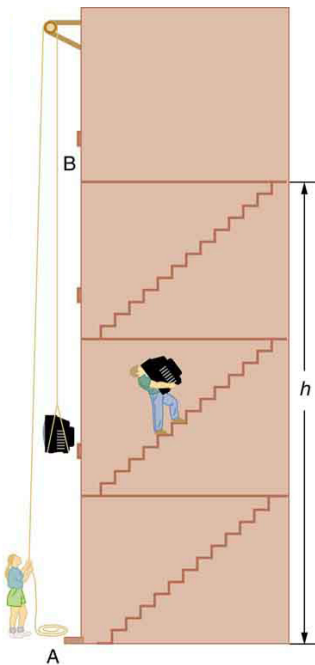
Equation:

$$\begin{aligned}
 mgh &= (0.500 \text{ kg}) (9.80 \text{ m/s}^2) (1.00 \text{ m}) \\
 &= 4.90 \text{ kg} \cdot \text{m}^2/\text{s}^2 = 4.90 \text{ J}.
 \end{aligned}$$

Note that the units of gravitational potential energy turn out to be joules, the same as for work and other forms of energy. As the clock runs, the mass is lowered. We can think of the mass as gradually giving up its 4.90 J of gravitational potential energy, *without directly considering the force of gravity that does the work*.

Using Potential Energy to Simplify Calculations

The equation $\Delta PE_g = mgh$ applies for any path that has a change in height of h , not just when the mass is lifted straight up. (See [\[link\]](#).) It is much easier to calculate mgh (a simple multiplication) than it is to calculate the work done along a complicated path. The idea of gravitational potential energy has the double advantage that it is very broadly applicable and it makes calculations easier. From now on, we will consider that any change in vertical position h of a mass m is accompanied by a change in gravitational potential energy mgh , and we will avoid the equivalent but more difficult task of calculating work done by or against the gravitational force.



The change in gravitational potential energy (ΔPE_g) between points A and B is independent of the path.

$\Delta PE_g = mgh$ for any path between the two points. Gravity is one of a small class of forces where the work done by or against the force depends only on the starting and ending points, not on the path between them.

Example:

The Force to Stop Falling

A 60.0-kg person jumps onto the floor from a height of 3.00 m. If he lands stiffly (with his knee joints compressing by 0.500 cm), calculate the force on the knee joints.

Strategy

This person's energy is brought to zero in this situation by the work done on him by the floor as he stops. The initial PE_g is transformed into KE as he falls. The work done by the floor reduces this kinetic energy to zero.

Solution

The work done on the person by the floor as he stops is given by

Equation:

$$W = Fd \cos \theta = -Fd,$$

with a minus sign because the displacement while stopping and the force from floor are in opposite directions ($\cos \theta = \cos 180^\circ = -1$). The floor removes energy from the system, so it does negative work.

The kinetic energy the person has upon reaching the floor is the amount of potential energy lost by falling through height h :

Equation:

$$\text{KE} = -\Delta\text{PE}_g = -mgh,$$

The distance d that the person's knees bend is much smaller than the height h of the fall, so the additional change in gravitational potential energy during the knee bend is ignored.

The work W done by the floor on the person stops the person and brings the person's kinetic energy to zero:

Equation:

$$W = -\text{KE} = mgh.$$

Combining this equation with the expression for W gives

Equation:

$$-Fd = mgh.$$

Recalling that h is negative because the person fell *down*, the force on the knee joints is given by

Equation:

$$F = -\frac{mgh}{d} = -\frac{(60.0 \text{ kg})(9.80 \text{ m/s}^2)(-3.00 \text{ m})}{5.00 \times 10^{-3} \text{ m}} = 3.53 \times 10^5 \text{ N}.$$

Discussion

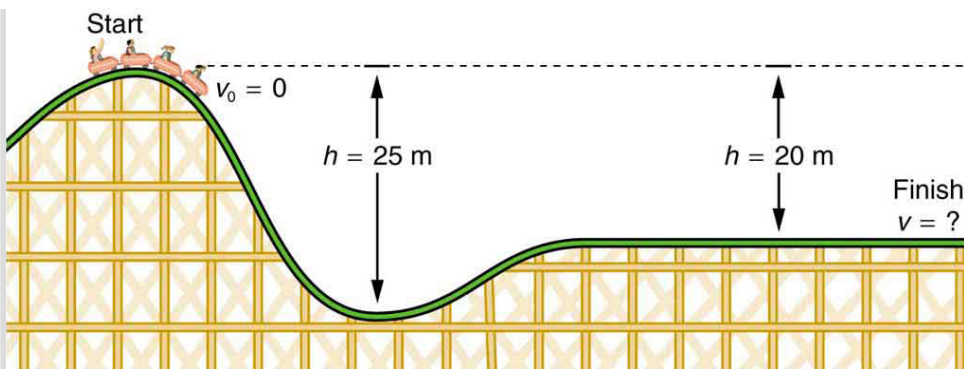
Such a large force (500 times more than the person's weight) over the short impact time is enough to break bones. A much better way to cushion the shock is by bending the legs or rolling on the ground, increasing the time over which the force acts. A bending motion of 0.5 m this way yields a force 100 times smaller than in the example. A kangaroo's hopping shows this method in action. The kangaroo is the only large animal to use hopping for locomotion, but the shock in hopping is cushioned by the bending of its hind legs in each jump.(See [\[link\]](#).)



The work done by the ground upon the kangaroo reduces its kinetic energy to zero as it lands. However, by applying the force of the ground on the hind legs over a longer distance, the impact on the bones is reduced.
(credit: Chris Samuel, Flickr)

Example:**Finding the Speed of a Roller Coaster from its Height**

(a) What is the final speed of the roller coaster shown in [\[link\]](#) if it starts from rest at the top of the 20.0 m hill and work done by frictional forces is negligible? (b) What is its final speed (again assuming negligible friction) if its initial speed is 5.00 m/s?



The speed of a roller coaster increases as gravity pulls it downhill and is greatest at its lowest point. Viewed in terms of energy, the roller-coaster-Earth system's gravitational potential energy is converted to kinetic energy. If work done by friction is negligible, all ΔPE_g is converted to KE.

Strategy

The roller coaster loses potential energy as it goes downhill. We neglect friction, so that the remaining force exerted by the track is the normal force, which is perpendicular to the direction of motion and does no work. The net work on the roller coaster is then done by gravity alone. The *loss* of gravitational potential energy from moving *downward* through a distance h equals the *gain* in kinetic energy. This can be written in equation form as $-\Delta PE_g = \Delta KE$. Using the equations for PE_g and KE, we can solve for the final speed v , which is the desired quantity.

Solution for (a)

Here the initial kinetic energy is zero, so that $\Delta KE = \frac{1}{2}mv^2$. The equation for change in potential energy states that $\Delta PE_g = mgh$. Since h is negative in this case, we will rewrite this as $\Delta PE_g = -mg |h|$ to show the minus sign clearly. Thus,

Equation:

$$-\Delta PE_g = \Delta KE$$

becomes

Equation:

$$mg |h| = \frac{1}{2}mv^2.$$

Solving for v , we find that mass cancels and that

Equation:

$$v = \sqrt{2g | h |}.$$

Substituting known values,

Equation:

$$\begin{aligned} v &= \sqrt{2(9.80 \text{ m/s}^2)(20.0 \text{ m})} \\ &= 19.8 \text{ m/s.} \end{aligned}$$

Solution for (b)

Again $-\Delta PE_g = \Delta KE$. In this case there is initial kinetic energy, so

$\Delta KE = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$. Thus,

Equation:

$$mg | h | = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2.$$

Rearranging gives

Equation:

$$\frac{1}{2}mv^2 = mg | h | + \frac{1}{2}mv_0^2.$$

This means that the final kinetic energy is the sum of the initial kinetic energy and the gravitational potential energy. Mass again cancels, and

Equation:

$$v = \sqrt{2g | h | + v_0^2}.$$

This equation is very similar to the kinematics equation $v = \sqrt{v_0^2 + 2ad}$, but it is more general—the kinematics equation is valid only for constant acceleration, whereas our equation above is valid for any path regardless of whether the object moves with a constant acceleration. Now, substituting known values gives

Equation:

$$\begin{aligned}
 v &= \sqrt{2(9.80 \text{ m/s}^2)(20.0 \text{ m}) + (5.00 \text{ m/s})^2} \\
 &= 20.4 \text{ m/s}.
 \end{aligned}$$

Discussion and Implications

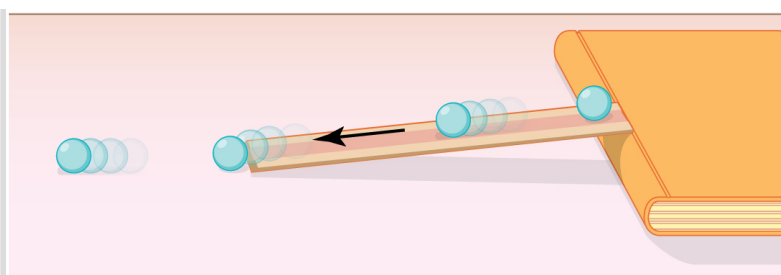
First, note that mass cancels. This is quite consistent with observations made in [Falling Objects](#) that all objects fall at the same rate if friction is negligible. Second, only the speed of the roller coaster is considered; there is no information about its direction at any point. This reveals another general truth. When friction is negligible, the speed of a falling body depends only on its initial speed and height, and not on its mass or the path taken. For example, the roller coaster will have the same final speed whether it falls 20.0 m straight down or takes a more complicated path like the one in the figure. Third, and perhaps unexpectedly, the final speed in part (b) is greater than in part (a), but by far less than 5.00 m/s. Finally, note that speed can be found at *any* height along the way by simply using the appropriate value of h at the point of interest.

We have seen that work done by or against the gravitational force depends only on the starting and ending points, and not on the path between, allowing us to define the simplifying concept of gravitational potential energy. We can do the same thing for a few other forces, and we will see that this leads to a formal definition of the law of conservation of energy.

Note:

Making Connections: Take-Home Investigation—Converting Potential to Kinetic Energy

One can study the conversion of gravitational potential energy into kinetic energy in this experiment. On a smooth, level surface, use a ruler of the kind that has a groove running along its length and a book to make an incline (see [link](#)). Place a marble at the 10-cm position on the ruler and let it roll down the ruler. When it hits the level surface, measure the time it takes to roll one meter. Now place the marble at the 20-cm and the 30-cm positions and again measure the times it takes to roll 1 m on the level surface. Find the velocity of the marble on the level surface for all three positions. Plot velocity squared versus the distance traveled by the marble. What is the shape of each plot? If the shape is a straight line, the plot shows that the marble's kinetic energy at the bottom is proportional to its potential energy at the release point.



A marble rolls down a ruler, and its speed on the level surface is measured.

Test Prep for AP Courses

Exercise:

Problem:

A 1.0 kg baseball is flying at 10 m/s. How much kinetic energy does it have? Potential energy?

- a. 10 J, 20 J
- b. 50 J, 20 J
- c. unknown, 50 J
- d. 50 J, unknown

Solution:

(d)

Exercise:

Problem:

A 2.0-kg potato has been launched out of a potato cannon at 9.0 m/s. What is the kinetic energy? If you then learn that it is 4.0 m above the ground, what is the total mechanical energy relative to the ground?

- a. 78 J, 3 J
- b. 160 J, 81 J

- c. 81 J, 160 J
- d. 81 J, 3 J

Exercise:

Problem:

You have a 120-g yo-yo that you are swinging at 0.9 m/s. How much energy does it have? How high can it get above the lowest point of the swing without your doing any additional work, on Earth? How high could it get on the Moon, where gravity is 1/6 Earth's?

Solution:

0.049 J; 0.041 m, 0.25 m

Section Summary

- Work done against gravity in lifting an object becomes potential energy of the object-Earth system.
- The change in gravitational potential energy, ΔPE_g , is $\Delta PE_g = mgh$, with h being the increase in height and g the acceleration due to gravity.
- The gravitational potential energy of an object near Earth's surface is due to its position in the mass-Earth system. Only differences in gravitational potential energy, ΔPE_g , have physical significance.
- As an object descends without friction, its gravitational potential energy changes into kinetic energy corresponding to increasing speed, so that $\Delta KE = -\Delta PE_g$.

Conceptual Questions

Exercise:

Problem:

In [\[link\]](#), we calculated the final speed of a roller coaster that descended 20 m in height and had an initial speed of 5 m/s downhill. Suppose the roller coaster had had an initial speed of 5 m/s uphill instead, and it coasted uphill, stopped, and then rolled back down to a final point 20 m below the start. We would find in that case that its final speed is the same as its initial. Explain in terms of conservation of energy.

Exercise:**Problem:**

Does the work you do on a book when you lift it onto a shelf depend on the path taken? On the time taken? On the height of the shelf? On the mass of the book?

Problems & Exercises**Exercise:****Problem:**

A hydroelectric power facility (see [\[link\]](#)) converts the gravitational potential energy of water behind a dam to electric energy. (a) What is the gravitational potential energy relative to the generators of a lake of volume 50.0 km^3 (mass $= 5.00 \times 10^{13} \text{ kg}$), given that the lake has an average height of 40.0 m above the generators? (b) Compare this with the energy stored in a 9-megaton fusion bomb.



Hydroelectric facility (credit: Denis Belevich, Wikimedia Commons)

Solution:

(a) $1.96 \times 10^{16} \text{ J}$

(b) The ratio of gravitational potential energy in the lake to the energy stored in the bomb is 0.52. That is, the energy stored in the lake is approximately half that in a 9-megaton fusion bomb.

Exercise:

Problem:

(a) How much gravitational potential energy (relative to the ground on which it is built) is stored in the Great Pyramid of Cheops, given that its mass is about 7×10^9 kg and its center of mass is 36.5 m above the surrounding ground? (b) How does this energy compare with the daily food intake of a person?

Exercise:

Problem:

Suppose a 350-g kookaburra (a large kingfisher bird) picks up a 75-g snake and raises it 2.5 m from the ground to a branch. (a) How much work did the bird do on the snake? (b) How much work did it do to raise its own center of mass to the branch?

Solution:

(a) 1.8 J

(b) 8.6 J

Exercise:

Problem:

In [\[link\]](#), we found that the speed of a roller coaster that had descended 20.0 m was only slightly greater when it had an initial speed of 5.00 m/s than when it started from rest. This implies that $\Delta PE \gg KE_i$. Confirm this statement by taking the ratio of ΔPE to KE_i . (Note that mass cancels.)

Exercise:

Problem:

A 100-g toy car is propelled by a compressed spring that starts it moving. The car follows the curved track in [\[link\]](#). Show that the final speed of the toy car is 0.687 m/s if its initial speed is 2.00 m/s and it coasts up the frictionless slope, gaining 0.180 m in altitude.



A toy car moves up a sloped track.
(credit: Leszek Leszczynski, Flickr)

Solution:

$$v_f = \sqrt{2gh + v_0^2} = \sqrt{2(9.80 \text{ m/s}^2)(-0.180 \text{ m}) + (2.00 \text{ m/s})^2} = 0.687 \text{ m/s}$$

Exercise:**Problem:**

In a downhill ski race, surprisingly, little advantage is gained by getting a running start. (This is because the initial kinetic energy is small compared with the gain in gravitational potential energy on even small hills.) To demonstrate this, find the final speed and the time taken for a skier who skies 70.0 m along a 30° slope neglecting friction: (a) Starting from rest. (b) Starting with an initial speed of 2.50 m/s. (c) Does the answer surprise you? Discuss why it is still advantageous to get a running start in very competitive events.

Glossary

gravitational potential energy

the energy an object has due to its position in a gravitational field

Conservative Forces and Potential Energy

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define conservative force, potential energy, and mechanical energy.
- Explain the potential energy of a spring in terms of its compression when Hooke's law applies.
- Use the work-energy theorem to show how having only conservative forces leads to conservation of mechanical energy.

The information presented in this section supports the following AP® learning objectives and science practices:

- **4.C.1.1** The student is able to calculate the total energy of a system and justify the mathematical routines used in the calculation of component types of energy within the system whose sum is the total energy. **(S.P. 1.4, 2.1, 2.2)**
- **4.C.2.1** The student is able to make predictions about the changes in the mechanical energy of a system when a component of an external force acts parallel or antiparallel to the direction of the displacement of the center of mass. **(S.P. 6.4)**
- **5.B.1.1** The student is able to set up a representation or model showing that a single object can only have kinetic energy and use information about that object to calculate its kinetic energy. **(S.P. 1.4, 2.2)**
- **5.B.1.2** The student is able to translate between a representation of a single object, which can only have kinetic

energy, and a system that includes the object, which may have both kinetic and potential energies. (S.P. 1.5)

- **5.B.3.1** The student is able to describe and make qualitative and/or quantitative predictions about everyday examples of systems with internal potential energy. (S.P. 2.2, 6.4, 7.2)
- **5.B.3.2** The student is able to make quantitative calculations of the internal potential energy of a system from a description or diagram of that system. (S.P. 1.4, 2.2)
- **5.B.3.3** The student is able to apply mathematical reasoning to create a description of the internal potential energy of a system from a description or diagram of the objects and interactions in that system. (S.P. 1.4, 2.2)

Potential Energy and Conservative Forces

Work is done by a force, and some forces, such as weight, have special characteristics. A **conservative force** is one, like the gravitational force, for which work done by or against it depends only on the starting and ending points of a motion and not on the path taken. We can define a **potential energy** (PE) for any conservative force, just as we did for the gravitational force. For example, when you wind up a toy, an egg timer, or an old-fashioned watch, you do work against its spring and store energy in it. (We treat these springs as ideal, in that we assume there is no friction and no production of thermal energy.) This stored energy is recoverable as work, and it is useful to think of it as potential energy contained in the spring. Indeed, the reason that the spring has this characteristic is that its force is *conservative*. That is, a conservative force results in stored or potential energy. Gravitational potential energy is one example, as is the energy stored in a spring. We will also see how conservative forces are related to the conservation of energy.

Note:**Potential Energy and Conservative Forces**

Potential energy is the energy a system has due to position, shape, or configuration. It is stored energy that is completely recoverable.

A conservative force is one for which work done by or against it depends only on the starting and ending points of a motion and not on the path taken.

We can define a potential energy (PE) for any conservative force. The work done against a conservative force to reach a final configuration depends on the configuration, not the path followed, and is the potential energy added.

Note:**Real World Connections: Energy of a Bowling Ball**

How much energy does a bowling ball have? (Just think about it for a minute.)

If you are thinking that you need more information, you're right. If we can measure the ball's velocity, then determining its kinetic energy is simple.

Note that this does require defining a reference frame in which to measure the velocity. Determining the ball's potential energy also requires more information. You need to know its height above the ground, which requires a reference frame of the ground. Without the ground—in other words, Earth—the ball does not classically have potential energy. Potential energy comes from the interaction between the ball and the ground. Another way of thinking about this is to compare the ball's potential energy on Earth and on the Moon. A bowling ball a certain height above Earth is going to have more potential energy than the same bowling ball the same height above the surface of the Moon, because Earth has greater mass than the Moon and therefore exerts more gravity on the ball. Thus, potential energy requires a system of at least two objects, or an object with an internal structure of at least two parts.

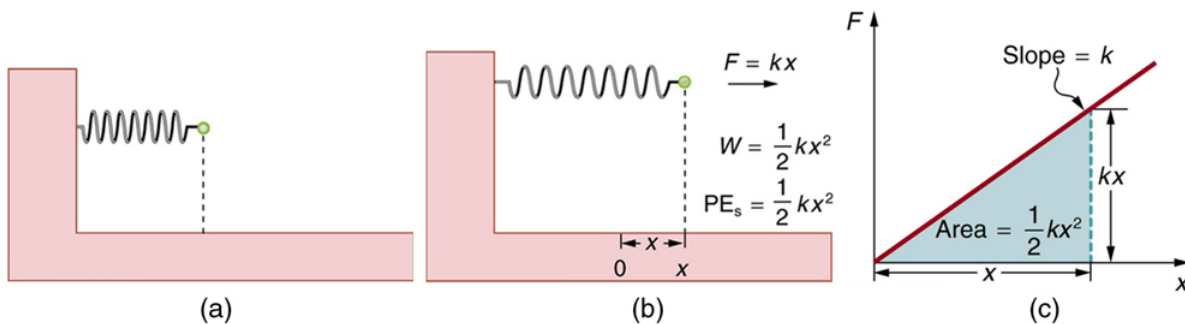
Potential Energy of a Spring

First, let us obtain an expression for the potential energy stored in a spring (PE_s). We calculate the work done to stretch or compress a spring that obeys Hooke's law. (Hooke's law was examined in [Elasticity: Stress and Strain](#), and states that the magnitude of force F on the spring and the resulting deformation ΔL are proportional, $F = k\Delta L$.) (See [\[link\]](#).) For our spring, we will replace ΔL (the amount of deformation produced by a force F) by the distance x that the spring is stretched or compressed along its length. So the force needed to stretch the spring has magnitude $F = kx$, where k is the spring's force constant. The force increases linearly from 0 at the start to kx in the fully stretched position. The average force is $kx/2$. Thus the work done in stretching or compressing the spring is $W_s = Fd = \left(\frac{kx}{2}\right)x = \frac{1}{2}kx^2$. Alternatively, we noted in [Kinetic Energy and the Work-Energy Theorem](#) that the area under a graph of F vs. x is the work done by the force. In [\[link\]](#)(c) we see that this area is also $\frac{1}{2}kx^2$. We therefore define the **potential energy of a spring**, PE_s , to be

Equation:

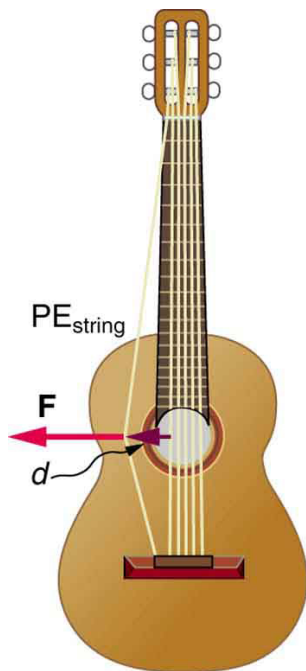
$$PE_s = \frac{1}{2}kx^2,$$

where k is the spring's force constant and x is the displacement from its undeformed position. The potential energy represents the work done *on* the spring and the energy stored in it as a result of stretching or compressing it a distance x . The potential energy of the spring PE_s does not depend on the path taken; it depends only on the stretch or squeeze x in the final configuration.



- (a) An undeformed spring has no PE_s stored in it. (b) The force needed to stretch (or compress) the spring a distance x has a magnitude $F = kx$, and the work done to stretch (or compress) it is $\frac{1}{2}kx^2$. Because the force is conservative, this work is stored as potential energy (PE_s) in the spring, and it can be fully recovered. (c) A graph of F vs. x has a slope of k , and the area under the graph is $\frac{1}{2}kx^2$. Thus the work done or potential energy stored is $\frac{1}{2}kx^2$.

The equation $PE_s = \frac{1}{2}kx^2$ has general validity beyond the special case for which it was derived. Potential energy can be stored in any elastic medium by deforming it. Indeed, the general definition of **potential energy** is energy due to position, shape, or configuration. For shape or position deformations, stored energy is $PE_s = \frac{1}{2}kx^2$, where k is the force constant of the particular system and x is its deformation. Another example is seen in [\[link\]](#) for a guitar string.



Work is done

to deform the
guitar string,
giving it
potential
energy.

When
released, the
potential
energy is
converted to
kinetic
energy and
back to
potential as
the string
oscillates
back and
forth. A very
small
fraction is
dissipated as
sound
energy,
slowly
removing
energy from
the string.

Conservation of Mechanical Energy

Let us now consider what form the work-energy theorem takes when only conservative forces are involved. This will lead us to the conservation of energy principle. The work-energy theorem states that the net work done by all forces acting on a system equals its change in kinetic energy. In equation form, this is

Equation:

$$W_{\text{net}} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \Delta\text{KE}.$$

If only conservative forces act, then

Equation:

$$W_{\text{net}} = W_{\text{c}},$$

where W_{c} is the total work done by all conservative forces. Thus,

Equation:

$$W_{\text{c}} = \Delta\text{KE}.$$

Now, if the conservative force, such as the gravitational force or a spring force, does work, the system loses potential energy. That is, $W_{\text{c}} = -\Delta\text{PE}$. Therefore,

Equation:

$$-\Delta\text{PE} = \Delta\text{KE}$$

or

Equation:

$$\Delta\text{KE} + \Delta\text{PE} = 0.$$

This equation means that the total kinetic and potential energy is constant for any process involving only conservative forces. That is,

Equation:

$$\text{KE} + \text{PE} = \text{constant}$$

or

(conservative forces only),

$$\text{KE}_i + \text{PE}_i = \text{KE}_f + \text{PE}_f$$

where i and f denote initial and final values. This equation is a form of the work-energy theorem for conservative forces; it is known as the **conservation of mechanical energy** principle. Remember that this applies to the extent that all the forces are conservative, so that friction is negligible. The total kinetic plus potential energy of a system is defined to be its **mechanical energy**, $(KE + PE)$. In a system that experiences only conservative forces, there is a potential energy associated with each force, and the energy only changes form between KE and the various types of PE, with the total energy remaining constant.

The internal energy of a system is the sum of the kinetic energies of all of its elements, plus the potential energy due to all of the interactions due to conservative forces between all of the elements.

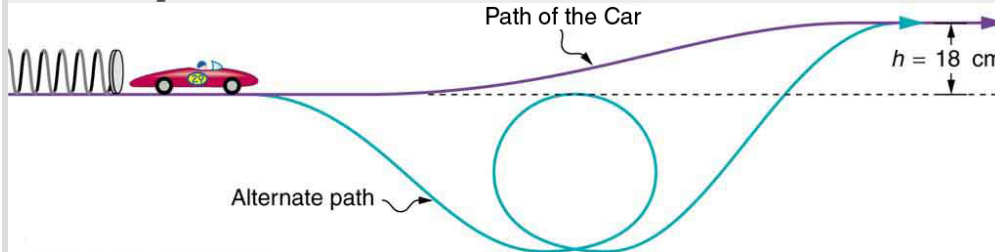
Note:**Real World Connections**

Consider a wind-up toy, such as a car. It uses a spring system to store energy. The amount of energy stored depends only on how many times it is wound, not how quickly or slowly the winding happens. Similarly, a dart gun using compressed air stores energy in its internal structure. In this case, the energy stored inside depends only on how many times it is pumped, not how quickly or slowly the pumping is done. The total energy put into the system, whether through winding or pumping, is equal to the total energy conserved in the system (minus any energy loss in the system due to interactions between its parts, such as air leaks in the dart gun). Since the internal energy of the system is conserved, you can calculate the amount of stored energy by measuring the kinetic energy of the system (the moving car or dart) when the potential energy is released.

Example:**Using Conservation of Mechanical Energy to Calculate the Speed of a Toy Car**

A 0.100-kg toy car is propelled by a compressed spring, as shown in [\[link\]](#). The car follows a track that rises 0.180 m above the starting point. The

spring is compressed 4.00 cm and has a force constant of 250.0 N/m. Assuming work done by friction to be negligible, find (a) how fast the car is going before it starts up the slope and (b) how fast it is going at the top of the slope.



A toy car is pushed by a compressed spring and coasts up a slope. Assuming negligible friction, the potential energy in the spring is first completely converted to kinetic energy, and then to a combination of kinetic and gravitational potential energy as the car rises. The details of the path are unimportant because all forces are conservative—the car would have the same final speed if it took the alternate path shown.

Strategy

The spring force and the gravitational force are conservative forces, so conservation of mechanical energy can be used. Thus,

Equation:

$$KE_i + PE_i = KE_f + PE_f$$

or

Equation:

$$\frac{1}{2}mv_i^2 + mgh_i + \frac{1}{2}kx_i^2 = \frac{1}{2}mv_f^2 + mgh_f + \frac{1}{2}kx_f^2,$$

where h is the height (vertical position) and x is the compression of the spring. This general statement looks complex but becomes much simpler when we start considering specific situations. First, we must identify the

initial and final conditions in a problem; then, we enter them into the last equation to solve for an unknown.

Solution for (a)

This part of the problem is limited to conditions just before the car is released and just after it leaves the spring. Take the initial height to be zero, so that both h_i and h_f are zero. Furthermore, the initial speed v_i is zero and the final compression of the spring x_f is zero, and so several terms in the conservation of mechanical energy equation are zero and it simplifies to

Equation:

$$\frac{1}{2} k x_i^2 = \frac{1}{2} m v_f^2.$$

In other words, the initial potential energy in the spring is converted completely to kinetic energy in the absence of friction. Solving for the final speed and entering known values yields

Equation:

$$\begin{aligned} v_f &= \sqrt{\frac{k}{m}} x_i \\ &= \sqrt{\frac{250.0 \text{ N/m}}{0.100 \text{ kg}}} (0.0400 \text{ m}) \\ &= 2.00 \text{ m/s.} \end{aligned}$$

Solution for (b)

One method of finding the speed at the top of the slope is to consider conditions just before the car is released and just after it reaches the top of the slope, completely ignoring everything in between. Doing the same type of analysis to find which terms are zero, the conservation of mechanical energy becomes

Equation:

$$\frac{1}{2} k x_i^2 = \frac{1}{2} m v_f^2 + m g h_f.$$

This form of the equation means that the spring's initial potential energy is converted partly to gravitational potential energy and partly to kinetic

energy. The final speed at the top of the slope will be less than at the bottom. Solving for v_f and substituting known values gives

Equation:

$$\begin{aligned}v_f &= \sqrt{\frac{kx_i^2}{m} - 2gh_f} \\&= \sqrt{\left(\frac{250.0 \text{ N/m}}{0.100 \text{ kg}}\right)(0.0400 \text{ m})^2 - 2(9.80 \text{ m/s}^2)(0.180 \text{ m})} \\&= 0.687 \text{ m/s.}\end{aligned}$$

Discussion

Another way to solve this problem is to realize that the car's kinetic energy before it goes up the slope is converted partly to potential energy—that is, to take the final conditions in part (a) to be the initial conditions in part (b).

Note:

Applying the Science Practices: Potential Energy in a Spring

Suppose you are running an experiment in which two 250 g carts connected by a spring (with spring constant 120 N/m) are run into a solid block, and the compression of the spring is measured. In one run of this experiment, the spring was measured to compress from its rest length of 5.0 cm to a minimum length of 2.0 cm. What was the potential energy stored in this system?

Note:

Answer

Note that the change in length of the spring is 3.0 cm. Hence we can apply Equation 7.42 to find that the potential energy is $PE = (1/2)(120 \text{ N/m})(0.030 \text{ m})^2 = 0.0541 \text{ J}$.

Note that, for conservative forces, we do not directly calculate the work they do; rather, we consider their effects through their corresponding potential energies, just as we did in [\[link\]](#). Note also that we do not consider details of the path taken—only the starting and ending points are important (as long as the path is not impossible). This assumption is usually a tremendous simplification, because the path may be complicated and forces may vary along the way.

Note:

PhET Explorations: Energy Skate Park

Learn about conservation of energy with a skater dude! Build tracks, ramps and jumps for the skater and view the kinetic energy, potential energy and friction as he moves. You can also take the skater to different planets or even space!

[Energy](#)
[y.](#)
[Skate](#)
[Park](#)

Test Prep for AP Courses

Exercise:

Problem:

Two 4.0 kg masses are connected to each other by a spring with a force constant of 25 N/m and a rest length of 1.0 m. If the spring has been compressed to 0.80 m in length and the masses are traveling toward each other at 0.50 m/s (each), what is the total energy in the system?

- a. 1.0 J
- b. 1.5 J
- c. 9.0 J
- d. 8.0 J

Exercise:

Problem:

A spring with a force constant of 5000 N/m and a rest length of 3.0 m is used in a catapult. When compressed to 1.0 m, it is used to launch a 50 kg rock. However, there is an error in the release mechanism, so the rock gets launched almost straight up. How high does it go, and how fast is it going when it hits the ground?

Solution:

20 m high, 20 m/s.

Exercise:

Problem:

What information do you need to calculate the kinetic energy and potential energy of a spring? Potential energy due to gravity? How many objects do you need information about for each of these cases?

Exercise:

Problem:

You are loading a toy dart gun, which has two settings, the more powerful with the spring compressed twice as far as the lower setting. If it takes 5.0 J of work to compress the dart gun to the lower setting, how much work does it take for the higher setting?

- a. 20 J
- b. 10 J
- c. 2.5 J
- d. 40 J

Solution:

(a)

Exercise:**Problem:**

Describe a system you use daily with internal potential energy.

Exercise:**Problem:**

Old-fashioned pendulum clocks are powered by masses that need to be wound back to the top of the clock about once a week to counteract energy lost due to friction and to the chimes. One particular clock has three masses: 4.0 kg, 4.0 kg, and 6.0 kg. They can drop 1.3 meters. How much energy does the clock use in a week?

- a. 51 J
- b. 76 J
- c. 127 J
- d. 178 J

Solution:

(d)

Exercise:**Problem:**

A water tower stores not only water, but (at least part of) the energy to move the water. How much? Make reasonable estimates for how much water is in the tower, and other quantities you need.

Exercise:

Problem:

Old-fashioned pocket watches needed to be wound daily so they wouldn't run down and lose time, due to the friction in the internal components. This required a large number of turns of the winding key, but not much force per turn, and it was possible to overwind and break the watch. How was the energy stored?

- a. A small mass raised a long distance
- b. A large mass raised a short distance
- c. A weak spring deformed a long way
- d. A strong spring deformed a short way

Solution:

(c)

Exercise:**Problem:**

Some of the very first clocks invented in China were powered by water. Describe how you think this was done.

Section Summary

- A conservative force is one for which work depends only on the starting and ending points of a motion, not on the path taken.
- We can define potential energy (PE) for any conservative force, just as we defined PE_g for the gravitational force.
- The potential energy of a spring is $PE_s = \frac{1}{2}kx^2$, where k is the spring's force constant and x is the displacement from its undeformed position.
- Mechanical energy is defined to be $KE + PE$ for a conservative force.
- When only conservative forces act on and within a system, the total mechanical energy is constant. In equation form,

Equation:

$$KE + PE = \text{constant}$$

or

$$KE_i + PE_i = KE_f + PE_f$$

where i and f denote initial and final values. This is known as the conservation of mechanical energy.

Conceptual Questions**Exercise:**

Problem: What is a conservative force?

Exercise:

Problem:

The force exerted by a diving board is conservative, provided the internal friction is negligible. Assuming friction is negligible, describe changes in the potential energy of a diving board as a swimmer dives from it, starting just before the swimmer steps on the board until just after his feet leave it.

Exercise:

Problem:

Define mechanical energy. What is the relationship of mechanical energy to nonconservative forces? What happens to mechanical energy if only conservative forces act?

Exercise:

Problem:

What is the relationship of potential energy to conservative force?

Problems & Exercises

Exercise:

Problem:

A 5.00×10^5 -kg subway train is brought to a stop from a speed of 0.500 m/s in 0.400 m by a large spring bumper at the end of its track. What is the force constant k of the spring?

Solution:

$$7.81 \times 10^5 \text{ N/m}$$

Exercise:

Problem:

A pogo stick has a spring with a force constant of 2.50×10^4 N/m, which can be compressed 12.0 cm. To what maximum height can a child jump on the stick using only the energy in the spring, if the child and stick have a total mass of 40.0 kg? Explicitly show how you follow the steps in the [\[link\]](#).

Glossary

conservative force

a force that does the same work for any given initial and final configuration, regardless of the path followed

potential energy

energy due to position, shape, or configuration

potential energy of a spring

the stored energy of a spring as a function of its displacement; when Hooke's law applies, it is given by the expression $\frac{1}{2} kx^2$ where x is the distance the spring is compressed or extended and k is the spring constant

conservation of mechanical energy

the rule that the sum of the kinetic energies and potential energies remains constant if only conservative forces act on and within a system

mechanical energy

the sum of kinetic energy and potential energy

Nonconservative Forces

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define nonconservative forces and explain how they affect mechanical energy.
- Show how the principle of conservation of energy can be applied by treating the conservative forces in terms of their potential energies and any nonconservative forces in terms of the work they do.

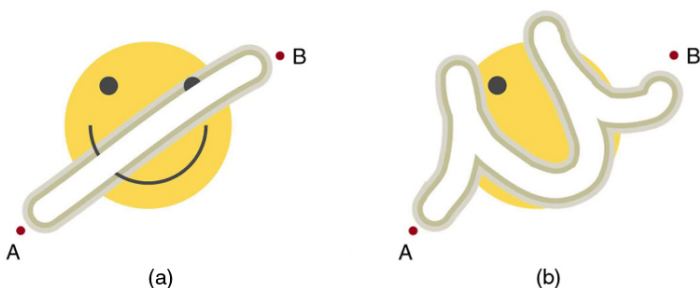
The information presented in this section supports the following AP® learning objectives and science practices:

- **4.C.1.2** The student is able to predict changes in the total energy of a system due to changes in position and speed of objects or frictional interactions within the system. **(S.P. 6.4)**
- **4.C.2.1** The student is able to make predictions about the changes in the mechanical energy of a system when a component of an external force acts parallel or antiparallel to the direction of the displacement of the center of mass. **(S.P. 6.4)**

Nonconservative Forces and Friction

Forces are either conservative or nonconservative. Conservative forces were discussed in [Conservative Forces and Potential Energy](#). A **nonconservative**

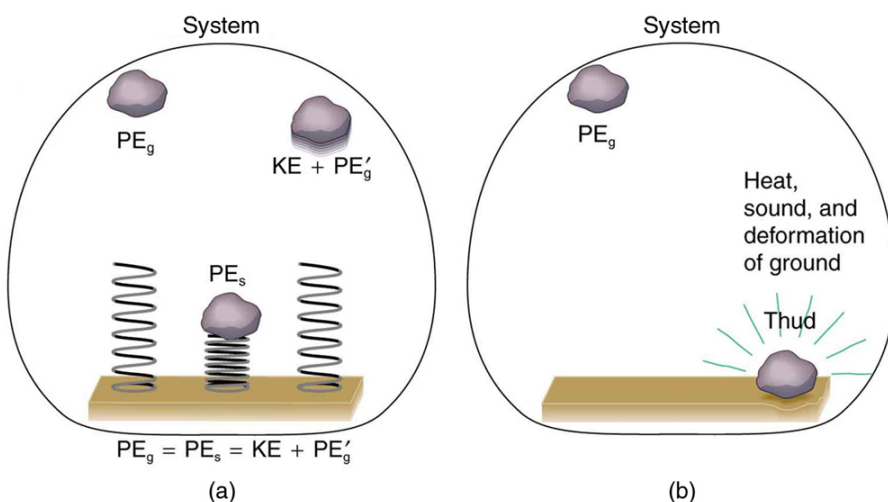
force is one for which work depends on the path taken. Friction is a good example of a nonconservative force. As illustrated in [\[link\]](#), work done against friction depends on the length of the path between the starting and ending points. Because of this dependence on path, there is no potential energy associated with nonconservative forces. An important characteristic is that the work done by a nonconservative force *adds or removes mechanical energy from a system*. **Friction**, for example, creates **thermal energy** that dissipates, removing energy from the system. Furthermore, even if the thermal energy is retained or captured, it cannot be fully converted back to work, so it is lost or not recoverable in that sense as well.



The amount of the happy face erased depends on the path taken by the eraser between points A and B, as does the work done against friction. Less work is done and less of the face is erased for the path in (a) than for the path in (b). The force here is friction, and most of the work goes into thermal energy that subsequently leaves the system (the happy face plus the eraser). The energy expended cannot be fully recovered.

How Nonconservative Forces Affect Mechanical Energy

Mechanical energy *may* not be conserved when nonconservative forces act. For example, when a car is brought to a stop by friction on level ground, it loses kinetic energy, which is dissipated as thermal energy, reducing its mechanical energy. [\[link\]](#) compares the effects of conservative and nonconservative forces. We often choose to understand simpler systems such as that described in [\[link\]](#)(a) first before studying more complicated systems as in [\[link\]](#)(b).



Comparison of the effects of conservative and nonconservative forces on the mechanical energy of a system. (a) A system with only conservative forces. When a rock is dropped onto a spring, its mechanical energy remains constant (neglecting air resistance) because the force in the spring is conservative. The spring can propel the rock back to its original height, where it once again has only potential energy due to gravity. (b) A system with nonconservative forces. When the same rock is dropped onto the ground, it is stopped by nonconservative forces that dissipate its mechanical energy as thermal energy, sound, and surface distortion. The rock has lost mechanical energy.

How the Work-Energy Theorem Applies

Now let us consider what form the work-energy theorem takes when both conservative and nonconservative forces act. We will see that the work done by nonconservative forces equals the change in the mechanical energy of a system. As noted in [Kinetic Energy and the Work-Energy Theorem](#), the work-energy theorem states that the net work on a system equals the change in its kinetic energy, or $W_{\text{net}} = \Delta KE$. The net work is the sum of the work by nonconservative forces plus the work by conservative forces. That is,

Equation:

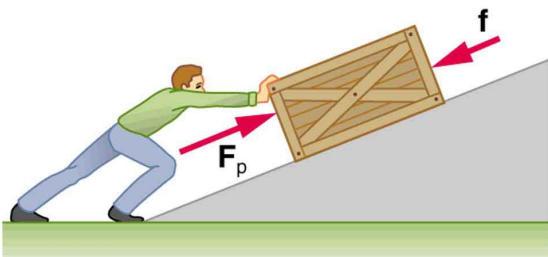
$$W_{\text{net}} = W_{\text{nc}} + W_{\text{c}},$$

so that

Equation:

$$W_{\text{nc}} + W_{\text{c}} = \Delta KE,$$

where W_{nc} is the total work done by all nonconservative forces and W_{c} is the total work done by all conservative forces.



A person pushes a crate up a ramp, doing work on the crate. Friction and gravitational force (not

shown) also do work on the crate; both forces oppose the person's push. As the crate is pushed up the ramp, it gains mechanical energy, implying that the work done by the person is greater than the work done by friction.

Consider [\[link\]](#), in which a person pushes a crate up a ramp and is opposed by friction. As in the previous section, we note that work done by a conservative force comes from a loss of gravitational potential energy, so that $W_c = -\Delta PE$. Substituting this equation into the previous one and solving for W_{nc} gives

Equation:

$$W_{nc} = \Delta KE + \Delta PE.$$

This equation means that the total mechanical energy ($KE + PE$) changes by exactly the amount of work done by nonconservative forces. In [\[link\]](#), this is the work done by the person minus the work done by friction. So even if energy is not conserved for the system of interest (such as the crate), we know that an equal amount of work was done to cause the change in total mechanical energy.

We rearrange $W_{nc} = \Delta KE + \Delta PE$ to obtain

Equation:

$$KE_i + PE_i + W_{nc} = KE_f + PE_f.$$

This means that the amount of work done by nonconservative forces adds to the mechanical energy of a system. If W_{nc} is positive, then mechanical energy is increased, such as when the person pushes the crate up the ramp in [\[link\]](#). If W_{nc} is negative, then mechanical energy is decreased, such as

when the rock hits the ground in [\[link\]](#)(b). If W_{nc} is zero, then mechanical energy is conserved, and nonconservative forces are balanced. For example, when you push a lawn mower at constant speed on level ground, your work done is removed by the work of friction, and the mower has a constant energy.

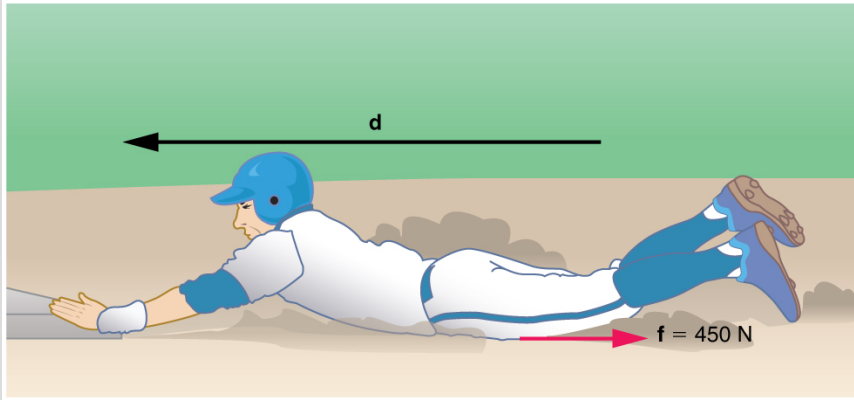
Applying Energy Conservation with Nonconservative Forces

When no change in potential energy occurs, applying $KE_i + PE_i + W_{nc} = KE_f + PE_f$ amounts to applying the work-energy theorem by setting the change in kinetic energy to be equal to the net work done on the system, which in the most general case includes both conservative and nonconservative forces. But when seeking instead to find a change in total mechanical energy in situations that involve changes in both potential and kinetic energy, the previous equation $KE_i + PE_i + W_{nc} = KE_f + PE_f$ says that you can start by finding the change in mechanical energy that would have resulted from just the conservative forces, including the potential energy changes, and add to it the work done, with the proper sign, by any nonconservative forces involved.

Example:

Calculating Distance Traveled: How Far a Baseball Player Slides

Consider the situation shown in [\[link\]](#), where a baseball player slides to a stop on level ground. Using energy considerations, calculate the distance the 65.0-kg baseball player slides, given that his initial speed is 6.00 m/s and the force of friction against him is a constant 450 N.



The baseball player slides to a stop in a distance d . In the process, friction removes the player's kinetic energy by doing an amount of work fd equal to the initial kinetic energy.

Strategy

Friction stops the player by converting his kinetic energy into other forms, including thermal energy. In terms of the work-energy theorem, the work done by friction, which is negative, is added to the initial kinetic energy to reduce it to zero. The work done by friction is negative, because \mathbf{f} is in the opposite direction of the motion (that is, $\theta = 180^\circ$, and so $\cos \theta = -1$).

Thus $W_{\text{nc}} = -fd$. The equation simplifies to

Equation:

$$\frac{1}{2}mv_i^2 - fd = 0$$

or

Equation:

$$fd = \frac{1}{2}mv_i^2.$$

This equation can now be solved for the distance d .

Solution

Solving the previous equation for d and substituting known values yields

Equation:

$$\begin{aligned}
 d &= \frac{mv_i^2}{2f} \\
 &= \frac{(65.0 \text{ kg})(6.00 \text{ m/s})^2}{(2)(450 \text{ N})} \\
 &= 2.60 \text{ m.}
 \end{aligned}$$

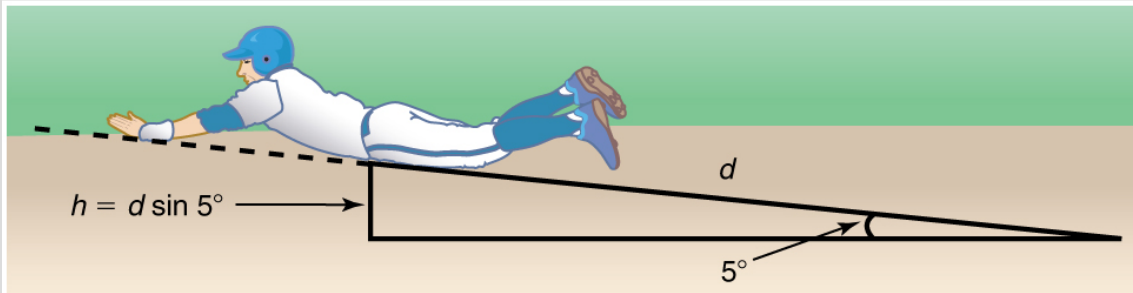
Discussion

The most important point of this example is that the amount of nonconservative work equals the change in mechanical energy. For example, you must work harder to stop a truck, with its large mechanical energy, than to stop a mosquito.

Example:

Calculating Distance Traveled: Sliding Up an Incline

Suppose that the player from [\[link\]](#) is running up a hill having a 5.00° incline upward with a surface similar to that in the baseball stadium. The player slides with the same initial speed, and the frictional force is still 450 N. Determine how far he slides.



The same baseball player slides to a stop on a 5.00° slope.

Strategy

In this case, the work done by the nonconservative friction force on the player reduces the mechanical energy he has from his kinetic energy at zero height, to the final mechanical energy he has by moving through distance d to reach height h along the hill, with $h = d \sin 5.00^\circ$. This is expressed by the equation

Equation:

$$\text{KE}_i + \text{PE}_i + W_{\text{nc}} = \text{KE}_f + \text{PE}_f.$$

Solution

The work done by friction is again $W_{\text{nc}} = -fd$; initially the potential energy is $\text{PE}_i = mg \cdot 0 = 0$ and the kinetic energy is $\text{KE}_i = \frac{1}{2}mv_i^2$; the final energy contributions are $\text{KE}_f = 0$ for the kinetic energy and $\text{PE}_f = mgh = mgd \sin \theta$ for the potential energy.

Substituting these values gives

Equation:

$$\frac{1}{2}mv_i^2 + 0 + (-fd) = 0 + mgd \sin \theta.$$

Solve this for d to obtain

Equation:

$$\begin{aligned} d &= \frac{\left(\frac{1}{2}\right)mv_i^2}{f + mg \sin \theta} \\ &= \frac{(0.5)(65.0 \text{ kg})(6.00 \text{ m/s})^2}{450 \text{ N} + (65.0 \text{ kg})(9.80 \text{ m/s}^2) \sin (5.00^\circ)} \\ &= 2.31 \text{ m.} \end{aligned}$$

Discussion

As might have been expected, the player slides a shorter distance by sliding uphill. Note that the problem could also have been solved in terms of the forces directly and the work energy theorem, instead of using the potential energy. This method would have required combining the normal force and force of gravity vectors, which no longer cancel each other because they point in different directions, and friction, to find the net force. You could then use the net force and the net work to find the distance d that reduces the kinetic energy to zero. By applying conservation of energy and using the potential energy instead, we need only consider the gravitational potential energy mgh , without combining and resolving force vectors. This simplifies the solution considerably.

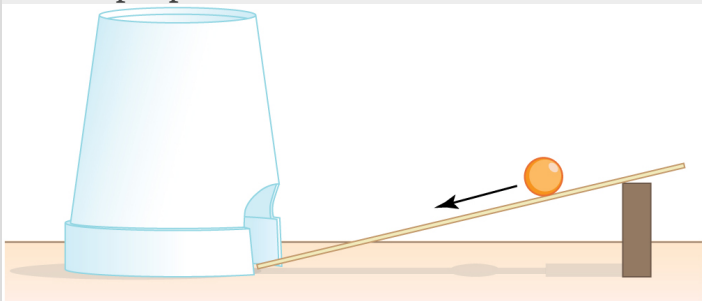
Note:

Making Connections: Take-Home Investigation—Determining Friction from the Stopping Distance

This experiment involves the conversion of gravitational potential energy into thermal energy. Use the ruler, book, and marble from [\[link\]](#). In addition, you will need a foam cup with a small hole in the side, as shown in [\[link\]](#). From the 10-cm position on the ruler, let the marble roll into the cup positioned at the bottom of the ruler. Measure the distance d the cup moves before stopping. What forces caused it to stop? What happened to the kinetic energy of the marble at the bottom of the ruler? Next, place the marble at the 20-cm and the 30-cm positions and again measure the distance the cup moves after the marble enters it. Plot the distance the cup moves versus the initial marble position on the ruler. Is this relationship linear?

With some simple assumptions, you can use these data to find the coefficient of kinetic friction μ_k of the cup on the table. The force of friction f on the cup is $\mu_k N$, where the normal force N is just the weight of the cup plus the marble. The normal force and force of gravity do no work because they are perpendicular to the displacement of the cup, which moves horizontally. The work done by friction is fd . You will need the mass of the marble as well to calculate its initial kinetic energy.

It is interesting to do the above experiment also with a steel marble (or ball bearing). Releasing it from the same positions on the ruler as you did with the glass marble, is the velocity of this steel marble the same as the velocity of the marble at the bottom of the ruler? Is the distance the cup moves proportional to the mass of the steel and glass marbles?



Rolling a marble down a ruler into a foam cup.

Note:**PhET Explorations: The Ramp**

Explore forces, energy and work as you push household objects up and down a ramp. Lower and raise the ramp to see how the angle of inclination affects the parallel forces acting on the file cabinet. Graphs show forces, energy and work.

[The
Ramp](#)
p

Test Prep for AP Courses**Exercise:****Problem:**

You are in a room in a basement with a smooth concrete floor (friction force equals 40 N) and a nice rug (friction force equals 55 N) that is 3 m by 4 m. However, you have to push a very heavy box from one corner of the rug to the opposite corner of the rug. Will you do more work against friction going around the floor or across the rug, and how much extra?

- a. Across the rug is 275 J extra
- b. Around the floor is 5 J extra
- c. Across the rug is 5 J extra
- d. Around the floor is 280 J extra

Solution:

(b)

Exercise:**Problem:**

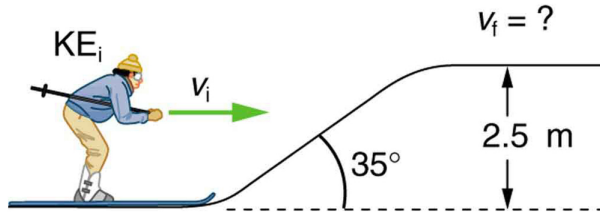
In the Appalachians, along the interstate, there are ramps of loose gravel for semis that have had their brakes fail to drive into to stop. Design an experiment to measure how effective this would be.

Section Summary

- A nonconservative force is one for which work depends on the path.
- Friction is an example of a nonconservative force that changes mechanical energy into thermal energy.
- Work W_{nc} done by a nonconservative force changes the mechanical energy of a system. In equation form, $W_{nc} = \Delta KE + \Delta PE$ or, equivalently, $KE_i + PE_i + W_{nc} = KE_f + PE_f$.
- When both conservative and nonconservative forces act, energy conservation can be applied and used to calculate motion in terms of the known potential energies of the conservative forces and the work done by nonconservative forces, instead of finding the net work from the net force, or having to directly apply Newton's laws.

Problems & Exercises**Exercise:****Problem:**

A 60.0-kg skier with an initial speed of 12.0 m/s coasts up a 2.50-m-high rise as shown in [\[link\]](#). Find her final speed at the top, given that the coefficient of friction between her skis and the snow is 0.0800. (Hint: Find the distance traveled up the incline assuming a straight-line path as shown in the figure.)



The skier's initial kinetic energy is partially used in coasting to the top of a rise.

Solution:

9.46 m/s

Exercise:

Problem:

(a) How high a hill can a car coast up (engine disengaged) if work done by friction is negligible and its initial speed is 110 km/h? (b) If, in actuality, a 750-kg car with an initial speed of 110 km/h is observed to coast up a hill to a height 22.0 m above its starting point, how much thermal energy was generated by friction? (c) What is the average force of friction if the hill has a slope 2.5° above the horizontal?

Glossary

nonconservative force

a force whose work depends on the path followed between the given initial and final configurations

friction

the force between surfaces that opposes one sliding on the other; friction changes mechanical energy into thermal energy

Conservation of Energy

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain the law of the conservation of energy.
- Describe some of the many forms of energy.
- Define efficiency of an energy conversion process as the fraction left as useful energy or work, rather than being transformed, for example, into thermal energy.

The information presented in this section supports the following AP® learning objectives and science practices:

- **4.C.1.2** The student is able to predict changes in the total energy of a system due to changes in position and speed of objects or frictional interactions within the system. **(S.P. 6.4)**
- **4.C.2.1** The student is able to make predictions about the changes in the mechanical energy of a system when a component of an external force acts parallel or antiparallel to the direction of the displacement of the center of mass. **(S.P. 6.4)**
- **4.C.2.2** The student is able to apply the concepts of conservation of energy and the work-energy theorem to determine qualitatively and/or quantitatively that work done on a two-object system in linear motion will change the kinetic energy of the center of mass of the system, the potential energy of the systems, and/or the internal energy of the system. **(S.P. 1.4, 2.2, 7.2)**
- **5.A.2.1** The student is able to define open and closed systems for everyday situations and apply conservation concepts for energy, charge, and linear momentum to those situations. **(S.P. 6.4, 7.2)**
- **5.B.5.4** The student is able to make claims about the interaction between a system and its environment in which the environment exerts a force on the system, thus doing work on the system and

changing the energy of the system (kinetic energy plus potential energy). (S.P. 6.4, 7.2)

- **5.B.5.5** The student is able to predict and calculate the energy transfer to (i.e., the work done on) an object or system from information about a force exerted on the object or system through a distance. (S.P. 2.2, 6.4)

Law of Conservation of Energy

Energy, as we have noted, is conserved, making it one of the most important physical quantities in nature. The **law of conservation of energy** can be stated as follows:

Total energy is constant in any process. It may change in form or be transferred from one system to another, but the total remains the same.

We have explored some forms of energy and some ways it can be transferred from one system to another. This exploration led to the definition of two major types of energy—mechanical energy ($KE + PE$) and energy transferred via work done by nonconservative forces (W_{nc}). But energy takes *many* other forms, manifesting itself in *many* different ways, and we need to be able to deal with all of these before we can write an equation for the above general statement of the conservation of energy.

Other Forms of Energy than Mechanical Energy

At this point, we deal with all other forms of energy by lumping them into a single group called other energy (OE). Then we can state the conservation of energy in equation form as

Equation:

$$KE_i + PE_i + W_{nc} + OE_i = KE_f + PE_f + OE_f.$$

All types of energy and work can be included in this very general statement of conservation of energy. Kinetic energy is KE , work done by a conservative force is represented by PE , work done by nonconservative forces is W_{nc} , and all other energies are included as OE . This equation applies to all previous examples; in those situations OE was constant, and so it subtracted out and was not directly considered.

Note:

Making Connections: Usefulness of the Energy Conservation Principle

The fact that energy is conserved and has many forms makes it very important. You will find that energy is discussed in many contexts, because it is involved in all processes. It will also become apparent that many situations are best understood in terms of energy and that problems are often most easily conceptualized and solved by considering energy.

When does OE play a role? One example occurs when a person eats. Food is oxidized with the release of carbon dioxide, water, and energy. Some of this chemical energy is converted to kinetic energy when the person moves, to potential energy when the person changes altitude, and to thermal energy (another form of OE).

Some of the Many Forms of Energy

What are some other forms of energy? You can probably name a number of forms of energy not yet discussed. Many of these will be covered in later chapters, but let us detail a few here. **Electrical energy** is a common form that is converted to many other forms and does work in a wide range of practical situations. Fuels, such as gasoline and food, carry **chemical energy** that can be transferred to a system through oxidation. Chemical fuel can also produce electrical energy, such as in batteries. Batteries can in turn produce light, which is a very pure form of energy. Most energy sources on Earth are in fact stored energy from the energy we receive from the Sun. We sometimes refer to this as **radiant energy**, or electromagnetic radiation, which includes visible light, infrared, and ultraviolet radiation. **Nuclear energy** comes from

processes that convert measurable amounts of mass into energy. Nuclear energy is transformed into the energy of sunlight, into electrical energy in power plants, and into the energy of the heat transfer and blast in weapons. Atoms and molecules inside all objects are in random motion. This internal mechanical energy from the random motions is called **thermal energy**, because it is related to the temperature of the object. These and all other forms of energy can be converted into one another and can do work.

Note:

Real World Connections: Open or Closed System?

Consider whether the following systems are open or closed: a car, a spring-operated dart gun, and the system shown in [\[link\]](#)(a).

A car is not a closed system. You add energy in the form of more gas in the tank (or charging the batteries), and energy is lost due to air resistance and friction.

A spring-operated dart gun is not a closed system. You have to initially compress the spring. Once that has been done, however, the dart gun and dart can be treated as a closed system. All of the energy remains in the system consisting of these two objects.

[\[link\]](#)(a) is an example of a closed system, once it has been started. All of the energy in the system remains there; none is brought in from outside or leaves.

[\[link\]](#) gives the amount of energy stored, used, or released from various objects and in various phenomena. The range of energies and the variety of types and situations is impressive.

Note:

Problem-Solving Strategies for Energy

You will find the following problem-solving strategies useful whenever you deal with energy. The strategies help in organizing and reinforcing energy concepts. In fact, they are used in the examples presented in this chapter. The familiar general problem-solving strategies presented earlier—involving

identifying physical principles, knowns, and unknowns, checking units, and so on—continue to be relevant here.

Step 1. Determine the system of interest and identify what information is given and what quantity is to be calculated. A sketch will help.

Step 2. Examine all the forces involved and determine whether you know or are given the potential energy from the work done by the forces. Then use step 3 or step 4.

Step 3. If you know the potential energies for the forces that enter into the problem, then forces are all conservative, and you can apply conservation of mechanical energy simply in terms of potential and kinetic energy. The equation expressing conservation of energy is

Equation:

$$KE_i + PE_i = KE_f + PE_f.$$

Step 4. If you know the potential energy for only some of the forces, possibly because some of them are nonconservative and do not have a potential energy, or if there are other energies that are not easily treated in terms of force and work, then the conservation of energy law in its most general form must be used.

Equation:

$$KE_i + PE_i + W_{nc} + OE_i = KE_f + PE_f + OE_f.$$

In most problems, one or more of the terms is zero, simplifying its solution. Do not calculate W_c , the work done by conservative forces; it is already incorporated in the PE terms.

Step 5. You have already identified the types of work and energy involved (in step 2). Before solving for the unknown, *eliminate terms wherever possible* to simplify the algebra. For example, choose $h = 0$ at either the initial or final point, so that PE_g is zero there. Then solve for the unknown in the customary manner.

Step 6. *Check the answer to see if it is reasonable.* Once you have solved a problem, reexamine the forms of work and energy to see if you have set up the conservation of energy equation correctly. For example, work done against friction should be negative, potential energy at the bottom of a hill should be less than that at the top, and so on. Also check to see that the numerical value obtained is reasonable. For example, the final speed of a

skateboarder who coasts down a 3-m-high ramp could reasonably be 20 km/h, but *not* 80 km/h.

Transformation of Energy

The transformation of energy from one form into others is happening all the time. The chemical energy in food is converted into thermal energy through metabolism; light energy is converted into chemical energy through photosynthesis. In a larger example, the chemical energy contained in coal is converted into thermal energy as it burns to turn water into steam in a boiler. This thermal energy in the steam in turn is converted to mechanical energy as it spins a turbine, which is connected to a generator to produce electrical energy. (In all of these examples, not all of the initial energy is converted into the forms mentioned. This important point is discussed later in this section.)

Another example of energy conversion occurs in a solar cell. Sunlight impinging on a solar cell (see [\[link\]](#)) produces electricity, which in turn can be used to run an electric motor. Energy is converted from the primary source of solar energy into electrical energy and then into mechanical energy.



Solar energy is converted into electrical energy by solar cells, which is used to run a motor in this solar-power aircraft. (credit: NASA)

Object/phenomenon	Energy in joules
Big Bang	10^{68}
Energy released in a supernova	10^{44}
Fusion of all the hydrogen in Earth's oceans	10^{34}
Annual world energy use	4×10^{20}
Large fusion bomb (9 megaton)	3.8×10^{16}
1 kg hydrogen (fusion to helium)	6.4×10^{14}
1 kg uranium (nuclear fission)	8.0×10^{13}
Hiroshima-size fission bomb (10 kiloton)	4.2×10^{13}

Object/phenomenon	Energy in joules
90,000-ton aircraft carrier at 30 knots	1.1×10^{10}
1 barrel crude oil	5.9×10^9
1 ton TNT	4.2×10^9
1 gallon of gasoline	1.2×10^8
Daily home electricity use (developed countries)	7×10^7
Daily adult food intake (recommended)	1.2×10^7
1000-kg car at 90 km/h	3.1×10^5
1 g fat (9.3 kcal)	3.9×10^4
ATP hydrolysis reaction	3.2×10^4

Object/phenomenon	Energy in joules
1 g carbohydrate (4.1 kcal)	1.7×10^4
1 g protein (4.1 kcal)	1.7×10^4
Tennis ball at 100 km/h	22
Mosquito (10^{-2} g at 0.5 m/s)	1.3×10^{-6}
Single electron in a TV tube beam	4.0×10^{-15}
Energy to break one DNA strand	10^{-19}

Energy of Various Objects and Phenomena

Efficiency

Even though energy is conserved in an energy conversion process, the output of *useful energy* or work will be less than the energy input. The **efficiency** Eff of an energy conversion process is defined as

Equation:

$$\text{Efficiency(Eff)} = \frac{\text{useful energy or work output}}{\text{total energy input}} = \frac{W_{\text{out}}}{E_{\text{in}}}.$$

[\[link\]](#) lists some efficiencies of mechanical devices and human activities. In a coal-fired power plant, for example, about 40% of the chemical energy in the coal becomes useful electrical energy. The other 60% transforms into other (perhaps less useful) energy forms, such as thermal energy, which is then released to the environment through combustion gases and cooling towers.

Activity/device	Efficiency (%) [footnote] Representative values
Cycling and climbing	20
Swimming, surface	2
Swimming, submerged	4
Shoveling	3
Weightlifting	9
Steam engine	17
Gasoline engine	30
Diesel engine	35
Nuclear power plant	35
Coal power plant	42
Electric motor	98
Compact fluorescent light	20

Activity/device	Efficiency (%) [footnote] Representative values
Gas heater (residential)	90
Solar cell	10

Efficiency of the Human Body and Mechanical Devices

Note:

PhET Explorations: Masses and Springs

A realistic mass and spring laboratory. Hang masses from springs and adjust the spring stiffness and damping. You can even slow time. Transport the lab to different planets. A chart shows the kinetic, potential, and thermal energies for each spring.

[Masses
and
Spring
s](#)

Test Prep for AP Courses

Exercise:

Problem:

You do 30 J of work to load a toy dart gun. However, the dart is 10 cm long and feels a frictional force of 10 N while going through the dart gun's barrel. What is the kinetic energy of the fired dart?

- a. 30 J

- b. 29 J
- c. 28 J
- d. 27 J

Solution:

(c)

Exercise:

Problem:

When an object is lifted by a crane, it begins and ends its motion at rest. The same is true of an object pushed across a rough surface. Explain why this happens. What are the differences between these systems?

Exercise:

Problem:

A child has two red wagons, with the rear one tied to the front by a stretchy rope (a spring). If the child pulls on the front wagon, the ____ increases.

- a. kinetic energy of the wagons
- b. potential energy stored in the spring
- c. both A and B
- d. not enough information

Solution:

(c)

Exercise:

Problem:

A child has two red wagons, with the rear one tied to the front by a stretchy rope (a spring). If the child pulls on the front wagon, the energy stored in the system increases. How do the relative amounts of potential and kinetic energy in this system change over time?

Exercise:

Problem: Which of the following are closed systems?

- a. Earth
- b. a car
- c. a frictionless pendulum
- d. a mass on a spring in a vacuum

Solution:

(c), (d)

Exercise:

Problem: Describe a real-world example of a closed system.

Exercise:

Problem:

A 5.0-kg rock falls off of a 10 m cliff. If air resistance exerts a force of 10 N, what is the kinetic energy when the rock hits the ground?

- a. 400 J
- b. 12.6 m/s
- c. 100 J
- d. 500 J

Solution:

(a)

Exercise:

Problem:

Hydroelectricity is generated by storing water behind a dam, and then letting some of it run through generators in the dam to turn them. If the system is the water, what is the environment that is doing work on it? If a dam has water 100 m deep behind it, how much energy was generated if 10,000 kg of water exited the dam at 2.0 m/s?

Exercise:**Problem:**

Before railroads were invented, goods often traveled along canals, with mules pulling barges from the bank. If a mule is exerting a 1200 N force for 10 km, and the rope connecting the mule to the barge is at a 20 degree angle from the direction of travel, how much work did the mule do on the barge?

- a. 12 MJ
- b. 11 MJ
- c. 4.1 MJ
- d. 6 MJ

Solution:

(b)

Exercise:**Problem:**

Describe an instance today in which you did work, by the scientific definition. Then calculate how much work you did in that instance, showing your work.

Section Summary

- The law of conservation of energy states that the total energy is constant in any process. Energy may change in form or be transferred from one

system to another, but the total remains the same.

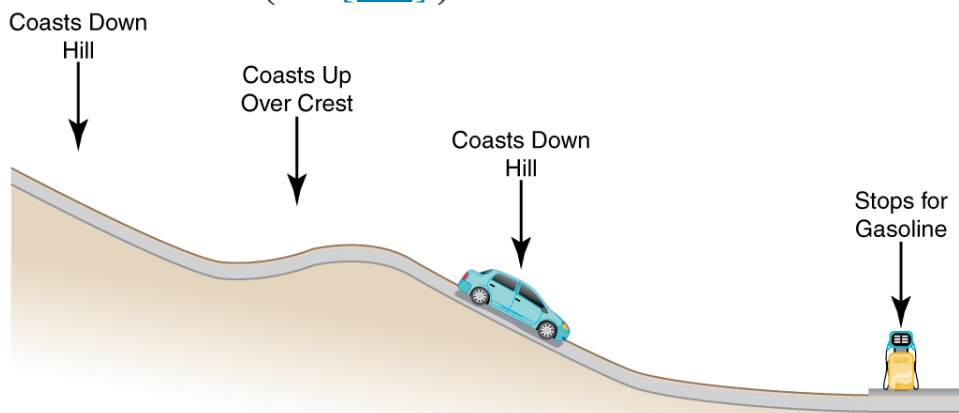
- When all forms of energy are considered, conservation of energy is written in equation form as $KE_i + PE_i + W_{nc} + OE_i = KE_f + PE_f + OE_f$, where OE is all **other forms of energy** besides mechanical energy.
- Commonly encountered forms of energy include electric energy, chemical energy, radiant energy, nuclear energy, and thermal energy.
- Energy is often utilized to do work, but it is not possible to convert all the energy of a system to work.
- The efficiency Eff of a machine or human is defined to be $Eff = \frac{W_{out}}{E_{in}}$, where W_{out} is useful work output and E_{in} is the energy consumed.

Conceptual Questions

Exercise:

Problem:

Consider the following scenario. A car for which friction is *not* negligible accelerates from rest down a hill, running out of gasoline after a short distance. The driver lets the car coast farther down the hill, then up and over a small crest. He then coasts down that hill into a gas station, where he brakes to a stop and fills the tank with gasoline. Identify the forms of energy the car has, and how they are changed and transferred in this series of events. (See [\[link\]](#).)



A car experiencing non-negligible friction coasts down a hill, over a small crest, then downhill again, and comes to a stop at a gas station.

Exercise:**Problem:**

Describe the energy transfers and transformations for a javelin, starting from the point at which an athlete picks up the javelin and ending when the javelin is stuck into the ground after being thrown.

Exercise:**Problem:**

Do devices with efficiencies of less than one violate the law of conservation of energy? Explain.

Exercise:**Problem:**

List four different forms or types of energy. Give one example of a conversion from each of these forms to another form.

Exercise:

Problem: List the energy conversions that occur when riding a bicycle.

Problems & Exercises**Exercise:****Problem:**

Using values from [\[link\]](#), how many DNA molecules could be broken by the energy carried by a single electron in the beam of an old-fashioned TV tube? (These electrons were not dangerous in themselves, but they did create dangerous x rays. Later model tube TVs had shielding that absorbed x rays before they escaped and exposed viewers.)

Solution:

4×10^4 molecules

Exercise:

Problem:

Using energy considerations and assuming negligible air resistance, show that a rock thrown from a bridge 20.0 m above water with an initial speed of 15.0 m/s strikes the water with a speed of 24.8 m/s independent of the direction thrown.

Solution:

Equating ΔPE_g and ΔKE , we obtain

$$v = \sqrt{2gh + v_0^2} = \sqrt{2(9.80 \text{ m/s}^2)(20.0 \text{ m}) + (15.0 \text{ m/s})^2} = 24.8 \text{ m/s}$$

Exercise:

Problem:

If the energy in fusion bombs were used to supply the energy needs of the world, how many of the 9-megaton variety would be needed for a year's supply of energy (using data from [\[link\]](#))? This is not as far-fetched as it may sound—there are thousands of nuclear bombs, and their energy can be trapped in underground explosions and converted to electricity, as natural geothermal energy is.

Exercise:

Problem:

(a) Use of hydrogen fusion to supply energy is a dream that may be realized in the next century. Fusion would be a relatively clean and almost limitless supply of energy, as can be seen from [\[link\]](#). To illustrate this, calculate how many years the present energy needs of the world could be supplied by one millionth of the oceans' hydrogen fusion energy. (b) How does this time compare with historically significant events, such as the duration of stable economic systems?

Solution:

(a) 25×10^6 years

(b) This is much, much longer than human time scales.

Glossary

law of conservation of energy

the general law that total energy is constant in any process; energy may change in form or be transferred from one system to another, but the total remains the same

electrical energy

the energy carried by a flow of charge

chemical energy

the energy in a substance stored in the bonds between atoms and molecules that can be released in a chemical reaction

radiant energy

the energy carried by electromagnetic waves

nuclear energy

energy released by changes within atomic nuclei, such as the fusion of two light nuclei or the fission of a heavy nucleus

thermal energy

the energy within an object due to the random motion of its atoms and molecules that accounts for the object's temperature

efficiency

a measure of the effectiveness of the input of energy to do work; useful energy or work divided by the total input of energy

Power

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Calculate power by calculating changes in energy over time.
- Examine power consumption and calculations of the cost of energy consumed.

What is Power?

Power—the word conjures up many images: a professional football player muscling aside his opponent, a dragster roaring away from the starting line, a volcano blowing its lava into the atmosphere, or a rocket blasting off, as in [\[link\]](#).



This powerful rocket on the Space Shuttle *Endeavor* did work and consumed energy at a very high rate. (credit: NASA)

These images of power have in common the rapid performance of work, consistent with the scientific definition of **power** (P) as the rate at which work is done.

Note:

Power

Power is the rate at which work is done.

Equation:

$$P = \frac{W}{t}$$

The SI unit for power is the **watt** (W), where 1 watt equals 1 joule/second ($1 \text{ W} = 1 \text{ J/s}$).

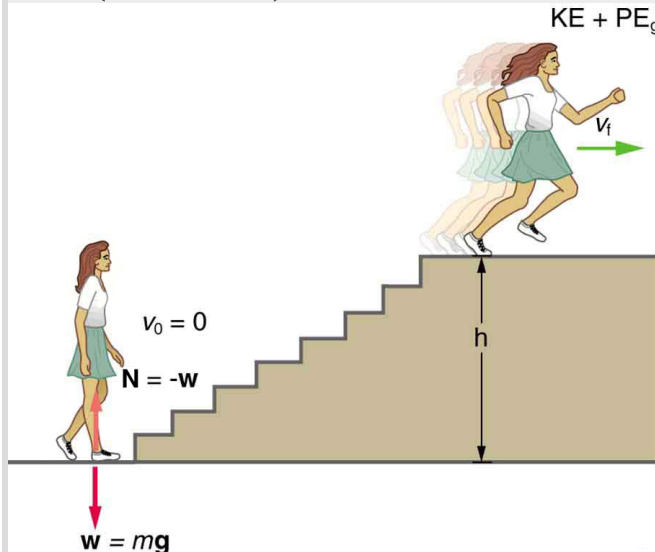
Because work is energy transfer, power is also the rate at which energy is expended. A 60-W light bulb, for example, expends 60 J of energy per second. Great power means a large amount of work or energy developed in a short time. For example, when a powerful car accelerates rapidly, it does a large amount of work and consumes a large amount of fuel in a short time.

Calculating Power from Energy

Example:

Calculating the Power to Climb Stairs

What is the power output for a 60.0-kg woman who runs up a 3.00 m high flight of stairs in 3.50 s, starting from rest but having a final speed of 2.00 m/s? (See [\[link\]](#).)



When this woman runs upstairs starting from rest, she converts the chemical energy originally from food into kinetic energy and gravitational potential energy. Her power output depends on how fast she does this.

Strategy and Concept

The work going into mechanical energy is $W = KE + PE$. At the bottom of the stairs, we take both KE and PE_g as initially zero; thus,

$W = KE_f + PE_g = \frac{1}{2}mv_f^2 + mgh$, where h is the vertical height of the stairs. Because all terms are given, we can calculate W and then divide it by time to get power.

Solution

Substituting the expression for W into the definition of power given in the previous equation, $P = W/t$ yields

Equation:

$$P = \frac{W}{t} = \frac{\frac{1}{2}mv_f^2 + mgh}{t}.$$

Entering known values yields

Equation:

$$\begin{aligned} P &= \frac{0.5(60.0 \text{ kg})(2.00 \text{ m/s})^2 + (60.0 \text{ kg})(9.80 \text{ m/s}^2)(3.00 \text{ m})}{3.50 \text{ s}} \\ &= \frac{120 \text{ J} + 1764 \text{ J}}{3.50 \text{ s}} \\ &= 538 \text{ W}. \end{aligned}$$

Discussion

The woman does 1764 J of work to move up the stairs compared with only 120 J to increase her kinetic energy; thus, most of her power output is required for climbing rather than accelerating.

It is impressive that this woman's useful power output is slightly less than 1 **horsepower** (1 hp = 746 W)! People can generate more than a horsepower with their leg muscles for short periods of time by rapidly converting available blood sugar and oxygen into work output. (A horse can put out 1 hp for hours on end.) Once oxygen is depleted, power output decreases and the person begins to breathe rapidly to obtain oxygen to metabolize more food—this is known as the *aerobic* stage of exercise. If the woman climbed the stairs slowly, then her power output would be much less, although the amount of work done would be the same.

Note:

Making Connections: Take-Home Investigation—Measure Your Power Rating

Determine your own power rating by measuring the time it takes you to climb a flight of stairs. We will ignore the gain in kinetic energy, as the above example showed that it was a small portion of the energy gain. Don't expect that your output will be more than about 0.5 hp.

Examples of Power

Examples of power are limited only by the imagination, because there are as many types as there are forms of work and energy. (See [\[link\]](#) for some examples.) Sunlight reaching Earth's surface carries a maximum power of about 1.3 kilowatts per square meter (kW/m^2). A tiny fraction of this is retained by Earth over the long term. Our consumption rate of fossil fuels is far greater than the rate at which they are stored, so it is inevitable that they will be depleted. Power implies that energy is transferred, perhaps changing form. It is never possible to change one form completely into another without losing some of it as thermal energy. For example, a 60-W incandescent bulb converts only 5 W of electrical power to light, with 55 W dissipating into thermal energy. Furthermore, the typical electric power plant converts only 35 to 40% of its fuel into electricity. The remainder becomes a huge amount of thermal energy that must be dispersed as heat transfer, as rapidly as it is created. A coal-fired power plant may produce 1000 megawatts; 1 megawatt (MW) is 10^6 W of electric power. But the power plant consumes chemical energy at a rate of about 2500 MW, creating heat transfer to the surroundings at a rate of 1500 MW. (See [\[link\]](#).)



Tremendous amounts of electric power are generated by coal-fired power plants such as this one in China, but an even larger

amount of power goes into heat transfer to the surroundings. The large cooling towers here are needed to transfer heat as rapidly as it is produced. The transfer of heat is not unique to coal plants but is an unavoidable consequence of generating electric power from any fuel—nuclear, coal, oil, natural gas, or the like. (credit: Kleinolive, Wikimedia Commons)

Object or Phenomenon	Power in Watts
Supernova (at peak)	5×10^{37}
Milky Way galaxy	10^{37}
Crab Nebula pulsar	10^{28}

Object or Phenomenon	Power in Watts
The Sun	4×10^{26}
Volcanic eruption (maximum)	4×10^{15}
Lightning bolt	2×10^{12}
Nuclear power plant (total electric and heat transfer)	3×10^9
Aircraft carrier (total useful and heat transfer)	10^8
Dragster (total useful and heat transfer)	2×10^6
Car (total useful and heat transfer)	8×10^4
Football player (total useful and heat transfer)	5×10^3
Clothes dryer	4×10^3

Object or Phenomenon	Power in Watts
Person at rest (all heat transfer)	100
Typical incandescent light bulb (total useful and heat transfer)	60
Heart, person at rest (total useful and heat transfer)	8
Electric clock	3
Pocket calculator	10^{-3}

Power Output or Consumption

Power and Energy Consumption

We usually have to pay for the energy we use. It is interesting and easy to estimate the cost of energy for an electrical appliance if its power consumption rate and time used are known. The higher the power consumption rate and the longer the appliance is used, the greater the cost of that appliance. The power consumption rate is $P = W/t = E/t$, where E is the energy supplied by the electricity company. So the energy consumed over a time t is

Equation:

$$E = Pt.$$

Electricity bills state the energy used in units of **kilowatt-hours** ($\text{kW} \cdot \text{h}$), which is the product of power in kilowatts and time in hours. This unit is convenient because electrical power consumption at the kilowatt level for hours at a time is typical.

Example:

Calculating Energy Costs

What is the cost of running a 0.200-kW computer 6.00 h per day for 30.0 d if the cost of electricity is \$0.120 per $\text{kW} \cdot \text{h}$?

Strategy

Cost is based on energy consumed; thus, we must find E from $E = Pt$ and then calculate the cost. Because electrical energy is expressed in $\text{kW} \cdot \text{h}$, at the start of a problem such as this it is convenient to convert the units into kW and hours.

Solution

The energy consumed in $\text{kW} \cdot \text{h}$ is

Equation:

$$\begin{aligned} E &= Pt = (0.200 \text{ kW})(6.00 \text{ h/d})(30.0 \text{ d}) \\ &= 36.0 \text{ kW} \cdot \text{h}, \end{aligned}$$

and the cost is simply given by

Equation:

$$\text{cost} = (36.0 \text{ kW} \cdot \text{h})(\$0.120 \text{ per kW} \cdot \text{h}) = \$4.32 \text{ per month.}$$

Discussion

The cost of using the computer in this example is neither exorbitant nor negligible. It is clear that the cost is a combination of power and time. When both are high, such as for an air conditioner in the summer, the cost is high.

The motivation to save energy has become more compelling with its ever-increasing price. Armed with the knowledge that energy consumed is the

product of power and time, you can estimate costs for yourself and make the necessary value judgments about where to save energy. Either power or time must be reduced. It is most cost-effective to limit the use of high-power devices that normally operate for long periods of time, such as water heaters and air conditioners. This would not include relatively high power devices like toasters, because they are on only a few minutes per day. It would also not include electric clocks, in spite of their 24-hour-per-day usage, because they are very low power devices. It is sometimes possible to use devices that have greater efficiencies—that is, devices that consume less power to accomplish the same task. One example is the compact fluorescent light bulb, which produces over four times more light per watt of power consumed than its incandescent cousin.

Modern civilization depends on energy, but current levels of energy consumption and production are not sustainable. The likelihood of a link between global warming and fossil fuel use (with its concomitant production of carbon dioxide), has made reduction in energy use as well as a shift to non-fossil fuels of the utmost importance. Even though energy in an isolated system is a conserved quantity, the final result of most energy transformations is waste heat transfer to the environment, which is no longer useful for doing work. As we will discuss in more detail in [Thermodynamics](#), the potential for energy to produce useful work has been “degraded” in the energy transformation.

Section Summary

- Power is the rate at which work is done, or in equation form, for the average power P for work W done over a time t , $P = W/t$.
- The SI unit for power is the watt (W), where $1 \text{ W} = 1 \text{ J/s}$.
- The power of many devices such as electric motors is also often expressed in horsepower (hp), where $1 \text{ hp} = 746 \text{ W}$.

Conceptual Questions

Exercise:

Problem:

Most electrical appliances are rated in watts. Does this rating depend on how long the appliance is on? (When off, it is a zero-watt device.) Explain in terms of the definition of power.

Exercise:**Problem:**

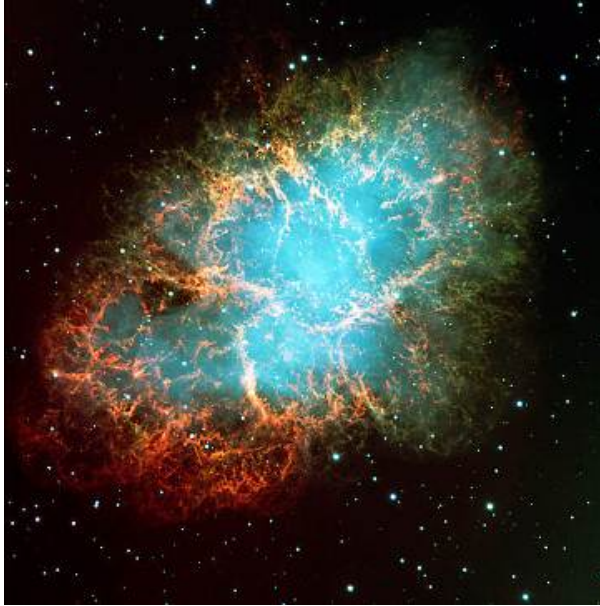
Explain, in terms of the definition of power, why energy consumption is sometimes listed in kilowatt-hours rather than joules. What is the relationship between these two energy units?

Exercise:**Problem:**

A spark of static electricity, such as that you might receive from a doorknob on a cold dry day, may carry a few hundred watts of power. Explain why you are not injured by such a spark.

Problems & Exercises**Exercise:****Problem:**

The Crab Nebula (see [\[link\]](#)) pulsar is the remnant of a supernova that occurred in A.D. 1054. Using data from [\[link\]](#), calculate the approximate factor by which the power output of this astronomical object has declined since its explosion.



Crab Nebula (credit: ESO, via
Wikimedia Commons)

Solution:

$$2 \times 10^{-10}$$

Exercise:

Problem:

Suppose a star 1000 times brighter than our Sun (that is, emitting 1000 times the power) suddenly goes supernova. Using data from [\[link\]](#): (a) By what factor does its power output increase? (b) How many times brighter than our entire Milky Way galaxy is the supernova? (c) Based on your answers, discuss whether it should be possible to observe supernovas in distant galaxies. Note that there are on the order of 10^{11} observable galaxies, the average brightness of which is somewhat less than our own galaxy.

Exercise:

Problem:

A person in good physical condition can put out 100 W of useful power for several hours at a stretch, perhaps by pedaling a mechanism that drives an electric generator. Neglecting any problems of generator efficiency and practical considerations such as resting time: (a) How many people would it take to run a 4.00-kW electric clothes dryer? (b) How many people would it take to replace a large electric power plant that generates 800 MW?

Solution:

(a) 40

(b) 8 million

Exercise:**Problem:**

What is the cost of operating a 3.00-W electric clock for a year if the cost of electricity is \$0.0900 per kW · h?

Exercise:**Problem:**

A large household air conditioner may consume 15.0 kW of power. What is the cost of operating this air conditioner 3.00 h per day for 30.0 d if the cost of electricity is \$0.110 per kW · h?

Solution:

\$149

Exercise:

Problem:

(a) What is the average power consumption in watts of an appliance that uses $5.00 \text{ kW} \cdot \text{h}$ of energy per day? (b) How many joules of energy does this appliance consume in a year?

Exercise:**Problem:**

(a) What is the average useful power output of a person who does $6.00 \times 10^6 \text{ J}$ of useful work in 8.00 h ? (b) Working at this rate, how long will it take this person to lift 2000 kg of bricks 1.50 m to a platform? (Work done to lift his body can be omitted because it is not considered useful output here.)

Solution:

(a) 208 W

(b) 141 s

Exercise:**Problem:**

A 500-kg dragster accelerates from rest to a final speed of 110 m/s in 400 m (about a quarter of a mile) and encounters an average frictional force of 1200 N . What is its average power output in watts and horsepower if this takes 7.30 s ?

Exercise:**Problem:**

(a) How long will it take an 850-kg car with a useful power output of 40.0 hp ($1 \text{ hp} = 746 \text{ W}$) to reach a speed of 15.0 m/s , neglecting friction? (b) How long will this acceleration take if the car also climbs a 3.00-m -high hill in the process?

Solution:

(a) 3.20 s

(b) 4.04 s

Exercise:

Problem:

(a) Find the useful power output of an elevator motor that lifts a 2500-kg load a height of 35.0 m in 12.0 s, if it also increases the speed from rest to 4.00 m/s. Note that the total mass of the counterbalanced system is 10,000 kg—so that only 2500 kg is raised in height, but the full 10,000 kg is accelerated. (b) What does it cost, if electricity is \$0.0900 per kW · h?

Exercise:

Problem:

(a) What is the available energy content, in joules, of a battery that operates a 2.00-W electric clock for 18 months? (b) How long can a battery that can supply 8.00×10^4 J run a pocket calculator that consumes energy at the rate of 1.00×10^{-3} W?

Solution:

(a) 9.46×10^7 J

(b) 2.54 y

Exercise:

Problem:

(a) How long would it take a 1.50×10^5 -kg airplane with engines that produce 100 MW of power to reach a speed of 250 m/s and an altitude of 12.0 km if air resistance were negligible? (b) If it actually takes 900 s, what is the power? (c) Given this power, what is the average force of air resistance if the airplane takes 1200 s? (Hint: You must find the distance the plane travels in 1200 s assuming constant acceleration.)

Exercise:

Problem:

Calculate the power output needed for a 950-kg car to climb a 2.00° slope at a constant 30.0 m/s while encountering wind resistance and friction totaling 600 N. Explicitly show how you follow the steps in the [Problem-Solving Strategies for Energy](#).

Solution:

Identify knowns: $m = 950$ kg, slope angle $\theta = 2.00^\circ$, $v = 30.0$ m/s, $f = 600$ N

Identify unknowns: power P of the car, force F that car applies to road

Solve for unknown:

$$P = \frac{W}{t} = \frac{Fd}{t} = F\left(\frac{d}{t}\right) = Fv,$$

where F is parallel to the incline and must oppose the resistive forces and the force of gravity:

$$F = f + w = 600 \text{ N} + mg \sin \theta$$

Insert this into the expression for power and solve:

$$\begin{aligned} P &= (f + mg \sin \theta)v \\ &= \left[600 \text{ N} + (950 \text{ kg})(9.80 \text{ m/s}^2) \sin 2^\circ \right] (30.0 \text{ m/s}) \\ &= 2.77 \times 10^4 \text{ W} \end{aligned}$$

About 28 kW (or about 37 hp) is reasonable for a car to climb a gentle incline.

Exercise:

Problem:

(a) Calculate the power per square meter reaching Earth's upper atmosphere from the Sun. (Take the power output of the Sun to be 4.00×10^{26} W.) (b) Part of this is absorbed and reflected by the atmosphere, so that a maximum of 1.30 kW/m^2 reaches Earth's surface. Calculate the area in km^2 of solar energy collectors needed to replace an electric power plant that generates 750 MW if the collectors convert an average of 2.00% of the maximum power into electricity. (This small conversion efficiency is due to the devices themselves, and the fact that the sun is directly overhead only briefly.) With the same assumptions, what area would be needed to meet the United States' energy needs (1.05×10^{20} J)? Australia's energy needs (5.4×10^{18} J)? China's energy needs (6.3×10^{19} J)? (These energy consumption values are from 2006.)

Glossary

power

the rate at which work is done

watt

(W) SI unit of power, with $1 \text{ W} = 1 \text{ J/s}$

horsepower

an older non-SI unit of power, with $1 \text{ hp} = 746 \text{ W}$

kilowatt-hour

(kW · h) unit used primarily for electrical energy provided by electric utility companies

Work, Energy, and Power in Humans

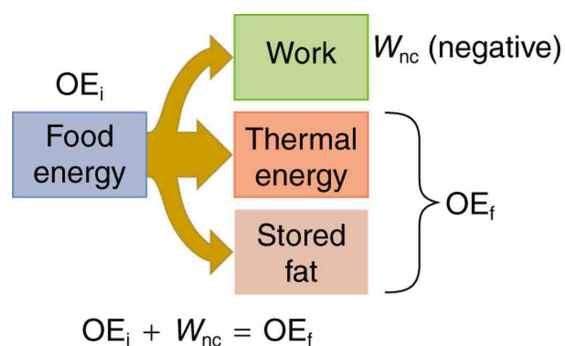
LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain the human body's consumption of energy when at rest versus when engaged in activities that do useful work.
- Calculate the conversion of chemical energy in food into useful work.

Energy Conversion in Humans

Our own bodies, like all living organisms, are energy conversion machines. Conservation of energy implies that the chemical energy stored in food is converted into work, thermal energy, and/or stored as chemical energy in fatty tissue. (See [\[link\]](#).) The fraction going into each form depends both on how much we eat and on our level of physical activity. If we eat more than is needed to do work and stay warm, the remainder goes into body fat.



Energy consumed by humans is converted to work, thermal energy, and stored fat. By far the largest fraction goes to thermal energy, although the fraction varies depending on the type of physical activity.

Power Consumed at Rest

The *rate* at which the body uses food energy to sustain life and to do different activities is called the **metabolic rate**. The total energy conversion rate of a person *at rest* is called the **basal metabolic rate** (BMR) and is divided among various systems in the body, as shown in [\[link\]](#). The largest fraction goes to the liver and spleen, with the brain coming next. Of course, during vigorous exercise, the energy consumption of the skeletal muscles and heart increase markedly. About 75% of the calories burned in a day go into these basic functions. The BMR is a function of age, gender, total body weight, and amount of muscle mass (which burns more calories than body fat). Athletes have a greater BMR due to this last factor.

Organ	Power consumed at rest (W)	Oxygen consumption (mL/min)	Percent of BMR
Liver & spleen	23	67	27

Organ	Power consumed at rest (W)	Oxygen consumption (mL/min)	Percent of BMR
Brain	16	47	19
Skeletal muscle	15	45	18
Kidney	9	26	10
Heart	6	17	7
Other	16	48	19
Totals	85 W	250 mL/min	100%

Basal Metabolic Rates (BMR)

Energy consumption is directly proportional to oxygen consumption because the digestive process is basically one of oxidizing food. We can measure the energy people use during various activities by measuring their oxygen use. (See [\[link\]](#).) Approximately 20 kJ of energy are produced for each liter of oxygen consumed, independent of the type of food. [\[link\]](#) shows energy and oxygen consumption rates (power expended) for a variety of activities.

Power of Doing Useful Work

Work done by a person is sometimes called **useful work**, which is *work done on the outside world*, such as lifting weights. Useful work requires a force exerted through a distance on the outside world, and so it excludes internal work, such as that done by the heart when pumping blood. Useful work does include that done in climbing stairs or accelerating to a full run, because these are accomplished by exerting forces on the outside world. Forces exerted by the body are nonconservative, so that they can change the

mechanical energy (KE + PE) of the system worked upon, and this is often the goal. A baseball player throwing a ball, for example, increases both the ball's kinetic and potential energy.

If a person needs more energy than they consume, such as when doing vigorous work, the body must draw upon the chemical energy stored in fat. So exercise can be helpful in losing fat. However, the amount of exercise needed to produce a loss in fat, or to burn off extra calories consumed that day, can be large, as [\[link\]](#) illustrates.

Example:**Calculating Weight Loss from Exercising**

If a person who normally requires an average of 12,000 kJ (3000 kcal) of food energy per day consumes 13,000 kJ per day, he will steadily gain weight. How much bicycling per day is required to work off this extra 1000 kJ?

Solution

[\[link\]](#) states that 400 W are used when cycling at a moderate speed. The time required to work off 1000 kJ at this rate is then

Equation:

$$\text{Time} = \frac{\text{energy}}{\left(\frac{\text{energy}}{\text{time}}\right)} = \frac{1000 \text{ kJ}}{400 \text{ W}} = 2500 \text{ s} = 42 \text{ min.}$$

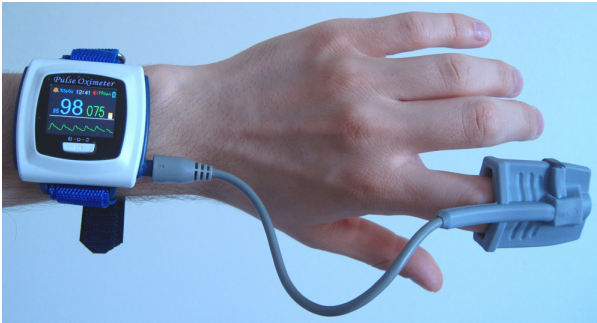
Discussion

If this person uses more energy than he or she consumes, the person's body will obtain the needed energy by metabolizing body fat. If the person uses 13,000 kJ but consumes only 12,000 kJ, then the amount of fat loss will be

Equation:

$$\text{Fat loss} = (1000 \text{ kJ}) \left(\frac{1.0 \text{ g fat}}{39 \text{ kJ}} \right) = 26 \text{ g,}$$

assuming the energy content of fat to be 39 kJ/g.



A pulse oxymeter is an apparatus that measures the amount of oxygen in blood.

Oxymeters can be used to determine a person's metabolic rate, which is the rate at which food energy is converted to another form. Such measurements can indicate the level of athletic conditioning as well as certain medical problems. (credit: UusiAjaja, Wikimedia Commons)

Activity	Energy consumption in watts	Oxygen consumption in liters O ₂ /min
Sleeping	83	0.24
Sitting at rest	120	0.34

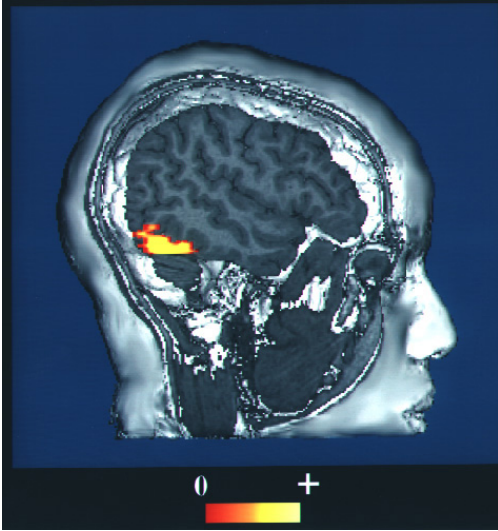
Activity	Energy consumption in watts	Oxygen consumption in liters O₂/min
Standing relaxed	125	0.36
Sitting in class	210	0.60
Walking (5 km/h)	280	0.80
Cycling (13–18 km/h)	400	1.14
Shivering	425	1.21
Playing tennis	440	1.26
Swimming breaststroke	475	1.36
Ice skating (14.5 km/h)	545	1.56
Climbing stairs (116/min)	685	1.96
Cycling (21 km/h)	700	2.00
Running cross-country	740	2.12
Playing basketball	800	2.28

Activity	Energy consumption in watts	Oxygen consumption in liters O₂/min
Cycling, professional racer	1855	5.30
Sprinting	2415	6.90

Energy and Oxygen Consumption Rates[\[footnote\]](#) (Power)
for an average 76-kg male

All bodily functions, from thinking to lifting weights, require energy. (See [\[link\]](#).) The many small muscle actions accompanying all quiet activity, from sleeping to head scratching, ultimately become thermal energy, as do less visible muscle actions by the heart, lungs, and digestive tract.

Shivering, in fact, is an involuntary response to low body temperature that pits muscles against one another to produce thermal energy in the body (and do no work). The kidneys and liver consume a surprising amount of energy, but the biggest surprise of all is that a full 25% of all energy consumed by the body is used to maintain electrical potentials in all living cells. (Nerve cells use this electrical potential in nerve impulses.) This bioelectrical energy ultimately becomes mostly thermal energy, but some is utilized to power chemical processes such as in the kidneys and liver, and in fat production.



This fMRI scan shows an increased level of energy consumption in the vision center of the brain. Here, the patient was being asked to recognize faces.
(credit: NIH via Wikimedia Commons)

Section Summary

- The human body converts energy stored in food into work, thermal energy, and/or chemical energy that is stored in fatty tissue.
- The *rate* at which the body uses food energy to sustain life and to do different activities is called the metabolic rate, and the corresponding rate when at rest is called the basal metabolic rate (BMR)
- The energy included in the basal metabolic rate is divided among various systems in the body, with the largest fraction going to the liver and spleen, and the brain coming next.
- About 75% of food calories are used to sustain basic body functions included in the basal metabolic rate.

- The energy consumption of people during various activities can be determined by measuring their oxygen use, because the digestive process is basically one of oxidizing food.

Conceptual Questions

Exercise:

Problem:

Explain why it is easier to climb a mountain on a zigzag path rather than one straight up the side. Is your increase in gravitational potential energy the same in both cases? Is your energy consumption the same in both?

Exercise:

Problem:

Do you do work on the outside world when you rub your hands together to warm them? What is the efficiency of this activity?

Exercise:

Problem:

Shivering is an involuntary response to lowered body temperature. What is the efficiency of the body when shivering, and is this a desirable value?

Exercise:

Problem:

Discuss the relative effectiveness of dieting and exercise in losing weight, noting that most athletic activities consume food energy at a rate of 400 to 500 W, while a single cup of yogurt can contain 1360 kJ (325 kcal). Specifically, is it likely that exercise alone will be sufficient to lose weight? You may wish to consider that regular exercise may increase the metabolic rate, whereas protracted dieting may reduce it.

Problems & Exercises

Exercise:

Problem:

(a) How long can you rapidly climb stairs (116/min) on the 93.0 kcal of energy in a 10.0-g pat of butter? (b) How many flights is this if each flight has 16 stairs?

Solution:

(a) 9.5 min

(b) 69 flights of stairs

Exercise:

Problem:

(a) What is the power output in watts and horsepower of a 70.0-kg sprinter who accelerates from rest to 10.0 m/s in 3.00 s? (b) Considering the amount of power generated, do you think a well-trained athlete could do this repetitively for long periods of time?

Exercise:

Problem:

Calculate the power output in watts and horsepower of a shot-putter who takes 1.20 s to accelerate the 7.27-kg shot from rest to 14.0 m/s, while raising it 0.800 m. (Do not include the power produced to accelerate his body.)



Shot putter at the
Dornoch Highland
Gathering in 2007.
(credit: John Haslam,
Flickr)

Solution:

641 W, 0.860 hp

Exercise:

Problem:

(a) What is the efficiency of an out-of-condition professor who does 2.10×10^5 J of useful work while metabolizing 500 kcal of food energy? (b) How many food calories would a well-conditioned athlete metabolize in doing the same work with an efficiency of 20%?

Exercise:

Problem:

Energy that is not utilized for work or heat transfer is converted to the chemical energy of body fat containing about 39 kJ/g. How many grams of fat will you gain if you eat 10,000 kJ (about 2500 kcal) one day and do nothing but sit relaxed for 16.0 h and sleep for the other 8.00 h? Use data from [\[link\]](#) for the energy consumption rates of these activities.

Solution:

31 g

Exercise:**Problem:**

Using data from [\[link\]](#), calculate the daily energy needs of a person who sleeps for 7.00 h, walks for 2.00 h, attends classes for 4.00 h, cycles for 2.00 h, sits relaxed for 3.00 h, and studies for 6.00 h. (Studying consumes energy at the same rate as sitting in class.)

Exercise:**Problem:**

What is the efficiency of a subject on a treadmill who puts out work at the rate of 100 W while consuming oxygen at the rate of 2.00 L/min? (Hint: See [\[link\]](#).)

Solution:

14.3%

Exercise:

Problem:

Shoveling snow can be extremely taxing because the arms have such a low efficiency in this activity. Suppose a person shoveling a footpath metabolizes food at the rate of 800 W. (a) What is her useful power output? (b) How long will it take her to lift 3000 kg of snow 1.20 m? (This could be the amount of heavy snow on 20 m of footpath.) (c) How much waste heat transfer in kilojoules will she generate in the process?

Exercise:**Problem:**

Very large forces are produced in joints when a person jumps from some height to the ground. (a) Calculate the magnitude of the force produced if an 80.0-kg person jumps from a 0.600-m-high ledge and lands stiffly, compressing joint material 1.50 cm as a result. (Be certain to include the weight of the person.) (b) In practice the knees bend almost involuntarily to help extend the distance over which you stop. Calculate the magnitude of the force produced if the stopping distance is 0.300 m. (c) Compare both forces with the weight of the person.

Solution:

(a) $3.21 \times 10^4 \text{ N}$

(b) $2.35 \times 10^3 \text{ N}$

(c) Ratio of net force to weight of person is 41.0 in part (a); 3.00 in part (b)

Exercise:

Problem:

Jogging on hard surfaces with insufficiently padded shoes produces large forces in the feet and legs. (a) Calculate the magnitude of the force needed to stop the downward motion of a jogger's leg, if his leg has a mass of 13.0 kg, a speed of 6.00 m/s, and stops in a distance of 1.50 cm. (Be certain to include the weight of the 75.0-kg jogger's body.) (b) Compare this force with the weight of the jogger.

Exercise:**Problem:**

(a) Calculate the energy in kJ used by a 55.0-kg woman who does 50 deep knee bends in which her center of mass is lowered and raised 0.400 m. (She does work in both directions.) You may assume her efficiency is 20%. (b) What is the average power consumption rate in watts if she does this in 3.00 min?

Solution:

(a) 108 kJ

(b) 599 W

Exercise:**Problem:**

Kanellos Kanellopoulos flew 119 km from Crete to Santorini, Greece, on April 23, 1988, in the *Daedalus 88*, an aircraft powered by a bicycle-type drive mechanism (see [\[link\]](#)). His useful power output for the 234-min trip was about 350 W. Using the efficiency for cycling from [\[link\]](#), calculate the food energy in kilojoules he metabolized during the flight.

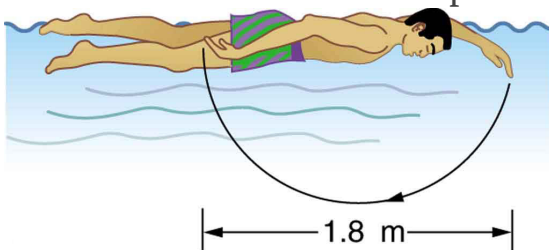


The Daedalus 88 in flight.
(credit: NASA photo by
Beasley)

Exercise:

Problem:

The swimmer shown in [\[link\]](#) exerts an average horizontal backward force of 80.0 N with his arm during each 1.80 m long stroke. (a) What is his work output in each stroke? (b) Calculate the power output of his arms if he does 120 strokes per minute.



Solution:

(a) 144 J

(b) 288 W

Exercise:

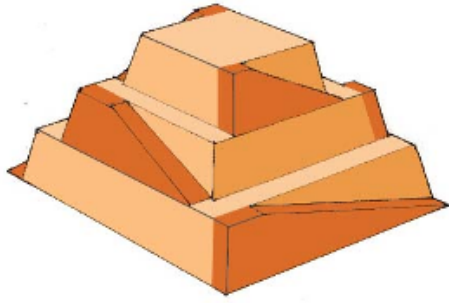
Problem:

Mountain climbers carry bottled oxygen when at very high altitudes.

(a) Assuming that a mountain climber uses oxygen at twice the rate for climbing 116 stairs per minute (because of low air temperature and winds), calculate how many liters of oxygen a climber would need for 10.0 h of climbing. (These are liters at sea level.) Note that only 40% of the inhaled oxygen is utilized; the rest is exhaled. (b) How much useful work does the climber do if he and his equipment have a mass of 90.0 kg and he gains 1000 m of altitude? (c) What is his efficiency for the 10.0-h climb?

Exercise:**Problem:**

The awe-inspiring Great Pyramid of Cheops was built more than 4500 years ago. Its square base, originally 230 m on a side, covered 13.1 acres, and it was 146 m high, with a mass of about 7×10^9 kg. (The pyramid's dimensions are slightly different today due to quarrying and some sagging.) Historians estimate that 20,000 workers spent 20 years to construct it, working 12-hour days, 330 days per year. (a) Calculate the gravitational potential energy stored in the pyramid, given its center of mass is at one-fourth its height. (b) Only a fraction of the workers lifted blocks; most were involved in support services such as building ramps (see [\[link\]](#)), bringing food and water, and hauling blocks to the site. Calculate the efficiency of the workers who did the lifting, assuming there were 1000 of them and they consumed food energy at the rate of 300 kcal/h. What does your answer imply about how much of their work went into block-lifting, versus how much work went into friction and lifting and lowering their own bodies? (c) Calculate the mass of food that had to be supplied each day, assuming that the average worker required 3600 kcal per day and that their diet was 5% protein, 60% carbohydrate, and 35% fat. (These proportions neglect the mass of bulk and nondigestible materials consumed.)



Ancient pyramids were probably constructed using ramps as simple machines.
(credit: Franck Monnier, Wikimedia Commons)

Solution:

- (a) 2.50×10^{12} J
- (b) 2.52%
- (c) 1.4×10^4 kg (14 metric tons)

Exercise:**Problem:**

(a) How long can you play tennis on the 800 kJ (about 200 kcal) of energy in a candy bar? (b) Does this seem like a long time? Discuss why exercise is necessary but may not be sufficient to cause a person to lose weight.

Glossary

metabolic rate

the rate at which the body uses food energy to sustain life and to do different activities

basal metabolic rate

the total energy conversion rate of a person at rest

useful work

work done on an external system

LEARNING OBJECTIVES

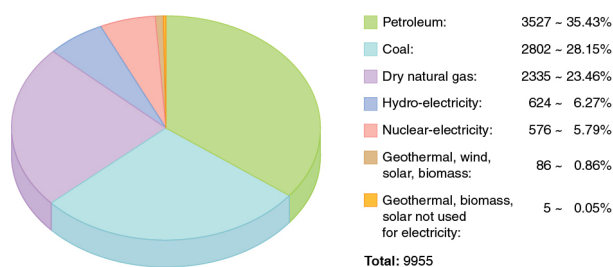
By the end of this section, you will be able to:

- Describe the distinction between renewable and nonrenewable energy sources.
- Explain why the inevitable conversion of energy to less useful forms makes it necessary to conserve energy resources.

Energy is an important ingredient in all phases of society. We live in a very interdependent world, and access to adequate and reliable energy resources is crucial for economic growth and for maintaining the quality of our lives. But current levels of energy consumption and production are not sustainable. About 40% of the world's energy comes from oil, and much of that goes to transportation uses. Oil prices are dependent as much upon new (or foreseen) discoveries as they are upon political events and situations around the world. The U.S., with 4.5% of the world's population, consumes 24% of the world's oil production per year; 66% of that oil is imported!

Renewable and Nonrenewable Energy Sources

The principal energy resources used in the world are shown in [\[link\]](#). The fuel mix has changed over the years but now is dominated by oil, although natural gas and solar contributions are increasing. **Renewable forms of energy** are those sources that cannot be used up, such as water, wind, solar, and biomass. About 85% of our energy comes from nonrenewable **fossil fuels**—oil, natural gas, coal. The likelihood of a link between global warming and fossil fuel use, with its production of carbon dioxide through combustion, has made, in the eyes of many scientists, a shift to non-fossil fuels of utmost importance—but it will not be easy.

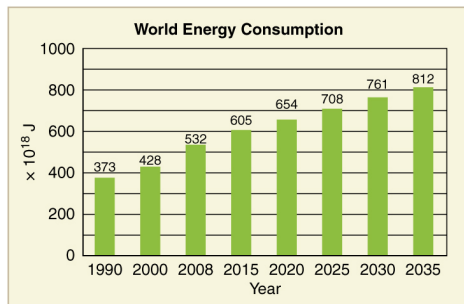


World energy consumption by source, in billions of kilowatt-hours: 2006. (credit: KVDP)

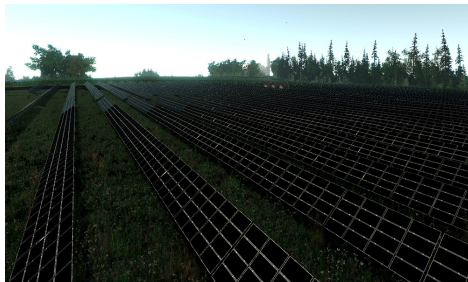
The World's Growing Energy Needs

World energy consumption continues to rise, especially in the developing countries. (See [\[link\]](#).) Global demand for energy has tripled in the past 50 years and might triple again in the next 30 years. While much of this growth will come from the rapidly booming economies of China and India, many of the developed countries, especially

those in Europe, are hoping to meet their energy needs by expanding the use of renewable sources. Although presently only a small percentage, renewable energy is growing very fast, especially wind energy. For example, Germany plans to meet 20% of its electricity and 10% of its overall energy needs with renewable resources by the year 2020. (See [\[link\]](#).) Energy is a key constraint in the rapid economic growth of China and India. In 2003, China surpassed Japan as the world's second largest consumer of oil. However, over 1/3 of this is imported. Unlike most Western countries, coal dominates the commercial energy resources of China, accounting for 2/3 of its energy consumption. In 2009 China surpassed the United States as the largest generator of CO₂. In India, the main energy resources are biomass (wood and dung) and coal. Half of India's oil is imported. About 70% of India's electricity is generated by highly polluting coal. Yet there are sizeable strides being made in renewable energy. India has a rapidly growing wind energy base, and it has the largest solar cooking program in the world.



Past and projected world energy use
(source: Based on data from U.S.
Energy Information Administration,
2011)



Solar cell arrays at a power plant in
Steindorf, Germany (credit: Michael
Betke, Flickr)

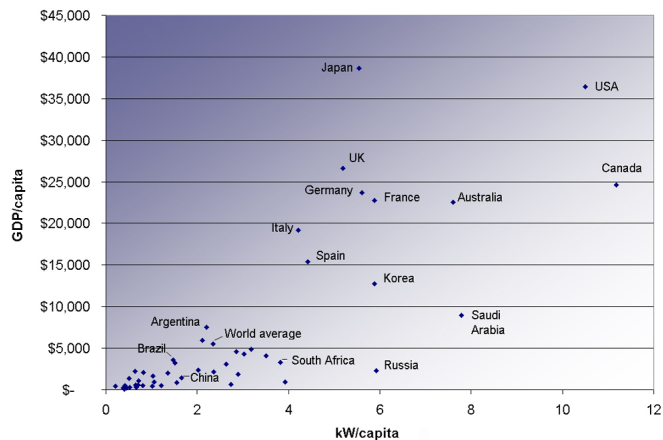
[\[link\]](#) displays the 2006 commercial energy mix by country for some of the prime energy users in the world. While non-renewable sources dominate, some countries get a sizeable percentage of their electricity from renewable resources. For example, about 67% of New Zealand's electricity demand is met by hydroelectric. Only 10% of the U.S. electricity is generated by renewable resources, primarily hydroelectric. It is difficult to determine total contributions of renewable energy in some countries with a large rural population, so these percentages in this table are left blank.

Country	Consumption, in EJ (10 ¹⁸ J)	Oil	Natural Gas	Coal	Nuclear	Hydro	Other Renewables
Australia	5.4	34%	17%	44%	0%	3%	1%
Brazil	9.6	48%	7%	5%	1%	35%	2%
China	63	22%	3%	69%	1%	6%	
Egypt	2.4	50%	41%	1%	0%	6%	
Germany	16	37%	24%	24%	11%	1%	3%
India	15	34%	7%	52%	1%	5%	
Indonesia	4.9	51%	26%	16%	0%	2%	3%
Japan	24	48%	14%	21%	12%	4%	1%
New Zealand	0.44	32%	26%	6%	0%	11%	19%
Russia	31	19%	53%	16%	5%	6%	
U.S.	105	40%	23%	22%	8%	3%	1%
World	432	39%	23%	24%	6%	6%	2%

Energy Consumption—Selected Countries (2006)

Energy and Economic Well-being

The last two columns in this table examine the energy and electricity use per capita. Economic well-being is dependent upon energy use, and in most countries higher standards of living, as measured by GDP (gross domestic product) per capita, are matched by higher levels of energy consumption per capita. This is borne out in [\[link\]](#). Increased efficiency of energy use will change this dependency. A global problem is balancing energy resource development against the harmful effects upon the environment in its extraction and use.



Power consumption per capita versus GDP per capita for various countries. Note the increase in energy usage with increasing GDP. (2007, credit: Frank van Mierlo, Wikimedia Commons)

Conserving Energy

As we finish this chapter on energy and work, it is relevant to draw some distinctions between two sometimes misunderstood terms in the area of energy use. As has been mentioned elsewhere, the “law of the conservation of energy” is a very useful principle in analyzing physical processes. It is a statement that cannot be proven from basic principles, but is a very good bookkeeping device, and no exceptions have ever been found. It states that the total amount of energy in an isolated system will always remain constant. Related to this principle, but remarkably different from it, is the important philosophy of energy conservation. This concept has to do with seeking to decrease the amount of energy used by an individual or group through (1) reduced activities (e.g., turning down thermostats, driving fewer kilometers) and/or (2) increasing conversion efficiencies in the performance of a particular task—such as developing and using more efficient room heaters, cars that have greater miles-per-gallon ratings, energy-efficient compact fluorescent lights, etc.

Since energy in an isolated system is not destroyed or created or generated, one might wonder why we need to be concerned about our energy resources, since energy is a conserved quantity. The problem is that the final result of most energy transformations is waste heat transfer to the environment and conversion to energy forms no longer useful for doing work. To state it in another way, the potential for energy to produce useful work has been “degraded” in the energy transformation. (This will be discussed in more detail in [Thermodynamics](#).)

Section Summary

- The relative use of different fuels to provide energy has changed over the years, but fuel use is currently dominated by oil, although natural gas and solar contributions are increasing.
- Although non-renewable sources dominate, some countries meet a sizeable percentage of their electricity needs from renewable resources.
- The United States obtains only about 10% of its energy from renewable sources, mostly hydroelectric power.
- Economic well-being is dependent upon energy use, and in most countries higher standards of living, as measured by GDP (Gross Domestic Product) per capita, are matched by higher levels of energy consumption per capita.
- Even though, in accordance with the law of conservation of energy, energy can never be created or destroyed, energy that can be used to do work is always partly converted to less useful forms, such as waste heat to the environment, in all of our uses of energy for practical purposes.

Conceptual Questions

Exercise:

Problem:

What is the difference between energy conservation and the law of conservation of energy? Give some examples of each.

Exercise:

Problem:

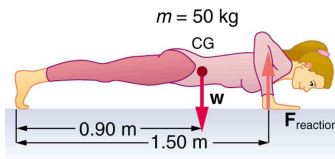
If the efficiency of a coal-fired electrical generating plant is 35%, then what do we mean when we say that energy is a conserved quantity?

Problems & Exercises

Exercise:

Problem: Integrated Concepts

(a) Calculate the force the woman in [\[link\]](#) exerts to do a push-up at constant speed, taking all data to be known to three digits. (b) How much work does she do if her center of mass rises 0.240 m? (c) What is her useful power output if she does 25 push-ups in 1 min? (Should work done lowering her body be included? See the discussion of useful work in [Work, Energy, and Power in Humans](#).



Forces involved in doing push-ups. The woman's weight acts as a force exerted downward on her center of gravity (CG).

Solution:

- (a) 294 N
- (b) 118 J
- (c) 49.0 W

Exercise:

Problem: Integrated Concepts

A 75.0-kg cross-country skier is climbing a 3.0° slope at a constant speed of 2.00 m/s and encounters air resistance of 25.0 N. Find his power output for work done against the gravitational force and air resistance. (b) What average force does he exert backward on the snow to accomplish this? (c) If he continues to exert this force and to experience the same air resistance when he reaches a level area, how long will it take him to reach a velocity of 10.0 m/s?

Exercise:**Problem: Integrated Concepts**

The 70.0-kg swimmer in [\[link\]](#) starts a race with an initial velocity of 1.25 m/s and exerts an average force of 80.0 N backward with his arms during each 1.80 m long stroke. (a) What is his initial acceleration if water resistance is 45.0 N? (b) What is the subsequent average resistance force from the water during the 5.00 s it takes him to reach his top velocity of 2.50 m/s? (c) Discuss whether water resistance seems to increase linearly with velocity.

Solution:

(a) 0.500 m/s^2

(b) 62.5 N

(c) Assuming the acceleration of the swimmer decreases linearly with time over the 5.00 s interval, the frictional force must therefore be increasing linearly with time, since $f = F - ma$. If the acceleration decreases linearly with time, the velocity will contain a term dependent on time squared (t^2). Therefore, the water resistance will not depend linearly on the velocity.

Exercise:**Problem: Integrated Concepts**

A toy gun uses a spring with a force constant of 300 N/m to propel a 10.0-g steel ball. If the spring is compressed 7.00 cm and friction is negligible: (a) How much force is needed to compress the spring? (b) To what maximum height can the ball be shot? (c) At what angles above the horizontal may a child aim to hit a target 3.00 m away at the same height as the gun? (d) What is the gun's maximum range on level ground?

Exercise:**Problem: Integrated Concepts**

(a) What force must be supplied by an elevator cable to produce an acceleration of 0.800 m/s^2 against a 200-N frictional force, if the mass of the loaded elevator is 1500 kg? (b) How much work is done by the cable in lifting the elevator 20.0 m? (c) What is the final speed of the elevator if it starts from rest? (d) How much work went into thermal energy?

Solution:

(a) $16.1 \times 10^3 \text{ N}$

(b) $3.22 \times 10^5 \text{ J}$

(c) 5.66 m/s

(d) 4.00 kJ

Exercise:**Problem: Unreasonable Results**

A car advertisement claims that its 900-kg car accelerated from rest to 30.0 m/s and drove 100 km, gaining 3.00 km in altitude, on 1.0 gal of gasoline. The average force of friction including air resistance was 700 N. Assume all values are known to three significant figures. (a) Calculate the car's efficiency. (b) What is unreasonable about the result? (c) Which premise is unreasonable, or which premises are inconsistent?

Exercise:**Problem: Unreasonable Results**

Body fat is metabolized, supplying 9.30 kcal/g, when dietary intake is less than needed to fuel metabolism. The manufacturers of an exercise bicycle claim that you can lose 0.500 kg of fat per day by vigorously exercising for 2.00 h per day on their machine. (a) How many kcal are supplied by the metabolization of 0.500 kg of fat? (b) Calculate the kcal/min that you would have to utilize to metabolize fat at the rate of 0.500 kg in 2.00 h. (c) What is unreasonable about the results? (d) Which premise is unreasonable, or which premises are inconsistent?

Solution:

(a) 4.65×10^3 kcal

(b) 38.8 kcal/min

(c) This power output is higher than the highest value on [\[link\]](#), which is about 35 kcal/min (corresponding to 2415 watts) for sprinting.

(d) It would be impossible to maintain this power output for 2 hours (imagine sprinting for 2 hours!).

Exercise:**Problem: Construct Your Own Problem**

Consider a person climbing and descending stairs. Construct a problem in which you calculate the long-term rate at which stairs can be climbed considering the mass of the person, his ability to generate power with his legs, and the height of a single stair step. Also consider why the same person can descend stairs at a faster rate for a nearly unlimited time in spite of the fact that very similar forces are exerted going down as going up. (This points to a fundamentally different process for descending versus climbing stairs.)

Exercise:**Problem: Construct Your Own Problem**

Consider humans generating electricity by pedaling a device similar to a stationary bicycle. Construct a problem in which you determine the number of people it would take to replace a large electrical generation facility. Among the things to consider are the power output that is reasonable using the legs, rest time, and the need for electricity 24 hours per day. Discuss the practical implications of your results.

Exercise:**Problem: Integrated Concepts**

A 105-kg basketball player crouches down 0.400 m while waiting to jump. After exerting a force on the floor through this 0.400 m, his feet leave the floor and his center of gravity rises 0.950 m above its normal standing erect position. (a) Using energy considerations, calculate his velocity when he leaves the floor. (b) What average force did he exert on the floor? (Do not neglect the force to support his weight as well as that to accelerate him.) (c) What was his power output during the acceleration phase?

Solution:

(a) 4.32 m/s

(b) 3.47×10^3 N

(c) 8.93 kW

Glossary

renewable forms of energy

those sources that cannot be used up, such as water, wind, solar, and biomass

fossil fuels

oil, natural gas, and coal

Connection for AP® courses
class="introduction"

"Each
rugby
player has
great
momentum
, which will
affect the
outcome of
their
collisions
with each
other and
the ground.
(credit:
vjpaul,
Flickr)"



In this chapter, you will learn about the concept of momentum and the relationship between momentum and force (both vector quantities) applied over a time interval. Have you ever considered why a glass dropped on a tile floor will often break, but a glass dropped on carpet will often remain intact? Both involve changes in momentum, but the actual collision with the floor is different in each case, just as an automobile collision without the benefit of an airbag can have a significantly different outcome than one with an airbag.

You will learn that the interaction of objects (like a glass and the floor or two automobiles) results in forces, which in turn result in changes in the momentum of each object. At the same time, you will see how the law of momentum conservation can be applied to a system to help determine the outcome of a collision.

The content in this chapter supports:

Big Idea 3 The interactions of an object with other objects can be described by forces.

Enduring Understanding 3.D A force exerted on an object can change the momentum of the object.

Essential Knowledge 3.D.2 The change in momentum of an object occurs over a time interval.

Big Idea 4: Interactions between systems can result in changes in those systems.

Enduring Understanding 4.B Interactions with other objects or systems can change the total linear momentum of a system.

Essential Knowledge 4.B.1 The change in linear momentum for a constant-mass system is the product of the mass of the system and the change in velocity of the center of mass.

Big Idea 5 Changes that occur as a result of interactions are constrained by conservation laws.

Enduring Understanding 5.A Certain quantities are conserved, in the sense that the changes of those quantities in a given system are always equal to the transfer of that quantity to or from the system by all possible interactions with other systems.

Essential Knowledge 5.A.2 For all systems under all circumstances, energy, charge, linear momentum, and angular momentum are conserved.

Essential Knowledge 5.D.1 In a collision between objects, linear momentum is conserved. In an elastic collision, kinetic energy is the same before and after.

Essential Knowledge 5.D.2 In a collision between objects, linear momentum is conserved. In an inelastic collision, kinetic energy is not the same before and after the collision.

Linear Momentum and Force

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define linear momentum.
- Explain the relationship between linear momentum and force.
- State Newton's second law of motion in terms of linear momentum.
- Calculate linear momentum given mass and velocity.

The information presented in this section supports the following AP® learning objectives and science practices:

- **3.D.1.1** The student is able to justify the selection of data needed to determine the relationship between the direction of the force acting on an object and the change in momentum caused by that force. **(S.P. 4.1)**

Linear Momentum

The scientific definition of linear momentum is consistent with most people's intuitive understanding of momentum: a large, fast-moving object has greater momentum than a smaller, slower object. **Linear momentum** is defined as the product of a system's mass multiplied by its velocity. In symbols, linear momentum is expressed as

Equation:

$$\mathbf{p} = m\mathbf{v}.$$

Momentum is directly proportional to the object's mass and also its velocity. Thus the greater an object's mass or the greater its velocity, the greater its momentum. Momentum \mathbf{p} is a vector having the same direction as the velocity \mathbf{v} . The SI unit for momentum is $\text{kg} \cdot \text{m/s}$.

Note:

Linear Momentum

Linear momentum is defined as the product of a system's mass multiplied by its velocity:

Equation:

$$\mathbf{p} = m\mathbf{v}.$$

Example:

Calculating Momentum: A Football Player and a Football

(a) Calculate the momentum of a 110-kg football player running at 8.00 m/s. (b) Compare the player's momentum with the momentum of a hard-thrown 0.410-kg football that has a speed of 25.0 m/s.

Strategy

No information is given regarding direction, and so we can calculate only the magnitude of the momentum, p . (As usual, a symbol that is in italics is a magnitude, whereas one that is italicized, boldfaced, and has an arrow is a vector.) In both parts of this example, the magnitude of momentum can be calculated directly from the definition of momentum given in the equation, which becomes

Equation:

$$p = mv$$

when only magnitudes are considered.

Solution for (a)

To determine the momentum of the player, substitute the known values for the player's mass and speed into the equation.

Equation:

$$p_{\text{player}} = (110 \text{ kg})(8.00 \text{ m/s}) = 880 \text{ kg} \cdot \text{m/s}$$

Solution for (b)

To determine the momentum of the ball, substitute the known values for the ball's mass and speed into the equation.

Equation:

$$p_{\text{ball}} = (0.410 \text{ kg})(25.0 \text{ m/s}) = 10.3 \text{ kg} \cdot \text{m/s}$$

The ratio of the player's momentum to that of the ball is

Equation:

$$\frac{p_{\text{player}}}{p_{\text{ball}}} = \frac{880}{10.3} = 85.9.$$

Discussion

Although the ball has greater velocity, the player has a much greater mass. Thus the momentum of the player is much greater than the momentum of the football, as you might guess. As a result, the player's motion is only slightly affected if he catches the ball. We shall quantify what happens in such collisions in terms of momentum in later sections.

Momentum and Newton's Second Law

The importance of momentum, unlike the importance of energy, was recognized early in the development of classical physics. Momentum was deemed so important that it was called the "quantity of motion." Newton actually stated his **second law of motion** in terms of momentum: The net external force equals the change in momentum of a system divided by the time over which it changes. Using symbols, this law is

Equation:

$$\mathbf{F}_{\text{net}} = \frac{\Delta \mathbf{p}}{\Delta t},$$

where \mathbf{F}_{net} is the net external force, $\Delta \mathbf{p}$ is the change in momentum, and Δt is the change in time.

Note:

Newton's Second Law of Motion in Terms of Momentum

The net external force equals the change in momentum of a system divided by the time over which it changes.

Equation:

$$\mathbf{F}_{\text{net}} = \frac{\Delta \mathbf{p}}{\Delta t}$$

Note:

Making Connections: Force and Momentum

Force and momentum are intimately related. Force acting over time can change momentum, and Newton's second law of motion, can be stated in its most broadly applicable form in terms of momentum. Momentum continues to be a key concept in the study of atomic and subatomic particles in quantum mechanics.

This statement of Newton's second law of motion includes the more familiar $\mathbf{F}_{\text{net}} = m\mathbf{a}$ as a special case. We can derive this form as follows. First, note that the change in momentum $\Delta \mathbf{p}$ is given by

Equation:

$$\Delta \mathbf{p} = \Delta(m\mathbf{v}).$$

If the mass of the system is constant, then

Equation:

$$\Delta(m\mathbf{v}) = m\Delta\mathbf{v}.$$

So that for constant mass, Newton's second law of motion becomes

Equation:

$$\mathbf{F}_{\text{net}} = \frac{\Delta\mathbf{p}}{\Delta t} = \frac{m\Delta\mathbf{v}}{\Delta t}.$$

Because $\frac{\Delta\mathbf{v}}{\Delta t} = \mathbf{a}$, we get the familiar equation

Equation:

$$\mathbf{F}_{\text{net}} = m\mathbf{a}$$

when the mass of the system is constant.

Newton's second law of motion stated in terms of momentum is more generally applicable because it can be applied to systems where the mass is changing, such as rockets, as well as to systems of constant mass. We will consider systems with varying mass in some detail; however, the relationship between momentum and force remains useful when mass is constant, such as in the following example.

Example:

Calculating Force: Venus Williams' Racquet

During the 2007 French Open, Venus Williams hit the fastest recorded serve in a premier women's match, reaching a speed of 58 m/s (209 km/h). What is the average force exerted on the 0.057-kg tennis ball by Venus Williams' racquet, assuming that the ball's speed just after impact is 58 m/s, that the initial horizontal component of the velocity before impact is negligible, and that the ball remained in contact with the racquet for 5.0 ms (milliseconds)?

Strategy

This problem involves only one dimension because the ball starts from having no horizontal velocity component before impact. Newton's second law stated in terms of momentum is then written as

Equation:

$$\mathbf{F}_{\text{net}} = \frac{\Delta \mathbf{p}}{\Delta t}.$$

As noted above, when mass is constant, the change in momentum is given by

Equation:

$$\Delta p = m\Delta v = m(v_f - v_i).$$

In this example, the velocity just after impact and the change in time are given; thus, once Δp is calculated, $F_{\text{net}} = \frac{\Delta p}{\Delta t}$ can be used to find the force.

Solution

To determine the change in momentum, substitute the values for the initial and final velocities into the equation above.

Equation:

$$\begin{aligned}\Delta p &= m(v_f - v_i) \\ &= (0.057 \text{ kg})(58 \text{ m/s} - 0 \text{ m/s}) \\ &= 3.306 \text{ kg} \cdot \text{m/s} \approx 3.3 \text{ kg} \cdot \text{m/s}\end{aligned}$$

Now the magnitude of the net external force can be determined by using

$$F_{\text{net}} = \frac{\Delta p}{\Delta t}:$$

Equation:

$$\begin{aligned}F_{\text{net}} &= \frac{\Delta p}{\Delta t} = \frac{3.306 \text{ kg} \cdot \text{m/s}}{5.0 \times 10^{-3} \text{ s}} \\ &= 661 \text{ N} \approx 660 \text{ N},\end{aligned}$$

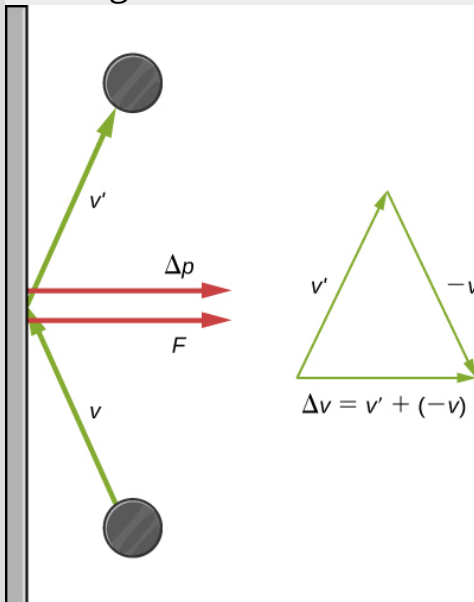
where we have retained only two significant figures in the final step.

Discussion

This quantity was the average force exerted by Venus Williams' racquet on the tennis ball during its brief impact (note that the ball also experienced the 0.56-N force of gravity, but that force was not due to the racquet). This problem could also be solved by first finding the acceleration and then using $F_{\text{net}} = ma$, but one additional step would be required compared with the strategy used in this example.

Note:

Making Connections: Illustrative Example



A puck has an elastic, glancing collision with the edge of an air hockey table.

In [\[link\]](#), a puck is shown colliding with the edge of an air hockey table at a glancing angle. During the collision, the edge of the table exerts a force \mathbf{F} on the puck, and the velocity of the puck changes as a result of the collision. The change in momentum is found by the equation:

Equation:

$$\Delta \mathbf{p} = m\Delta \mathbf{v} = m\mathbf{v}' - m\mathbf{v} = m(\mathbf{v}' + (-\mathbf{v}))$$

As shown, the direction of the change in velocity is the same as the direction of the change in momentum, which in turn is in the same direction as the force exerted by the edge of the table. Note that there is only a horizontal change in velocity. There is no difference in the vertical components of the initial and final velocity vectors; therefore, there is no vertical component to the change in velocity vector or the change in momentum vector. This is consistent with the fact that the force exerted by the edge of the table is purely in the horizontal direction.

Test Prep for AP Courses

Exercise:

Problem:

A boy standing on a frictionless ice rink is initially at rest. He throws a snowball in the $+x$ -direction, and it travels on a ballistic trajectory, hitting the ground some distance away. Which of the following is true about the boy while he is in the act of throwing the snowball?

- a. He feels an upward force to compensate for the downward trajectory of the snowball.
- b. He feels a backward force exerted by the snowball he is throwing.
- c. He feels no net force.
- d. He feels a forward force, the same force that propels the snowball.

Solution:

(b)

Exercise:

Problem:

A 150-g baseball is initially moving 80 mi/h in the $-x$ -direction. After colliding with a baseball bat for 20 ms, the baseball moves 80 mi/h in the $+x$ -direction. What is the magnitude and direction of the average force exerted by the bat on the baseball?

Section Summary

- Linear momentum (*momentum* for brevity) is defined as the product of a system's mass multiplied by its velocity.
- In symbols, linear momentum \mathbf{p} is defined to be

Equation:

$$\mathbf{p} = m\mathbf{v},$$

where m is the mass of the system and \mathbf{v} is its velocity.

- The SI unit for momentum is $\text{kg} \cdot \text{m/s}$.
- Newton's second law of motion in terms of momentum states that the net external force equals the change in momentum of a system divided by the time over which it changes.
- In symbols, Newton's second law of motion is defined to be

Equation:

$$\mathbf{F}_{\text{net}} = \frac{\Delta \mathbf{p}}{\Delta t},$$

\mathbf{F}_{net} is the net external force, $\Delta \mathbf{p}$ is the change in momentum, and Δt is the change time.

Conceptual Questions

Exercise:

Problem:

An object that has a small mass and an object that has a large mass have the same momentum. Which object has the largest kinetic energy?

Exercise:**Problem:**

An object that has a small mass and an object that has a large mass have the same kinetic energy. Which mass has the largest momentum?

Exercise:**Problem: Professional Application**

Football coaches advise players to block, hit, and tackle with their feet on the ground rather than by leaping through the air. Using the concepts of momentum, work, and energy, explain how a football player can be more effective with his feet on the ground.

Exercise:**Problem:**

How can a small force impart the same momentum to an object as a large force?

Problems & Exercises**Exercise:**

Problem:

(a) Calculate the momentum of a 2000-kg elephant charging a hunter at a speed of 7.50 m/s. (b) Compare the elephant's momentum with the momentum of a 0.0400-kg tranquilizer dart fired at a speed of 600 m/s. (c) What is the momentum of the 90.0-kg hunter running at 7.40 m/s after missing the elephant?

Solution:

(a) $1.50 \times 10^4 \text{ kg} \cdot \text{m/s}$

(b) 625 to 1

(c) $6.66 \times 10^2 \text{ kg} \cdot \text{m/s}$

Exercise:**Problem:**

(a) What is the mass of a large ship that has a momentum of $1.60 \times 10^9 \text{ kg} \cdot \text{m/s}$, when the ship is moving at a speed of 48.0 km/h? (b) Compare the ship's momentum to the momentum of a 1100-kg artillery shell fired at a speed of 1200 m/s.

Exercise:**Problem:**

(a) At what speed would a 2.00×10^4 -kg airplane have to fly to have a momentum of $1.60 \times 10^9 \text{ kg} \cdot \text{m/s}$ (the same as the ship's momentum in the problem above)? (b) What is the plane's momentum when it is taking off at a speed of 60.0 m/s? (c) If the ship is an aircraft carrier that launches these airplanes with a catapult, discuss the implications of your answer to (b) as it relates to recoil effects of the catapult on the ship.

Solution:

(a) $8.00 \times 10^4 \text{ m/s}$

(b) $1.20 \times 10^6 \text{ kg} \cdot \text{m/s}$

(c) Because the momentum of the airplane is 3 orders of magnitude smaller than of the ship, the ship will not recoil very much. The recoil would be -0.0100 m/s , which is probably not noticeable.

Exercise:

Problem:

(a) What is the momentum of a garbage truck that is $1.20 \times 10^4 \text{ kg}$ and is moving at 10.0 m/s ? (b) At what speed would an 8.00-kg trash can have the same momentum as the truck?

Exercise:

Problem:

A runaway train car that has a mass of $15,000 \text{ kg}$ travels at a speed of 5.4 m/s down a track. Compute the time required for a force of 1500 N to bring the car to rest.

Solution:

54 s

Exercise:

Problem:

The mass of Earth is $5.972 \times 10^{24} \text{ kg}$ and its orbital radius is an average of $1.496 \times 10^{11} \text{ m}$. Calculate its linear momentum.

Glossary

linear momentum

the product of mass and velocity

second law of motion

physical law that states that the net external force equals the change in momentum of a system divided by the time over which it changes

Impulse

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define impulse.
- Describe effects of impulses in everyday life.
- Determine the average effective force using graphical representation.
- Calculate average force and impulse given mass, velocity, and time.

The information presented in this section supports the following AP[®] learning objectives and science practices:

- **3.D.2.1** The student is able to justify the selection of routines for the calculation of the relationships between changes in momentum of an object, average force, impulse, and time of interaction. **(S.P. 2.1)**
- **3.D.2.2** The student is able to predict the change in momentum of an object from the average force exerted on the object and the interval of time during which the force is exerted. **(S.P. 6.4)**
- **3.D.2.3** The student is able to analyze data to characterize the change in momentum of an object from the average force exerted on the object and the interval of time during which the force is exerted. **(S.P. 5.1)**
- **3.D.2.4** The student is able to design a plan for collecting data to investigate the relationship between changes in momentum and the average force exerted on an object over time. **(S.P. 4.1)**
- **4.B.2.1** The student is able to apply mathematical routines to calculate the change in momentum of a system by analyzing

the average force exerted over a certain time on the system.
(S.P. 2.2)

- 4.B.2.2 The student is able to perform analysis on data presented as a force-time graph and predict the change in momentum of a system. (S.P. 5.1)

The effect of a force on an object depends on how long it acts, as well as how great the force is. In [\[link\]](#), a very large force acting for a short time had a great effect on the momentum of the tennis ball. A small force could cause the same **change in momentum**, but it would have to act for a much longer time. For example, if the ball were thrown upward, the gravitational force (which is much smaller than the tennis racquet's force) would eventually reverse the momentum of the ball. Quantitatively, the effect we are talking about is the change in momentum $\Delta \mathbf{p}$.

By rearranging the equation $\mathbf{F}_{\text{net}} = \frac{\Delta \mathbf{p}}{\Delta t}$ to be

Equation:

$$\Delta \mathbf{p} = \mathbf{F}_{\text{net}} \Delta t,$$

we can see how the change in momentum equals the average net external force multiplied by the time this force acts. The quantity $\mathbf{F}_{\text{net}} \Delta t$ is given the name **impulse**. Impulse is the same as the change in momentum.

Note:

Impulse: Change in Momentum

Change in momentum equals the average net external force multiplied by the time this force acts.

Equation:

$$\Delta \mathbf{p} = \mathbf{F}_{\text{net}} \Delta t$$

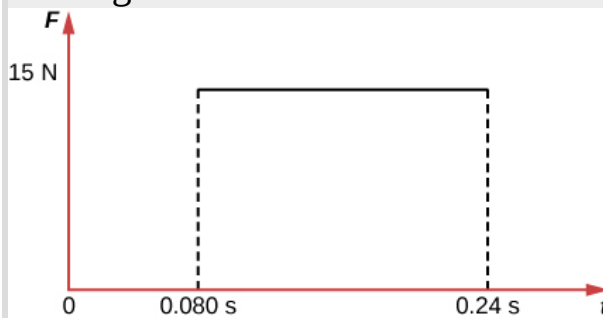
The quantity $\mathbf{F}_{\text{net}} \Delta t$ is given the name impulse.

There are many ways in which an understanding of impulse can save lives, or at least limbs. The dashboard padding in a car, and certainly the airbags, allow the net force on the occupants in the car to act over a much longer time when there is a sudden stop. The momentum change is the same for an occupant, whether an air bag is deployed or not, but the force (to bring the occupant to a stop) will be much less if it acts over a larger time. Cars today have many plastic components. One advantage of plastics is their lighter weight, which results in better gas mileage. Another advantage is that a car will crumple in a collision, especially in the event of a head-on collision. A longer collision time means the force on the car will be less. Deaths during car races decreased dramatically when the rigid frames of racing cars were replaced with parts that could crumple or collapse in the event of an accident.

Bones in a body will fracture if the force on them is too large. If you jump onto the floor from a table, the force on your legs can be immense if you land stiff-legged on a hard surface. Rolling on the ground after jumping from the table, or landing with a parachute, extends the time over which the force (on you from the ground) acts.

Note:

Making Connections: Illustrations of Force Exerted



This is a graph showing the force exerted by a fixed barrier on a block versus time.

A 1.2-kg block slides across a horizontal, frictionless surface with a constant speed of 3.0 m/s before striking a fixed barrier and coming to a stop. In [\[link\]](#), the force exerted by the barrier is assumed to be a constant 15 N during the 0.24-s collision. The impulse can be calculated using the area under the curve.

Equation:

$$\Delta p = F\Delta t = (15 \text{ N})(0.24 \text{ s}) = 3.6 \text{ kg} \cdot \text{m/s}$$

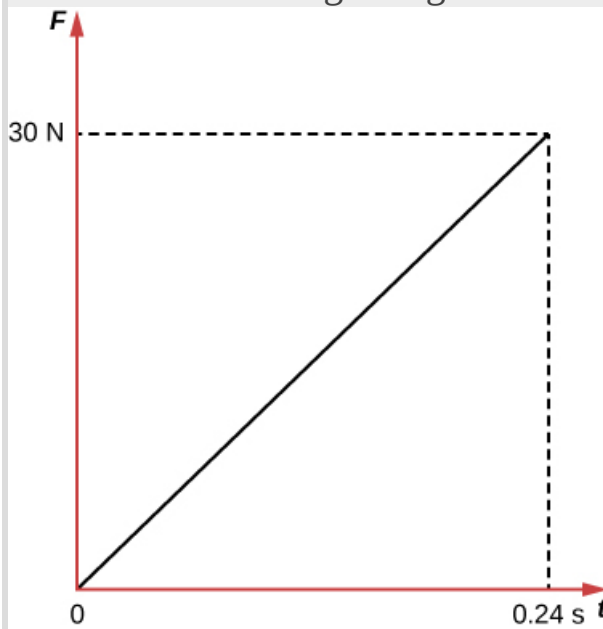
Note that the initial momentum of the block is:

Equation:

$$p_{\text{initial}} = mv_{\text{initial}} = (1.2 \text{ kg})(-3.0 \text{ m/s}) = -3.6 \text{ kg} \cdot \text{m/s}$$

We are assuming that the initial velocity is -3.0 m/s . We have established that the force exerted by the barrier is in the positive direction, so the initial velocity of the block must be in the negative direction. Since the final momentum of the block is zero, the impulse is equal to the change in momentum of the block.

Suppose that, instead of striking a fixed barrier, the block is instead stopped by a spring. Consider the force exerted by the spring over the time interval from the beginning of the collision until the block comes to rest.



This is a graph showing the

force exerted by a spring on a block versus time.

In this case, the impulse can be calculated again using the area under the curve (the area of a triangle):

Equation:

$$\Delta p = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(0.24 \text{ s})(30 \text{ N}) = 3.6 \text{ kg} \cdot \text{m/s}$$

Again, this is equal to the difference between the initial and final momentum of the block, so the impulse is equal to the change in momentum.

Example:

Calculating Magnitudes of Impulses: Two Billiard Balls Striking a Rigid Wall

Two identical billiard balls strike a rigid wall with the same speed, and are reflected without any change of speed. The first ball strikes perpendicular to the wall. The second ball strikes the wall at an angle of 30° from the perpendicular, and bounces off at an angle of 30° from perpendicular to the wall.

- (a) Determine the direction of the force on the wall due to each ball.
- (b) Calculate the ratio of the magnitudes of impulses on the two balls by the wall.

Strategy for (a)

In order to determine the force on the wall, consider the force on the ball due to the wall using Newton's second law and then apply Newton's third law to determine the direction. Assume the x -axis to be normal to the wall and to be positive in the initial direction of motion. Choose the y -axis to be along the wall in the plane of the second ball's motion. The momentum direction and the velocity direction are the same.

Solution for (a)

The first ball bounces directly into the wall and exerts a force on it in the $+x$ direction. Therefore the wall exerts a force on the ball in the $-x$ direction. The second ball continues with the same momentum component in the y direction, but reverses its x -component of momentum, as seen by sketching a diagram of the angles involved and keeping in mind the proportionality between velocity and momentum.

These changes mean the change in momentum for both balls is in the $-x$ direction, so the force of the wall on each ball is along the $-x$ direction.

Strategy for (b)

Calculate the change in momentum for each ball, which is equal to the impulse imparted to the ball.

Solution for (b)

Let u be the speed of each ball before and after collision with the wall, and m the mass of each ball. Choose the x -axis and y -axis as previously described, and consider the change in momentum of the first ball which strikes perpendicular to the wall.

Equation:

$$p_{xi} = mu; p_{yi} = 0$$

Equation:

$$p_{xf} = -mu; p_{yf} = 0$$

Impulse is the change in momentum vector. Therefore the x -component of impulse is equal to $-2mu$ and the y -component of impulse is equal to zero.

Now consider the change in momentum of the second ball.

Equation:

$$p_{xi} = mu \cos 30^\circ; p_{yi} = -mu \sin 30^\circ$$

Equation:

$$p_{xf} = -mu \cos 30^\circ; p_{yf} = -mu \sin 30^\circ$$

It should be noted here that while p_x changes sign after the collision, p_y does not. Therefore the x -component of impulse is equal to $-2mu \cos 30^\circ$

and the y -component of impulse is equal to zero.

The ratio of the magnitudes of the impulse imparted to the balls is

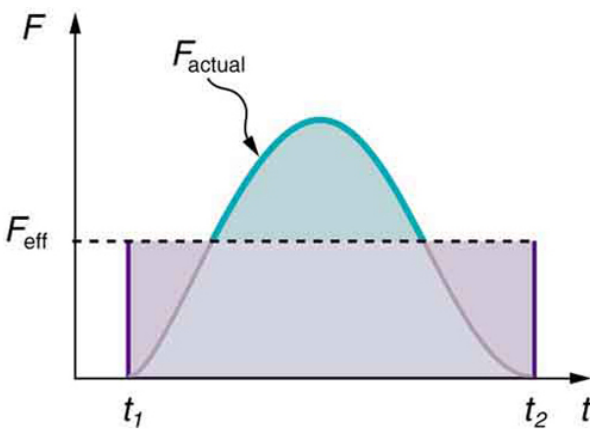
Equation:

$$\frac{2mu}{2mu \cos 30^\circ} = \frac{2}{\sqrt{3}} = 1.155.$$

Discussion

The direction of impulse and force is the same as in the case of (a); it is normal to the wall and along the negative x -direction. Making use of Newton's third law, the force on the wall due to each ball is normal to the wall along the positive x -direction.

Our definition of impulse includes an assumption that the force is constant over the time interval Δt . *Forces are usually not constant.* Forces vary considerably even during the brief time intervals considered. It is, however, possible to find an average effective force F_{eff} that produces the same result as the corresponding time-varying force. [\[link\]](#) shows a graph of what an actual force looks like as a function of time for a ball bouncing off the floor. The area under the curve has units of momentum and is equal to the impulse or change in momentum between times t_1 and t_2 . That area is equal to the area inside the rectangle bounded by F_{eff} , t_1 , and t_2 . Thus the impulses and their effects are the same for both the actual and effective forces.



A graph of force versus time

with time along the x -axis and force along the y -axis for an actual force and an equivalent effective force. The areas under the two curves are equal.

Note:

Making Connections: Baseball

In most real-life collisions, the forces acting on an object are not constant. For example, when a bat strikes a baseball, the force is very small at the beginning of the collision since only a small portion of the ball is initially in contact with the bat. As the collision continues, the ball deforms so that a greater fraction of the ball is in contact with the bat, resulting in a greater force. As the ball begins to leave the bat, the force drops to zero, much like the force curve in [\[link\]](#). Although the changing force is difficult to precisely calculate at each instant, the average force can be estimated very well in most cases.

Suppose that a 150-g baseball experiences an average force of 480 N in a direction opposite the initial 32 m/s speed of the baseball over a time interval of 0.017 s. What is the final velocity of the baseball after the collision?

Equation:

$$\Delta p = F\Delta t = (480)(0.017) = 8.16 \text{ kg} \cdot \text{m/s}$$

Equation:

$$mv_f - mv_i = 8.16 \text{ kg} \cdot \text{m/s}$$

Equation:

$$(0.150 \text{ kg})v_f - (0.150 \text{ kg})(-32 \text{ m/s}) = 8.16 \text{ kg} \cdot \text{m/s}$$

Equation:

$$v_f = 22 \text{ m/s}$$

Note in the above example that the initial velocity of the baseball prior to the collision is negative, consistent with the assumption we initially made that the force exerted by the bat is positive and in the direction opposite the initial velocity of the baseball. In this case, even though the force acting on the baseball varies with time, the average force is a good approximation of the effective force acting on the ball for the purposes of calculating the impulse and the change in momentum.

Note:**Making Connections: Take-Home Investigation—Hand Movement and Impulse**

Try catching a ball while “giving” with the ball, pulling your hands toward your body. Then, try catching a ball while keeping your hands still. Hit water in a tub with your full palm. After the water has settled, hit the water again by diving your hand with your fingers first into the water. (Your full palm represents a swimmer doing a belly flop and your diving hand represents a swimmer doing a dive.) Explain what happens in each case and why. Which orientations would you advise people to avoid and why?

Note:**Making Connections: Constant Force and Constant Acceleration**

The assumption of a constant force in the definition of impulse is analogous to the assumption of a constant acceleration in kinematics. In both cases, nature is adequately described without the use of calculus.

Note:**Applying the Science Practices: Verifying the Relationship between Force and Change in Linear Momentum**

Design an experiment in order to experimentally verify the relationship between the impulse of a force and change in linear momentum. For

simplicity, it would be best to ensure that frictional forces are very small or zero in your experiment so that the effect of friction can be neglected. As you design your experiment, consider the following:

- Would it be easier to analyze a one-dimensional collision or a two-dimensional collision?
- How will you measure the force?
- Should you have two objects in motion or one object bouncing off a rigid surface?
- How will you measure the duration of the collision?
- How will you measure the initial and final velocities of the object(s)?
- Would it be easier to analyze an elastic or inelastic collision?
- Should you verify the relationship mathematically or graphically?

Test Prep for AP Courses

Exercise:

Problem:

A 1.0-kg ball of putty is released from rest and falls vertically 1.5 m until it strikes a hard floor, where it comes to rest in a 0.045-s time interval. What is the magnitude and direction of the average force exerted on the ball by the floor during the collision?

- a. 33 N, up
- b. 120 N, up
- c. 120 N, down
- d. 240 N, down

Solution:

(b)

Exercise:

Problem:

A 75-g ball is dropped from rest from a height of 2.2 m. It bounces off the floor and rebounds to a maximum height of 1.7 m. If the ball is in contact with the floor for 0.024 s, what is the magnitude and direction of the average force exerted on the ball by the floor during the collision?

Exercise:**Problem:**

A 2.4-kg ceramic bowl falls to the floor. During the 0.018-s impact, the bowl experiences an average force of 750 N from the floor. The bowl is at rest after the impact. From what initial height did the bowl fall?

- a. 1.6 m
- b. 2.8 m
- c. 3.2 m
- d. 5.6 m

Solution:

(a)

Exercise:**Problem:**

Whether or not an object (such as a plate, glass, or bone) breaks upon impact depends on the average force exerted on that object by the surface. When a 1.2-kg glass figure hits the floor, it will break if it experiences an average force of 330 N. When it hits a tile floor, the glass comes to a stop in 0.015 s. From what minimum height must the glass fall to experience sufficient force to break? How would your answer change if the figure were falling to a padded or carpeted surface? Explain.

Exercise:**Problem:**

A 2.5-kg block slides across a frictionless table toward a horizontal spring. As the block bounces off the spring, a probe measures the velocity of the block (initially negative, moving away from the probe) over time as follows:

Velocity (m/s)	Time (s)
-12.0	0
-10.0	0.10
-6.0	0.20
0	0.30
6.0	0.40
10.0	0.50
12.0	0.60

What is the average force exerted on the block by the spring over the entire 0.60-s time interval of the collision?

- a. 50 N
- b. 60 N
- c. 100 N
- d. 120 N

Solution:

(c) (based on calculation of $F = \frac{m\Delta v}{\Delta t}$)

Exercise:**Problem:**

During an automobile crash test, the average force exerted by a solid wall on a 2500-kg car that hits the wall is measured to be 740,000 N over a 0.22-s time interval. What was the initial speed of the car prior to the collision, assuming the car is at rest at the end of the time interval?

Exercise:**Problem:**

A test car is driving toward a solid crash-test barrier with a speed of 45 mi/h. Two seconds prior to impact, the car begins to brake, but it is still moving when it hits the wall. After the collision with the wall, the car crumples somewhat and comes to a complete stop. In order to estimate the average force exerted by the wall on the car, what information would you need to collect?

- The (negative) acceleration of the car before it hits the wall and the distance the car travels while braking.
- The (negative) acceleration of the car before it hits the wall and the velocity of the car just before impact.
- The velocity of the car just before impact and the duration of the collision with the wall.
- The duration of the collision with the wall and the distance the car travels while braking.

Solution:

(c)

Exercise:

Problem:

Design an experiment to verify the relationship between the average force exerted on an object and the change in momentum of that object. As part of your explanation, list the equipment you would use and describe your experimental setup. What would you measure and how? How exactly would you verify the relationship? Explain.

Exercise:**Problem:**

A 22-g puck hits the wall of an air hockey table perpendicular to the wall with an initial speed of 14 m/s. The puck is in contact with the wall for 0.0055 s, and it rebounds from the wall with a speed of 14 m/s in the opposite direction. What is the magnitude of the average force exerted by the wall on the puck?

- a. 0.308 N
- b. 0.616 N
- c. 56 N
- d. 112 N

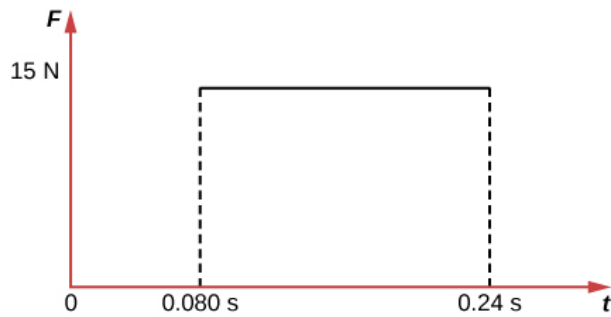
Solution:

(d)

Exercise:**Problem:**

A 22-g puck hits the wall of an air hockey table perpendicular to the wall with an initial speed of 7 m/s. The puck is in contact with the wall for 0.011 s, and the wall exerts an average force of 28 N on the puck during that time. Calculate the magnitude and direction of the change in momentum of the puck.

Exercise:**Problem:**



This is a graph showing the force exerted by a rigid wall versus time.

The graph in [\[link\]](#) represents the force exerted on a particle during a collision. What is the magnitude of the change in momentum of the particle as a result of the collision?

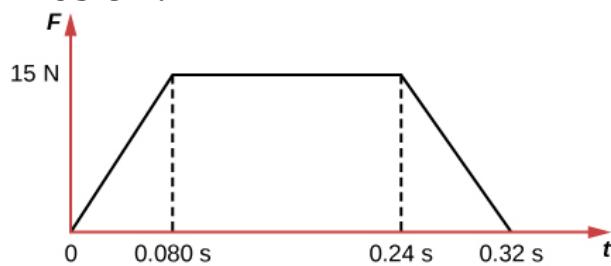
- a. $1.2 \text{ kg} \cdot \text{m/s}$
- b. $2.4 \text{ kg} \cdot \text{m/s}$
- c. $3.6 \text{ kg} \cdot \text{m/s}$
- d. $4.8 \text{ kg} \cdot \text{m/s}$

Solution:

(b)

Exercise:

Problem:



This is a graph showing the

force exerted by a rigid wall
versus time.

The graph in [\[link\]](#) represents the force exerted on a particle during a collision. What is the magnitude of the change in momentum of the particle as a result of the collision?

Section Summary

- Impulse, or change in momentum, equals the average net external force multiplied by the time this force acts:

Equation:

$$\Delta \mathbf{p} = \mathbf{F}_{\text{net}} \Delta t.$$

- Forces are usually not constant over a period of time.

Conceptual Questions

Exercise:

Problem: Professional Application

Explain in terms of impulse how padding reduces forces in a collision. State this in terms of a real example, such as the advantages of a carpeted vs. tile floor for a day care center.

Exercise:

Problem:

While jumping on a trampoline, sometimes you land on your back and other times on your feet. In which case can you reach a greater height and why?

Exercise:

Problem: Professional Application

Tennis racquets have “sweet spots.” If the ball hits a sweet spot then the player's arm is not jarred as much as it would be otherwise. Explain why this is the case.

Problems & Exercises**Exercise:****Problem:**

A bullet is accelerated down the barrel of a gun by hot gases produced in the combustion of gun powder. What is the average force exerted on a 0.0300-kg bullet to accelerate it to a speed of 600 m/s in a time of 2.00 ms (milliseconds)?

Solution:

$$9.00 \times 10^3 \text{ N}$$

Exercise:**Problem: Professional Application**

A car moving at 10 m/s crashes into a tree and stops in 0.26 s. Calculate the force the seat belt exerts on a passenger in the car to bring him to a halt. The mass of the passenger is 70 kg.

Exercise:

Problem:

A person slaps her leg with her hand, bringing her hand to rest in 2.50 milliseconds from an initial speed of 4.00 m/s. (a) What is the average force exerted on the leg, taking the effective mass of the hand and forearm to be 1.50 kg? (b) Would the force be any different if the woman clapped her hands together at the same speed and brought them to rest in the same time? Explain why or why not.

Solution:

a) 2.40×10^3 N toward the leg

b) The force on each hand would have the same magnitude as that found in part (a) (but in opposite directions by Newton's third law) because the change in momentum and the time interval are the same.

Exercise:**Problem: Professional Application**

A professional boxer hits his opponent with a 1000-N horizontal blow that lasts for 0.150 s. (a) Calculate the impulse imparted by this blow. (b) What is the opponent's final velocity, if his mass is 105 kg and he is motionless in midair when struck near his center of mass? (c) Calculate the recoil velocity of the opponent's 10.0-kg head if hit in this manner, assuming the head does not initially transfer significant momentum to the boxer's body. (d) Discuss the implications of your answers for parts (b) and (c).

Exercise:**Problem: Professional Application**

Suppose a child drives a bumper car head on into the side rail, which exerts a force of 4000 N on the car for 0.200 s. (a) What impulse is imparted by this force? (b) Find the final velocity of the bumper car if

its initial velocity was 2.80 m/s and the car plus driver have a mass of 200 kg. You may neglect friction between the car and floor.

Solution:

- a) 800 kg · m/s away from the wall
- b) 1.20 m/s away from the wall

Exercise:

Problem: Professional Application

One hazard of space travel is debris left by previous missions. There are several thousand objects orbiting Earth that are large enough to be detected by radar, but there are far greater numbers of very small objects, such as flakes of paint. Calculate the force exerted by a 0.100-mg chip of paint that strikes a spacecraft window at a relative speed of 4.00×10^3 m/s, given the collision lasts 6.00×10^{-8} s.

Exercise:

Problem: Professional Application

A 75.0-kg person is riding in a car moving at 20.0 m/s when the car runs into a bridge abutment. (a) Calculate the average force on the person if he is stopped by a padded dashboard that compresses an average of 1.00 cm. (b) Calculate the average force on the person if he is stopped by an air bag that compresses an average of 15.0 cm.

Solution:

- (a) 1.50×10^6 N away from the dashboard
- (b) 1.00×10^5 N away from the dashboard

Exercise:

Problem: Professional Application

Military rifles have a mechanism for reducing the recoil forces of the gun on the person firing it. An internal part recoils over a relatively large distance and is stopped by damping mechanisms in the gun. The larger distance reduces the average force needed to stop the internal part. (a) Calculate the recoil velocity of a 1.00-kg plunger that directly interacts with a 0.0200-kg bullet fired at 600 m/s from the gun. (b) If this part is stopped over a distance of 20.0 cm, what average force is exerted upon it by the gun? (c) Compare this to the force exerted on the gun if the bullet is accelerated to its velocity in 10.0 ms (milliseconds).

Exercise:**Problem:**

A cruise ship with a mass of 1.00×10^7 kg strikes a pier at a speed of 0.750 m/s. It comes to rest 6.00 m later, damaging the ship, the pier, and the tugboat captain's finances. Calculate the average force exerted on the pier using the concept of impulse. (Hint: First calculate the time it took to bring the ship to rest.)

Solution:

4.69×10^5 N in the boat's original direction of motion

Exercise:**Problem:**

Calculate the final speed of a 110-kg rugby player who is initially running at 8.00 m/s but collides head-on with a padded goalpost and experiences a backward force of 1.76×10^4 N for 5.50×10^{-2} s.

Exercise:

Problem:

Water from a fire hose is directed horizontally against a wall at a rate of 50.0 kg/s and a speed of 42.0 m/s. Calculate the magnitude of the force exerted on the wall, assuming the water's horizontal momentum is reduced to zero.

Solution:

2.10×10^3 N away from the wall

Exercise:**Problem:**

A 0.450-kg hammer is moving horizontally at 7.00 m/s when it strikes a nail and comes to rest after driving the nail 1.00 cm into a board. (a) Calculate the duration of the impact. (b) What was the average force exerted on the nail?

Exercise:**Problem:**

Starting with the definitions of momentum and kinetic energy, derive an equation for the kinetic energy of a particle expressed as a function of its momentum.

Solution:

$$\begin{aligned}\mathbf{p} &= m\mathbf{v} \Rightarrow p^2 = m^2v^2 \Rightarrow \frac{p^2}{m} = mv^2 \\ \Rightarrow \frac{p^2}{2m} &= \frac{1}{2}mv^2 = \text{KE} \\ KE &= \frac{p^2}{2m}\end{aligned}$$

Exercise:

Problem:

A ball with an initial velocity of 10 m/s moves at an angle 60° above the $+x$ -direction. The ball hits a vertical wall and bounces off so that it is moving 60° above the $-x$ -direction with the same speed. What is the impulse delivered by the wall?

Exercise:**Problem:**

When serving a tennis ball, a player hits the ball when its velocity is zero (at the highest point of a vertical toss). The racquet exerts a force of 540 N on the ball for 5.00 ms, giving it a final velocity of 45.0 m/s. Using these data, find the mass of the ball.

Solution:

60.0 g

Exercise:**Problem:**

A punter drops a ball from rest vertically 1 meter down onto his foot. The ball leaves the foot with a speed of 18 m/s at an angle 55° above the horizontal. What is the impulse delivered by the foot (magnitude and direction)?

Glossary

change in momentum

the difference between the final and initial momentum; the mass times the change in velocity

impulse

the average net external force times the time it acts; equal to the change in momentum

Conservation of Momentum

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Describe the law of conservation of linear momentum.
- Derive an expression for the conservation of momentum.
- Explain conservation of momentum with examples.
- Explain the law of conservation of momentum as it relates to atomic and subatomic particles.

The information presented in this section supports the following AP® learning objectives and science practices:

- **5.A.2.1** The student is able to define open and closed systems for everyday situations and apply conservation concepts for energy, charge, and linear momentum to those situations. (S.P. 6.4, 7.2)
- **5.D.1.4** The student is able to design an experimental test of an application of the principle of the conservation of linear momentum, predict an outcome of the experiment using the principle, analyze data generated by that experiment whose uncertainties are expressed numerically, and evaluate the match between the prediction and the outcome. (S.P. 4.2, 5.1, 5.3, 6.4)
- **5.D.2.1** The student is able to qualitatively predict, in terms of linear momentum and kinetic energy, how the outcome of a collision between two objects changes depending on whether the collision is elastic or inelastic. (S.P. 6.4, 7.2)
- **5.D.2.2** The student is able to plan data collection strategies to test the law of conservation of momentum in a two-object

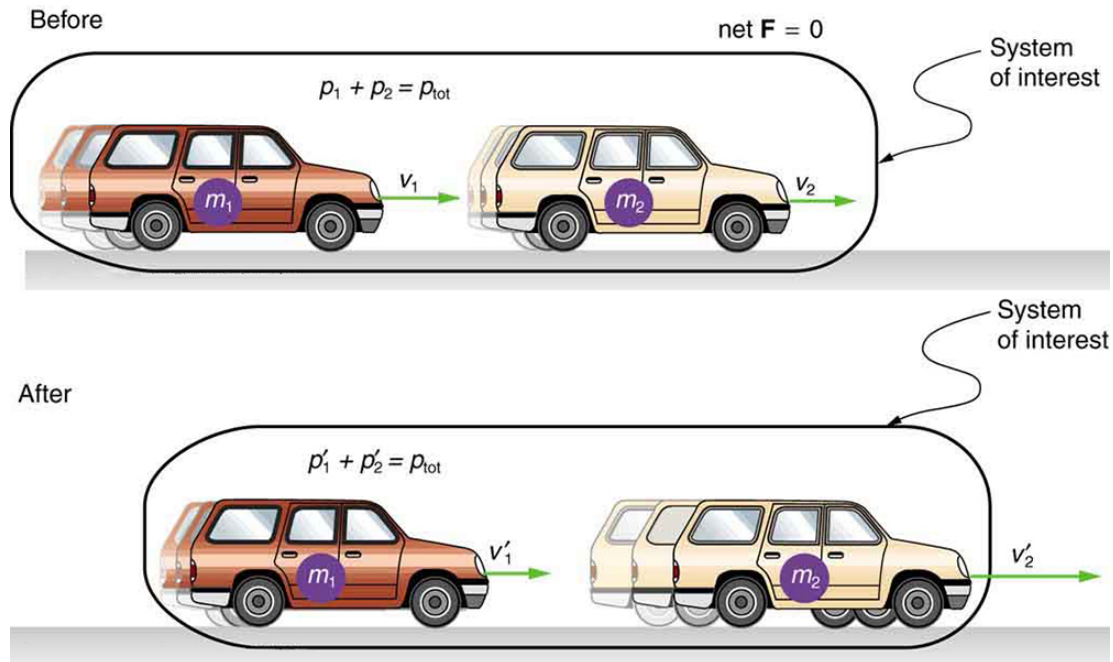
collision that is elastic or inelastic and analyze the resulting data graphically. (S.P.4.1, 4.2, 5.1)

- **5.D.3.1** The student is able to predict the velocity of the center of mass of a system when there is no interaction outside of the system but there is an interaction within the system (i.e., the student simply recognizes that interactions within a system do not affect the center of mass motion of the system and is able to determine that there is no external force). (S.P. 6.4)

Momentum is an important quantity because it is conserved. Yet it was not conserved in the examples in [Impulse](#) and [Linear Momentum and Force](#), where large changes in momentum were produced by forces acting on the system of interest. Under what circumstances is momentum conserved?

The answer to this question entails considering a sufficiently large system. It is always possible to find a larger system in which total momentum is constant, even if momentum changes for components of the system. If a football player runs into the goalpost in the end zone, there will be a force on him that causes him to bounce backward. However, the Earth also recoils —conserving momentum—because of the force applied to it through the goalpost. Because Earth is many orders of magnitude more massive than the player, its recoil is immeasurably small and can be neglected in any practical sense, but it is real nevertheless.

Consider what happens if the masses of two colliding objects are more similar than the masses of a football player and Earth—for example, one car bumping into another, as shown in [\[link\]](#). Both cars are coasting in the same direction when the lead car (labeled m_2) is bumped by the trailing car (labeled m_1). The only unbalanced force on each car is the force of the collision. (Assume that the effects due to friction are negligible.) Car 1 slows down as a result of the collision, losing some momentum, while car 2 speeds up and gains some momentum. We shall now show that the total momentum of the two-car system remains constant.



A car of mass m_1 moving with a velocity of v_1 bumps into another car of mass m_2 and velocity v_2 that it is following. As a result, the first car slows down to a velocity of v'_1 and the second speeds up to a velocity of v'_2 . The momentum of each car is changed, but the total momentum p_{tot} of the two cars is the same before and after the collision (if you assume friction is negligible).

Using the definition of impulse, the change in momentum of car 1 is given by

Equation:

$$\Delta p_1 = F_1 \Delta t,$$

where F_1 is the force on car 1 due to car 2, and Δt is the time the force acts (the duration of the collision). Intuitively, it seems obvious that the collision time is the same for both cars, but it is only true for objects traveling at ordinary speeds. This assumption must be modified for objects travelling

near the speed of light, without affecting the result that momentum is conserved.

Similarly, the change in momentum of car 2 is

Equation:

$$\Delta p_2 = F_2 \Delta t,$$

where F_2 is the force on car 2 due to car 1, and we assume the duration of the collision Δt is the same for both cars. We know from Newton's third law that $F_2 = -F_1$, and so

Equation:

$$\Delta p_2 = -F_1 \Delta t = -\Delta p_1.$$

Thus, the changes in momentum are equal and opposite, and

Equation:

$$\Delta p_1 + \Delta p_2 = 0.$$

Because the changes in momentum add to zero, the total momentum of the two-car system is constant. That is,

Equation:

$$p_1 + p_2 = \text{constant},$$

Equation:

$$p_1 + p_2 = p'_1 + p'_2,$$

where p'_1 and p'_2 are the momenta of cars 1 and 2 after the collision. (We often use primes to denote the final state.)

This result—that momentum is conserved—has validity far beyond the preceding one-dimensional case. It can be similarly shown that total momentum is conserved for any isolated system, with any number of

objects in it. In equation form, the **conservation of momentum principle** for an isolated system is written

Equation:

$$\mathbf{p}_{\text{tot}} = \text{constant},$$

or

Equation:

$$\mathbf{p}_{\text{tot}} = \mathbf{p}'_{\text{tot}},$$

where \mathbf{p}_{tot} is the total momentum (the sum of the momenta of the individual objects in the system) and \mathbf{p}'_{tot} is the total momentum some time later. (The total momentum can be shown to be the momentum of the center of mass of the system.) An **isolated system** is defined to be one for which the net external force is zero ($\mathbf{F}_{\text{net}} = 0$).

Note:

Conservation of Momentum Principle

Equation:

$$\begin{aligned}\mathbf{p}_{\text{tot}} &= \text{constant} \\ \mathbf{p}_{\text{tot}} &= \mathbf{p}'_{\text{tot}} \text{ (isolated system)}\end{aligned}$$

Note:

Isolated System

An isolated system is defined to be one for which the net external force is zero ($\mathbf{F}_{\text{net}} = 0$).

Note:

Making Connections: Cart Collisions

Consider two air carts with equal mass (m) on a linear track. The first cart moves with a speed v towards the second cart, which is initially at rest. We will take the initial direction of motion of the first cart as the positive direction.

The momentum of the system will be conserved in the collision. If the collision is elastic, then the first cart will stop after the collision.

Conservation of momentum therefore tells us that the second cart will have a final velocity v after the collision in the same direction as the initial velocity of the first cart.

The kinetic energy of the system will be conserved since the masses are equal and the final velocity of cart 2 is equal to the initial velocity of cart 1.

What would a graph of total momentum vs. time look like in this case?

What would a graph of total kinetic energy vs. time look like in this case?

Consider the center of mass of this system as the frame of reference. As cart 1 approaches cart 2, the center of mass remains exactly halfway between the two carts. The center of mass moves toward the stationary cart 2 at a speed $\frac{v}{2}$. After the collision, the center of mass continues moving in the same direction, away from (now stationary) cart 1 at a speed $\frac{v}{2}$. How would a graph of center-of-mass velocity vs. time compare to a graph of momentum vs. time?

Suppose instead that the two carts move with equal speeds v in opposite directions towards the center of mass. Again, they have an elastic collision, so after the collision, they exchange velocities (each cart moving in the opposite direction of its initial motion with the same speed). As the two carts approach, the center of mass is exactly between the two carts, at the point where they will collide. In this case, how would a graph of center-of-mass velocity vs. time compare to a graph of the momentum of the system vs. time?

Let us return to the example where the first cart is moving with a speed v toward the second cart, initially at rest. Suppose the second cart has some putty on one end so that, when the collision occurs, the two carts stick together in an inelastic collision. In this case, conservation of momentum tells us that the final velocity of the two-cart system will be half the initial velocity of the first cart, in the same direction as the first cart's initial motion. Kinetic energy will not be conserved in this case, however.

Compared to the moving cart before the collision, the overall moving mass after the collision is doubled, but the velocity is halved.

The initial kinetic energy of the system is:

Equation:

$$k_i = \frac{1}{2}mv^2(\text{1st cart}) + 0(\text{2nd cart}) = \frac{1}{2}mv^2$$

The final kinetic energy of the two carts ($2m$) moving together (at speed $v/2$) is:

Equation:

$$k_f = \frac{1}{2}(2m)\left(\frac{v}{2}\right)^2 = \frac{1}{4}mv^2$$

What would a graph of total momentum vs. time look like in this case?

What would a graph of total kinetic energy vs. time look like in this case?

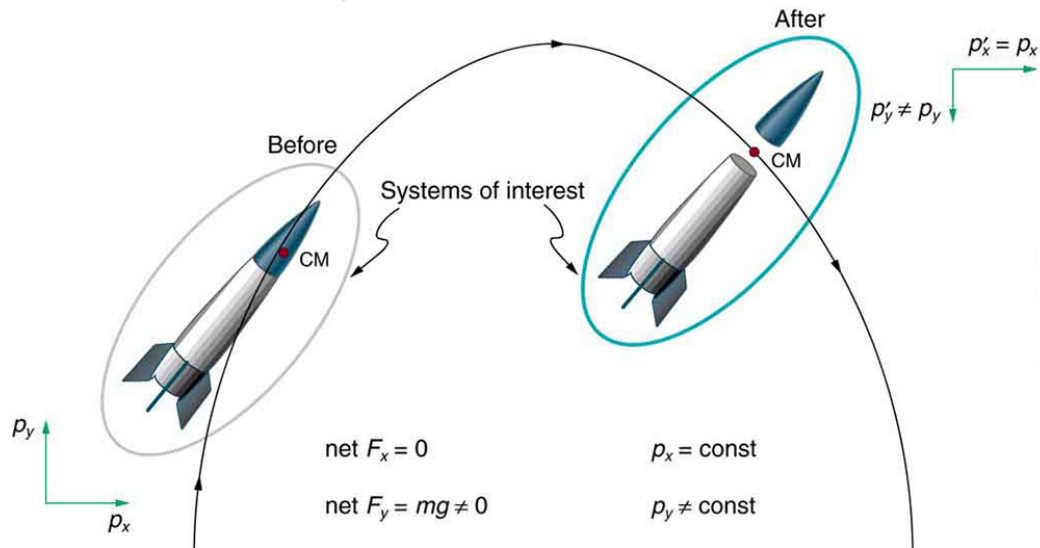
Consider the center of mass of this system. As cart 1 approaches cart 2, the center of mass remains exactly halfway between the two carts. The center of mass moves toward the stationary cart 2 at a speed $\frac{v}{2}$. After the collision, the two carts move together at a speed $\frac{v}{2}$. How would a graph of center-of-mass velocity vs. time compare to a graph of momentum vs. time?

Suppose instead that the two carts move with equal speeds v in opposite directions towards the center of mass. They have putty on the end of each cart so that they stick together after the collision. As the two carts approach, the center of mass is exactly between the two carts, at the point where they will collide. In this case, how would a graph of center-of-mass velocity vs. time compare to a graph of the momentum of the system vs. time?

Perhaps an easier way to see that momentum is conserved for an isolated system is to consider Newton's second law in terms of momentum,

$\mathbf{F}_{\text{net}} = \frac{\Delta \mathbf{p}_{\text{tot}}}{\Delta t}$. For an isolated system, ($\mathbf{F}_{\text{net}} = 0$); thus, $\Delta \mathbf{p}_{\text{tot}} = 0$, and \mathbf{p}_{tot} is constant.

We have noted that the three length dimensions in nature— x , y , and z —are independent, and it is interesting to note that momentum can be conserved in different ways along each dimension. For example, during projectile motion and where air resistance is negligible, momentum is conserved in the horizontal direction because horizontal forces are zero and momentum is unchanged. But along the vertical direction, the net vertical force is not zero and the momentum of the projectile is not conserved. (See [\[link\]](#).) However, if the momentum of the projectile-Earth system is considered in the vertical direction, we find that the total momentum is conserved.



The horizontal component of a projectile's momentum is conserved if air resistance is negligible, even in this case where a space probe separates. The forces causing the separation are internal to the system, so that the net external horizontal force $F_{x-\text{net}}$ is still zero. The vertical component of the momentum is not conserved, because the net vertical force $F_{y-\text{net}}$ is not zero. In the vertical direction, the space probe-Earth system needs to be considered and we find that the total momentum is conserved. The center of mass of the space probe takes the same path it would if the separation did not occur.

The conservation of momentum principle can be applied to systems as different as a comet striking Earth and a gas containing huge numbers of atoms and molecules. Conservation of momentum is violated only when the net external force is not zero. But another larger system can always be considered in which momentum is conserved by simply including the source of the external force. For example, in the collision of two cars considered above, the two-car system conserves momentum while each one-car system does not.

Note:

Making Connections: Take-Home Investigation—Drop of Tennis Ball and a Basketball

Hold a tennis ball side by side and in contact with a basketball. Drop the balls together. (Be careful!) What happens? Explain your observations. Now hold the tennis ball above and in contact with the basketball. What happened? Explain your observations. What do you think will happen if the basketball ball is held above and in contact with the tennis ball?

Note:

Making Connections: Take-Home Investigation—Two Tennis Balls in a Ballistic Trajectory

Tie two tennis balls together with a string about a foot long. Hold one ball and let the other hang down and throw it in a ballistic trajectory. Explain your observations. Now mark the center of the string with bright ink or attach a brightly colored sticker to it and throw again. What happened? Explain your observations.

Some aquatic animals such as jellyfish move around based on the principles of conservation of momentum. A jellyfish fills its umbrella section with water and then pushes the water out resulting in motion in the opposite direction to that of the jet of water. Squids propel themselves in a similar manner but, in contrast with jellyfish, are able to control the direction in which they move by aiming their nozzle forward or backward. Typical squids can move at speeds of 8 to 12 km/h.

The ballistocardiograph (BCG) was a diagnostic tool used in the second half of the 20th century to study the strength of the heart. About once a second, your heart beats, forcing blood into the aorta. A force in the opposite direction is exerted on the rest of your body (recall Newton's third law). A ballistocardiograph is a device that can measure this reaction force. This measurement is done by using a sensor (resting on the person) or by using a moving table suspended from the ceiling. This technique can gather information on the strength of the heart beat and the volume of blood passing from the heart. However, the electrocardiogram (ECG or EKG) and the echocardiogram (cardiac ECHO or ECHO; a technique that uses ultrasound to see an image of the heart) are more widely used in the practice of cardiology.

Note:

Applying Science Practices: Verifying the Conservation of Linear Momentum

Design an experiment to verify the conservation of linear momentum in a one-dimensional collision, both elastic and inelastic. For simplicity, try to ensure that friction is minimized so that it has a negligible effect on your experiment. As you consider your experiment, consider the following questions:

- Predict how the final momentum of the system will compare to the initial momentum of the system that you will measure. Justify your prediction.
- How will you measure the momentum of each object?
- Should you have two objects in motion or one object bouncing off a rigid surface?
- Should you verify the relationship mathematically or graphically?
- How will you estimate the uncertainty of your measurements? How will you express this uncertainty in your data?

When you have completed each experiment, compare the outcome to your prediction about the initial and final momentum of the system and evaluate your results.

Note:**Making Connections: Conservation of Momentum and Collision**

Conservation of momentum is quite useful in describing collisions. Momentum is crucial to our understanding of atomic and subatomic particles because much of what we know about these particles comes from collision experiments.

Subatomic Collisions and Momentum

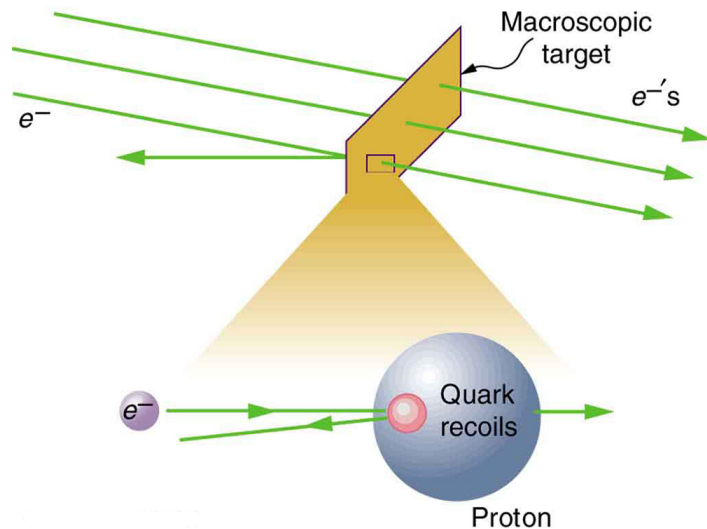
The conservation of momentum principle not only applies to the macroscopic objects, it is also essential to our explorations of atomic and subatomic particles. Giant machines hurl subatomic particles at one another, and researchers evaluate the results by assuming conservation of momentum (among other things).

On the small scale, we find that particles and their properties are invisible to the naked eye but can be measured with our instruments, and models of these subatomic particles can be constructed to describe the results.

Momentum is found to be a property of all subatomic particles including massless particles such as photons that compose light. Momentum being a property of particles hints that momentum may have an identity beyond the description of an object's mass multiplied by the object's velocity. Indeed, momentum relates to wave properties and plays a fundamental role in what measurements are taken and how we take these measurements.

Furthermore, we find that the conservation of momentum principle is valid when considering systems of particles. We use this principle to analyze the masses and other properties of previously undetected particles, such as the nucleus of an atom and the existence of quarks that make up particles of nuclei. [\[link\]](#) below illustrates how a particle scattering backward from another implies that its target is massive and dense. Experiments seeking evidence that **quarks** make up protons (one type of particle that makes up nuclei) scattered high-energy electrons off of protons (nuclei of hydrogen atoms). Electrons occasionally scattered straight backward in a manner that implied a very small and very dense particle makes up the proton—this observation is considered nearly direct evidence of quarks. The analysis

was based partly on the same conservation of momentum principle that works so well on the large scale.



A subatomic particle scatters straight backward from a target particle. In experiments seeking evidence for quarks, electrons were observed to occasionally scatter straight backward from a proton.

Test Prep for AP Courses

Exercise:

Problem: Which of the following is an example of an open system?

- a. Two air cars colliding on a track elastically.
- b. Two air cars colliding on a track and sticking together.
- c. A bullet being fired into a hanging wooden block and becoming embedded in the block, with the system then acting as a ballistic

pendulum.

- d. A bullet being fired into a hillside and becoming buried in the earth.

Solution:

(d)

Exercise:

Problem:

A 40-kg girl runs across a mat with a speed of 5.0 m/s and jumps onto a 120-kg hanging platform initially at rest, causing the girl and platform to swing back and forth like a pendulum together after her jump. What is the combined velocity of the girl and platform after the jump? What is the combined momentum of the girl and platform both before and after the collision?

A 50-kg boy runs across a mat with a speed of 6.0 m/s and collides with a soft barrier on the wall, rebounding off the wall and falling to the ground. The boy is at rest after the collision. What is the momentum of the boy before and after the collision? Is momentum conserved in this collision? Explain. Which of these is an example of an open system and which is an example of a closed system? Explain your answer.

Exercise:

Problem:

A student sets up an experiment to measure the momentum of a system of two air cars, A and B, of equal mass, moving on a linear, frictionless track. Before the collision, car A has a certain speed, and car B is at rest. Which of the following will be true about the total momentum of the two cars?

- a. It will be greater before the collision.
- b. It will be equal before and after the collision.

- c. It will be greater after the collision.
 - d. The answer depends on whether the collision is elastic or inelastic.
-

Solution:

(b)

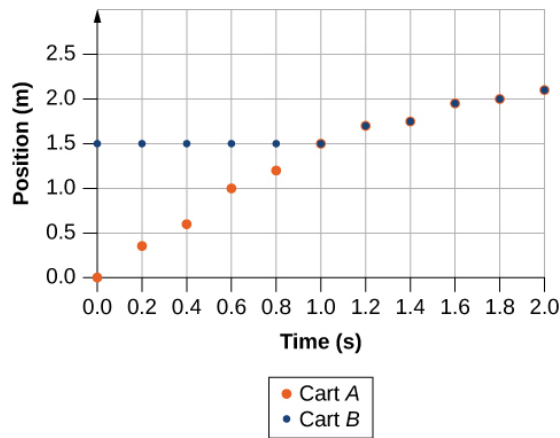
Exercise:

Problem:

A group of students has two carts, *A* and *B*, with wheels that turn with negligible friction. The carts can travel along a straight horizontal track. Cart *A* has known mass m_A . The students are asked to use a one-dimensional collision between the carts to determine the mass of cart *B*. Before the collision, cart *A* travels to the right and cart *B* is initially at rest. After the collision, the carts stick together.

- a. Describe an experimental procedure to determine the velocities of the carts before and after a collision, including all the additional equipment you would need. You may include a labeled diagram of your setup to help in your description. Indicate what measurements you would take and how you would take them. Include enough detail so that another student could carry out your procedure.
- b. There will be sources of error in the measurements taken in the experiment, both before and after the collision. For your experimental procedure, will the uncertainty in the calculated value of the mass of cart *B* be affected more by the error in the measurements taken before the collision or by those taken after the collision, or will it be equally affected by both sets of measurements? Justify your answer.

A group of students took measurements for one collision. A graph of the students' data is shown below.



The image shows a graph with position in meters on the vertical axis and time in seconds on the horizontal axis.

- c. Given $m_A = 0.50$ kg, use the graph to calculate the mass of cart B . Explicitly indicate the principles used in your calculations.
- d. The students are now asked to Consider the kinetic energy changes in an inelastic collision, specifically whether the initial values of one of the physical quantities affect the fraction of mechanical energy dissipated in the collision. How could you modify the experiment to investigate this question? Be sure to explicitly describe the calculations you would make, specifying all equations you would use (but do not actually do any algebra or arithmetic).

Exercise:

Problem:

Cart A is moving with an initial velocity $+v$ (in the positive direction) toward cart B, initially at rest. Both carts have equal mass and are on a frictionless surface. Which of the following statements correctly characterizes the velocity of the center of mass of the system before and after the collision?

- a. $\frac{+v}{2}$ before, $\frac{-v}{2}$ after
- b. $\frac{+v}{2}$ before, 0 after
- c. $\frac{+v}{2}$ before, $\frac{+v}{2}$ after
- d. 0 before, 0 after

Solution:

(c)

Exercise:**Problem:**

Cart A is moving with a velocity of $+10$ m/s toward cart B, which is moving with a velocity of $+4$ m/s. Both carts have equal mass and are moving on a frictionless surface. The two carts have an inelastic collision and stick together after the collision. Calculate the velocity of the center of mass of the system before and after the collision. If there were friction present in this problem, how would this external force affect the center-of-mass velocity both before and after the collision?

Section Summary

- The conservation of momentum principle is written

Equation:

$$\mathbf{p}_{\text{tot}} = \text{constant}$$

or

Equation:

$$\mathbf{p}_{\text{tot}} = \mathbf{p}'_{\text{tot}} \text{ (isolated system),}$$

\mathbf{p}_{tot} is the initial total momentum and \mathbf{p}'_{tot} is the total momentum some time later.

- An isolated system is defined to be one for which the net external force is zero ($\mathbf{F}_{\text{net}} = 0$).
- During projectile motion and where air resistance is negligible, momentum is conserved in the horizontal direction because horizontal forces are zero.
- Conservation of momentum applies only when the net external force is zero.
- The conservation of momentum principle is valid when considering systems of particles.

Conceptual Questions**Exercise:****Problem: Professional Application**

If you dive into water, you reach greater depths than if you do a belly flop. Explain this difference in depth using the concept of conservation of energy. Explain this difference in depth using what you have learned in this chapter.

Exercise:

Problem: Under what circumstances is momentum conserved?

Exercise:**Problem:**

Can momentum be conserved for a system if there are external forces acting on the system? If so, under what conditions? If not, why not?

Exercise:

Problem:

Momentum for a system can be conserved in one direction while not being conserved in another. What is the angle between the directions? Give an example.

Exercise:**Problem: Professional Application**

Explain in terms of momentum and Newton's laws how a car's air resistance is due in part to the fact that it pushes air in its direction of motion.

Exercise:**Problem:**

Can objects in a system have momentum while the momentum of the system is zero? Explain your answer.

Exercise:**Problem:**

Must the total energy of a system be conserved whenever its momentum is conserved? Explain why or why not.

Problems & Exercises**Exercise:****Problem: Professional Application**

Train cars are coupled together by being bumped into one another. Suppose two loaded train cars are moving toward one another, the first having a mass of 150,000 kg and a velocity of 0.300 m/s, and the second having a mass of 110,000 kg and a velocity of -0.120 m/s. (The minus indicates direction of motion.) What is their final velocity?

Solution:

0.122 m/s

Exercise:**Problem:**

Suppose a clay model of a koala bear has a mass of 0.200 kg and slides on ice at a speed of 0.750 m/s. It runs into another clay model, which is initially motionless and has a mass of 0.350 kg. Both being soft clay, they naturally stick together. What is their final velocity?

Exercise:**Problem: Professional Application**

Consider the following question: *A car moving at 10 m/s crashes into a tree and stops in 0.26 s. Calculate the force the seatbelt exerts on a passenger in the car to bring him to a halt. The mass of the passenger is 70 kg.* Would the answer to this question be different if the car with the 70-kg passenger had collided with a car that has a mass equal to and is traveling in the opposite direction and at the same speed? Explain your answer.

Solution:

In a collision with an identical car, momentum is conserved. Afterwards $v_f = 0$ for both cars. The change in momentum will be the same as in the crash with the tree. However, the force on the body is not determined since the time is not known. A padded stop will reduce injurious force on body.

Exercise:

Problem:

What is the velocity of a 900-kg car initially moving at 30.0 m/s, just after it hits a 150-kg deer initially running at 12.0 m/s in the same direction? Assume the deer remains on the car.

Exercise:**Problem:**

A 1.80-kg falcon catches a 0.650-kg dove from behind in midair. What is their velocity after impact if the falcon's velocity is initially 28.0 m/s and the dove's velocity is 7.00 m/s in the same direction?

Solution:

22.4 m/s in the same direction as the original motion

Glossary

conservation of momentum principle

when the net external force is zero, the total momentum of the system is conserved or constant

isolated system

a system in which the net external force is zero

quark

fundamental constituent of matter and an elementary particle

Elastic Collisions in One Dimension

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Describe an elastic collision of two objects in one dimension.
- Define internal kinetic energy.
- Derive an expression for conservation of internal kinetic energy in a one-dimensional collision.
- Determine the final velocities in an elastic collision given masses and initial velocities.

The information presented in this section supports the following AP® learning objectives and science practices:

- **4.B.1.1** The student is able to calculate the change in linear momentum of a two-object system with constant mass in linear motion from a representation of the system (data, graphs, etc.). (S.P. 1.4, 2.2)
- **4.B.1.2** The student is able to analyze data to find the change in linear momentum for a constant-mass system using the product of the mass and the change in velocity of the center of mass. (S.P. 5.1)
- **5.A.2.1** The student is able to define open and closed systems for everyday situations and apply conservation concepts for energy, charge, and linear momentum to those situations. (S.P. 6.4, 7.2)
- **5.D.1.1** The student is able to make qualitative predictions about natural phenomena based on conservation of linear momentum and restoration of kinetic energy in elastic collisions. (S.P. 6.4, 7.2)
- **5.D.1.2** The student is able to apply the principles of conservation of momentum and restoration of kinetic energy to reconcile a situation that appears to be isolated and elastic, but in

which data indicate that linear momentum and kinetic energy are not the same after the interaction, by refining a scientific question to identify interactions that have not been considered. Students will be expected to solve qualitatively and/or quantitatively for one-dimensional situations and only qualitatively in two-dimensional situations. **(S.P. 2.2, 3.2, 5.1, 5.3)**

- **5.D.1.5** The student is able to classify a given collision situation as elastic or inelastic, justify the selection of conservation of linear momentum and restoration of kinetic energy as the appropriate principles for analyzing an elastic collision, solve for missing variables, and calculate their values. **(S.P. 2.1, 2.2)**
- **5.D.1.6** The student is able to make predictions of the dynamical properties of a system undergoing a collision by application of the principle of linear momentum conservation and the principle of the conservation of energy in situations in which an elastic collision may also be assumed. **(S.P. 6.4)**
- **5.D.1.7** The student is able to classify a given collision situation as elastic or inelastic, justify the selection of conservation of linear momentum and restoration of kinetic energy as the appropriate principles for analyzing an elastic collision, solve for missing variables, and calculate their values. **(S.P. 2.1, 2.2)**
- **5.D.2.1** The student is able to qualitatively predict, in terms of linear momentum and kinetic energy, how the outcome of a collision between two objects changes depending on whether the collision is elastic or inelastic. **(S.P. 6.4, 7.2)**
- **5.D.2.2** The student is able to plan data collection strategies to test the law of conservation of momentum in a two-object collision that is elastic or inelastic and analyze the resulting data graphically. **(S.P.4.1, 4.2, 5.1)**
- **5.D.3.2** The student is able to make predictions about the velocity of the center of mass for interactions within a defined one-dimensional system. **(S.P. 6.4)**

Let us consider various types of two-object collisions. These collisions are the easiest to analyze, and they illustrate many of the physical principles involved in collisions. The conservation of momentum principle is very useful here, and it can be used whenever the net external force on a system is zero.

We start with the elastic collision of two objects moving along the same line—a one-dimensional problem. An **elastic collision** is one that also conserves internal kinetic energy. **Internal kinetic energy** is the sum of the kinetic energies of the objects in the system. [\[link\]](#) illustrates an elastic collision in which internal kinetic energy and momentum are conserved.

Truly elastic collisions can only be achieved with subatomic particles, such as electrons striking nuclei. Macroscopic collisions can be very nearly, but not quite, elastic—some kinetic energy is always converted into other forms of energy such as heat transfer due to friction and sound. One macroscopic collision that is nearly elastic is that of two steel blocks on ice. Another nearly elastic collision is that between two carts with spring bumpers on an air track. Icy surfaces and air tracks are nearly frictionless, more readily allowing nearly elastic collisions on them.

Note:

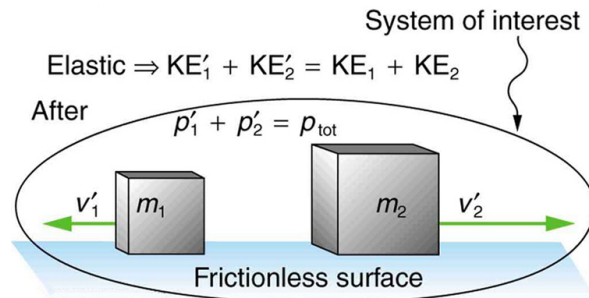
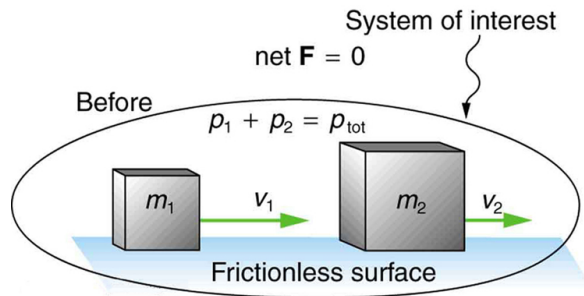
Elastic Collision

An **elastic collision** is one that conserves internal kinetic energy.

Note:

Internal Kinetic Energy

Internal kinetic energy is the sum of the kinetic energies of the objects in the system.



An elastic one-dimensional
two-object collision.
Momentum and internal kinetic
energy are conserved.

Now, to solve problems involving one-dimensional elastic collisions between two objects we can use the equations for conservation of momentum and conservation of internal kinetic energy. First, the equation for conservation of momentum for two objects in a one-dimensional collision is

Equation:

$$p_1 + p_2 = p'_1 + p'_2 \quad (F_{\text{net}} = 0)$$

or

Equation:

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2 \quad (F_{\text{net}} = 0),$$

where the primes (') indicate values after the collision. By definition, an elastic collision conserves internal kinetic energy, and so the sum of kinetic energies before the collision equals the sum after the collision. Thus,

Equation:

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2 \text{ (two-object elastic collision)}$$

expresses the equation for conservation of internal kinetic energy in a one-dimensional collision.

Note:

Making Connections: Collisions

Suppose data are collected on a collision between two masses sliding across a frictionless surface. Mass A (1.0 kg) moves with a velocity of +12 m/s, and mass B (2.0 kg) moves with a velocity of −12 m/s. The two masses collide and stick together after the collision. The table below shows the measured velocities of each mass at times before and after the collision:

Time (s)	Velocity A (m/s)	Velocity B (m/s)
0	+12	−12
1.0 s	+12	−12
2.0 s	−4.0	−4.0
3.0 s	−4.0	−4.0

The total mass of the system is 3.0 kg. The velocity of the center of mass of this system can be determined from the conservation of momentum. Consider

the system before the collision:

Equation:

$$(m_A + m_B)v_{cm} = m_A v_A + m_B v_B$$

Equation:

$$(3.0)v_{cm} = (1)(12) + (2)(-12)$$

Equation:

$$v_{cm} = -4.0 \text{ m/s}$$

After the collision, the center-of-mass velocity is the same:

Equation:

$$(m_A + m_B)v_{cm} = (m_A + m_B)v_{final}$$

Equation:

$$(3.0)v_{cm} = (3)(-4.0)$$

Equation:

$$v_{cm} = -4.0 \text{ m/s}$$

The total momentum of the system before the collision is:

Equation:

$$m_A v_A + m_B v_B = (1)(12) + (2)(-12) = -12 \text{ kg} \bullet \text{ m/s}$$

The total momentum of the system after the collision is:

Equation:

$$(m_A + m_B)v_{final} = (3)(-4) = -12 \text{ kg} \bullet \text{ m/s}$$

Thus, the change in momentum of the system is zero when measured this way. We get a similar result when we calculate the momentum using the center-of-mass velocity. Since the center-of-mass velocity is the same both

before and after the collision, we calculate the same momentum for the system using this method both before and after the collision.

Example:

Calculating Velocities Following an Elastic Collision

Calculate the velocities of two objects following an elastic collision, given that

Equation:

$$m_1 = 0.500 \text{ kg}, m_2 = 3.50 \text{ kg}, v_1 = 4.00 \text{ m/s}, \text{ and } v_2 = 0.$$

Strategy and Concept

First, visualize what the initial conditions mean—a small object strikes a larger object that is initially at rest. This situation is slightly simpler than the situation shown in [\[link\]](#) where both objects are initially moving. We are asked to find two unknowns (the final velocities v'_1 and v'_2). To find two unknowns, we must use two independent equations. Because this collision is elastic, we can use the above two equations. Both can be simplified by the fact that object 2 is initially at rest, and thus $v_2 = 0$. Once we simplify these equations, we combine them algebraically to solve for the unknowns.

Solution

For this problem, note that $v_2 = 0$ and use conservation of momentum. Thus,

Equation:

$$p_1 = p'_1 + p'_2$$

or

Equation:

$$m_1 v_1 = m_1 v'_1 + m_2 v'_2.$$

Using conservation of internal kinetic energy and that $v_2 = 0$,

Equation:

$$\frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 v'^2_1 + \frac{1}{2} m_2 v'^2_2.$$

Solving the first equation (momentum equation) for v'_2 , we obtain

Equation:

$$v'_2 = \frac{m_1}{m_2}(v_1 - v'_1).$$

Substituting this expression into the second equation (internal kinetic energy equation) eliminates the variable v'_2 , leaving only v'_1 as an unknown (the algebra is left as an exercise for the reader). There are two solutions to any quadratic equation; in this example, they are

Equation:

$$v'_1 = 4.00 \text{ m/s}$$

and

Equation:

$$v'_1 = -3.00 \text{ m/s}.$$

As noted when quadratic equations were encountered in earlier chapters, both solutions may or may not be meaningful. In this case, the first solution is the same as the initial condition. The first solution thus represents the situation before the collision and is discarded. The second solution ($v'_1 = -3.00 \text{ m/s}$) is negative, meaning that the first object bounces backward. When this negative value of v'_1 is used to find the velocity of the second object after the collision, we get

Equation:

$$v'_2 = \frac{m_1}{m_2}(v_1 - v'_1) = \frac{0.500 \text{ kg}}{3.50 \text{ kg}}[4.00 - (-3.00)] \text{ m/s}$$

or

Equation:

$$v'_2 = 1.00 \text{ m/s}.$$

Discussion

The result of this example is intuitively reasonable. A small object strikes a larger one at rest and bounces backward. The larger one is knocked forward, but with a low speed. (This is like a compact car bouncing backward off a full-size SUV that is initially at rest.) As a check, try calculating the internal

kinetic energy before and after the collision. You will see that the internal kinetic energy is unchanged at 4.00 J. Also check the total momentum before and after the collision; you will find it, too, is unchanged.

The equations for conservation of momentum and internal kinetic energy as written above can be used to describe any one-dimensional elastic collision of two objects. These equations can be extended to more objects if needed.

Note:

Making Connections: Take-Home Investigation—Ice Cubes and Elastic Collision

Find a few ice cubes which are about the same size and a smooth kitchen tabletop or a table with a glass top. Place the ice cubes on the surface several centimeters away from each other. Flick one ice cube toward a stationary ice cube and observe the path and velocities of the ice cubes after the collision. Try to avoid edge-on collisions and collisions with rotating ice cubes. Have you created approximately elastic collisions? Explain the speeds and directions of the ice cubes using momentum.

Note:

PhET Explorations: Collision Lab

Investigate collisions on an air hockey table. Set up your own experiments: vary the number of discs, masses and initial conditions. Is momentum conserved? Is kinetic energy conserved? Vary the elasticity and see what happens.

[Collision Lab](#)

Exercise:**Problem:**

Two cars (A and B) of mass 1.5 kg collide. Car A is initially moving at 12 m/s, and car B is initially moving in the same direction with a speed of 6 m/s. The two cars are moving along a straight line before and after the collision. What will be the change in momentum of this system after the collision?

- a. $-27 \text{ kg} \cdot \text{m/s}$
- b. zero
- c. $+27 \text{ kg} \cdot \text{m/s}$
- d. It depends on whether the collision is elastic or inelastic.

Solution:

(b)

Exercise:**Problem:**

Two cars (A and B) of mass 1.5 kg collide. Car A is initially moving at 24 m/s, and car B is initially moving in the opposite direction with a speed of 12 m/s. The two cars are moving along a straight line before and after the collision. (a) If the two cars have an elastic collision, calculate the change in momentum of the two-car system. (b) If the two cars have a completely inelastic collision, calculate the change in momentum of the two-car system.

Exercise:**Problem:**

Puck A (200 g) slides across a frictionless surface to collide with puck B (800 g), initially at rest. The velocity of each puck is measured during the experiment as follows:

Time	Velocity A	Velocity B
0	+8.0 m/s	0
1.0 s	+8.0 m/s	0
2.0 s	−2.0 m/s	+2.5 m/s
3.0 s	−2.0 m/s	+2.5 m/s

What is the change in momentum of the center of mass of the system as a result of the collision?

- a. +1.6 kg•m/s
- b. +0.8 kg•m/s
- c. 0
- d. −1.6 kg•m/s

Solution:

(c)

Exercise:

Problem:

For the table above, calculate the center-of-mass velocity of the system both before and after the collision, then calculate the center-of-mass momentum of the system both before and after the collision. From this, determine the change in the momentum of the system as a result of the collision.

Exercise:

Problem:

Two cars (A and B) of equal mass have an elastic collision. Prior to the collision, car A is moving at 15 m/s in the $+x$ -direction, and car B is moving at 10 m/s in the $-x$ -direction. Assuming that both cars continue moving along the x -axis after the collision, what will be the velocity of car A after the collision?

- a. same as the original 15 m/s speed, opposite direction
 - b. equal to car B's velocity prior to the collision
 - c. equal to the average of the two velocities, in its original direction
 - d. equal to the average of the two velocities, in the opposite direction
-

Solution:

(b)

Exercise:**Problem:**

Two cars (A and B) of equal mass have an elastic collision. Prior to the collision, car A is moving at 20 m/s in the $+x$ -direction, and car B is moving at 10 m/s in the $-x$ -direction. Assuming that both cars continue moving along the x -axis after the collision, what will be the velocities of each car after the collision?

Exercise:**Problem:**

A rubber ball is dropped from rest at a fixed height. It bounces off a hard floor and rebounds upward, but it only reaches 90% of its original fixed height. What is the best way to explain the loss of kinetic energy of the ball during the collision?

- a. Energy was required to deform the ball's shape during the collision with the floor.
- b. Energy was lost due to work done by the ball pushing on the floor during the collision.

- c. Energy was lost due to friction between the ball and the floor.
 - d. Energy was lost due to the work done by gravity during the motion.
-

Solution:

(a)

Exercise:

Problem:

A tennis ball strikes a wall with an initial speed of 15 m/s. The ball bounces off the wall but rebounds with slightly less speed (14 m/s) after the collision. Explain (a) what else changed its momentum in response to the ball's change in momentum so that overall momentum is conserved, and (b) how some of the ball's kinetic energy was lost.

Exercise:

Problem:

Two objects, A and B, have equal mass. Prior to the collision, mass A is moving 10 m/s in the $+x$ -direction, and mass B is moving 4 m/s in the $+x$ -direction. Which of the following results represents an inelastic collision between A and B?

- a. After the collision, mass A is at rest, and mass B moves 14 m/s in the $+x$ -direction.
 - b. After the collision, mass A moves 4 m/s in the $-x$ -direction, and mass B moves 18 m/s in the $+x$ -direction.
 - c. After the collision, the two masses stick together and move 7 m/s in the $+x$ -direction.
 - d. After the collision, mass A moves 4 m/s in the $+x$ -direction, and mass B moves 10 m/s in the $+x$ -direction.
-

Solution:

(c)

Exercise:

Problem:

Mass A is three times more massive than mass B. Mass A is initially moving 12 m/s in the $+x$ -direction. Mass B is initially moving 12 m/s in the $-x$ -direction. Assuming that the collision is elastic, calculate the final velocity of both masses after the collision. Show that your results are consistent with conservation of momentum and conservation of kinetic energy.

Exercise:**Problem:**

Two objects (A and B) of equal mass collide elastically. Mass A is initially moving 5.0 m/s in the $+x$ -direction prior to the collision. Mass B is initially moving 3.0 m/s in the $-x$ -direction prior to the collision. After the collision, mass A will be moving with a velocity of 3.0 m/s in the $-x$ -direction. What will be the velocity of mass B after the collision?

- a. 3.0 m/s in the $+x$ -direction
- b. 5.0 m/s in the $+x$ -direction
- c. 3.0 m/s in the $-x$ -direction
- d. 5.0 m/s in the $-x$ -direction

Solution:

(b)

Exercise:**Problem:**

Two objects (A and B) of equal mass collide elastically. Mass A is initially moving 4.0 m/s in the $+x$ -direction prior to the collision. Mass B is initially moving 8.0 m/s in the $-x$ -direction prior to the collision. After the collision, mass A will be moving with a velocity of 8.0 m/s in the $-x$ -direction. (a) Use the principle of conservation of momentum to predict the velocity of mass B after the collision. (b) Use the fact that kinetic energy is conserved in elastic collisions to predict the velocity of mass B after the collision.

Exercise:**Problem:**

Two objects of equal mass collide. Object A is initially moving in the $+x$ -direction with a speed of 12 m/s, and object B is initially at rest. After the collision, object A is at rest, and object B is moving away with some unknown velocity. There are no external forces acting on the system of two masses. What statement can we make about this collision?

- a. Both momentum and kinetic energy are conserved.
- b. Momentum is conserved, but kinetic energy is not conserved.
- c. Neither momentum nor kinetic energy is conserved.
- d. More information is needed in order to determine which is conserved.

Solution:

(a)

Exercise:**Problem:**

Two objects of equal mass collide. Object A is initially moving with a velocity of 15 m/s in the $+x$ -direction, and object B is initially at rest. After the collision, object A is at rest. There are no external forces acting on the system of two masses. (a) Use momentum conservation to deduce the velocity of object B after the collision. (b) Is this collision elastic? Justify your answer.

Exercise:**Problem:**

Which of the following statements is true about an inelastic collision?

- a. Momentum is conserved, and kinetic energy is conserved.
- b. Momentum is conserved, and kinetic energy is not conserved.
- c. Momentum is not conserved, and kinetic energy is conserved.
- d. Momentum is not conserved, and kinetic energy is not conserved.

Solution:

(b)

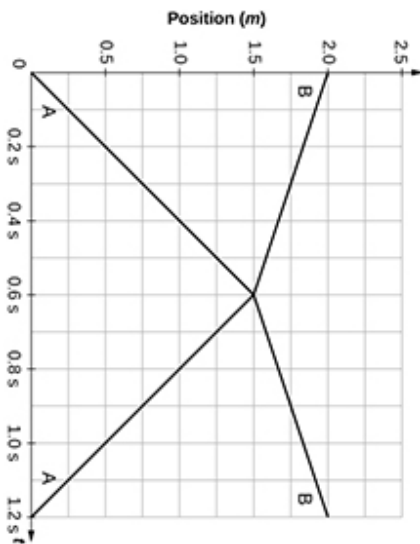
Exercise:

Problem:

Explain how the momentum and kinetic energy of a system of two colliding objects changes as a result of (a) an elastic collision and (b) an inelastic collision.

Exercise:

Problem:



This figure shows the positions of two colliding objects measured before, during, and after a collision. Mass A is 1.0 kg. Mass B is 3.0 kg. Which of the following statements is true?

- a. This is an elastic collision, with a total momentum of $0 \text{ kg} \cdot \text{m/s}$.
- b. This is an elastic collision, with a total momentum of $1.67 \text{ kg} \cdot \text{m/s}$.
- c. This is an inelastic collision, with a total momentum of $0 \text{ kg} \cdot \text{m/s}$.

- d. This is an inelastic collision, with a total momentum of $1.67 \text{ kg} \cdot \text{m/s}$.
-

Solution:

(a)

Exercise:

Problem:

For the above graph, determine the initial and final momentum for both objects, assuming mass A is 1.0 kg and mass B is 3.0 kg . Also, determine the initial and final kinetic energies for both objects. Based on your results, explain whether momentum is conserved in this collision, and state whether the collision is elastic or inelastic.

Exercise:

Problem:

Mass A (1.0 kg) slides across a frictionless surface with a velocity of 8 m/s in the positive direction. Mass B (3.0 kg) is initially at rest. The two objects collide and stick together. What will be the change in the center-of-mass velocity of the system as a result of the collision?

- a. There will be no change in the center-of-mass velocity.
 - b. The center-of-mass velocity will decrease by 2 m/s .
 - c. The center-of-mass velocity will decrease by 6 m/s .
 - d. The center-of-mass velocity will decrease by 8 m/s .
-

Solution:

(a)

Exercise:

Problem:

Mass A (1.0 kg) slides across a frictionless surface with a velocity of 4 m/s in the positive direction. Mass B (1.0 kg) slides across the same surface in the opposite direction with a velocity of -8 m/s. The two objects collide and stick together after the collision. Predict how the center-of-mass velocity will change as a result of the collision, and explain your prediction. Calculate the center-of-mass velocity of the system both before and after the collision and explain why it remains the same or why it has changed.

Section Summary

- An elastic collision is one that conserves internal kinetic energy.
- Conservation of kinetic energy and momentum together allow the final velocities to be calculated in terms of initial velocities and masses in one dimensional two-body collisions.

Conceptual Questions**Exercise:**

Problem: What is an elastic collision?

Problems & Exercises**Exercise:****Problem:**

Two identical objects (such as billiard balls) have a one-dimensional collision in which one is initially motionless. After the collision, the moving object is stationary and the other moves with the same speed as the other originally had. Show that both momentum and kinetic energy are conserved.

Exercise:

Problem: Professional Application

Two manned satellites approach one another at a relative speed of 0.250 m/s, intending to dock. The first has a mass of 4.00×10^3 kg, and the second a mass of 7.50×10^3 kg. If the two satellites collide elastically rather than dock, what is their final relative velocity?

Solution:

0.250 m/s

Exercise:

Problem:

A 70.0-kg ice hockey goalie, originally at rest, catches a 0.150-kg hockey puck slapped at him at a velocity of 35.0 m/s. Suppose the goalie and the ice puck have an elastic collision and the puck is reflected back in the direction from which it came. What would their final velocities be in this case?

Glossary

elastic collision

a collision that also conserves internal kinetic energy

internal kinetic energy

the sum of the kinetic energies of the objects in a system

Inelastic Collisions in One Dimension

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define inelastic collision.
- Explain perfectly inelastic collisions.
- Apply an understanding of collisions to sports.
- Determine recoil velocity and loss in kinetic energy given mass and initial velocity.

The information presented in this section supports the following AP® learning objectives and science practices:

- **4.B.1.1** The student is able to calculate the change in linear momentum of a two-object system with constant mass in linear motion from a representation of the system (data, graphs, etc.). (S.P. 1.4, 2.2)
- **5.A.2.1** The student is able to define open and closed systems for everyday situations and apply conservation concepts for energy, charge, and linear momentum to those situations. (S.P. 6.4, 7.2)
- **5.D.1.3** The student is able to apply mathematical routines appropriately to problems involving elastic collisions in one dimension and justify the selection of those mathematical routines based on conservation of momentum and restoration of kinetic energy. (S.P. 2.1, 2.2)
- **5.D.1.5** The student is able to classify a given collision situation as elastic or inelastic, justify the selection of conservation of linear momentum and restoration of kinetic energy as the appropriate principles for analyzing an elastic

collision, solve for missing variables, and calculate their values. (S.P. 2.1, 2.2)

- **5.D.2.1** The student is able to qualitatively predict, in terms of linear momentum and kinetic energy, how the outcome of a collision between two objects changes depending on whether the collision is elastic or inelastic. (S.P. 6.4, 7.2)
- **5.D.2.2** The student is able to plan data collection strategies to test the law of conservation of momentum in a two-object collision that is elastic or inelastic and analyze the resulting data graphically. (S.P.4.1, 4.2, 5.1)
- **5.D.2.3** The student is able to apply the conservation of linear momentum to a closed system of objects involved in an inelastic collision to predict the change in kinetic energy. (S.P. 6.4, 7.2)
- **5.D.2.4** The student is able to analyze data that verify conservation of momentum in collisions with and without an external friction force. (S.P. 4.1, 4.2, 4.4, 5.1, 5.3)
- **5.D.2.5** The student is able to classify a given collision situation as elastic or inelastic, justify the selection of conservation of linear momentum as the appropriate solution method for an inelastic collision, recognize that there is a common final velocity for the colliding objects in the totally inelastic case, solve for missing variables, and calculate their values. (S.P. 2.1 2.2)
- **5.D.2.6** The student is able to apply the conservation of linear momentum to an isolated system of objects involved in an inelastic collision to predict the change in kinetic energy. (S.P. 6.4, 7.2)

We have seen that in an elastic collision, internal kinetic energy is conserved. An **inelastic collision** is one in which the internal kinetic energy changes (it is not conserved). This lack of conservation means that the forces between colliding objects may remove or add internal kinetic energy.

Work done by internal forces may change the forms of energy within a system. For inelastic collisions, such as when colliding objects stick together, this internal work may transform some internal kinetic energy into heat transfer. Or it may convert stored energy into internal kinetic energy, such as when exploding bolts separate a satellite from its launch vehicle.

Note:

Inelastic Collision

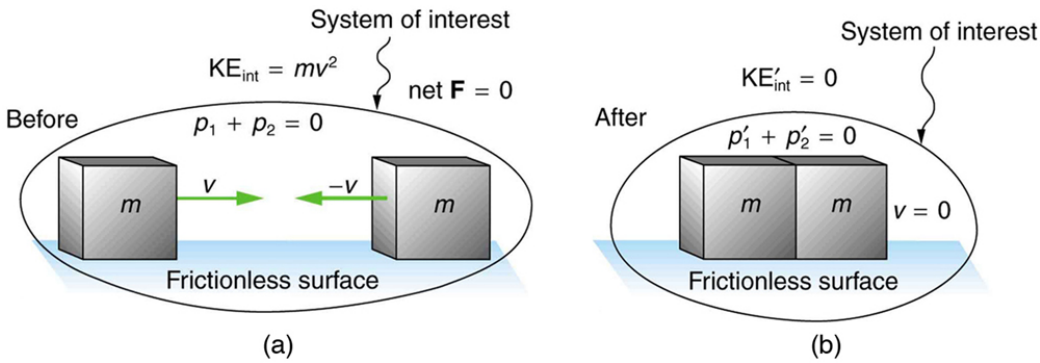
An inelastic collision is one in which the internal kinetic energy changes (it is not conserved).

[\[link\]](#) shows an example of an inelastic collision. Two objects that have equal masses head toward one another at equal speeds and then stick together. Their total internal kinetic energy is initially $\frac{1}{2}mv^2 + \frac{1}{2}mv^2 = mv^2$. The two objects come to rest after sticking together, conserving momentum. But the internal kinetic energy is zero after the collision. A collision in which the objects stick together is sometimes called a **perfectly inelastic collision** because it reduces internal kinetic energy more than does any other type of inelastic collision. In fact, such a collision reduces internal kinetic energy to the minimum it can have while still conserving momentum.

Note:

Perfectly Inelastic Collision

A collision in which the objects stick together is sometimes called “perfectly inelastic.”

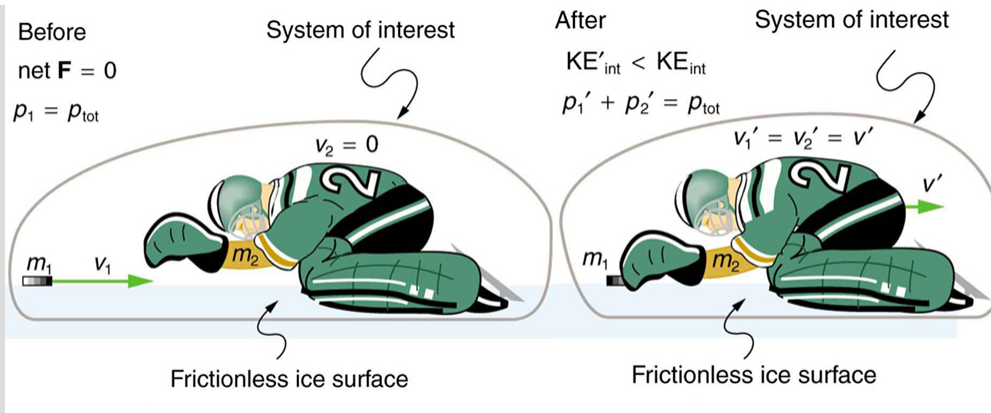


An inelastic one-dimensional two-object collision. Momentum is conserved, but internal kinetic energy is not conserved. (a) Two objects of equal mass initially head directly toward one another at the same speed. (b) The objects stick together (a perfectly inelastic collision), and so their final velocity is zero. The internal kinetic energy of the system changes in any inelastic collision and is reduced to zero in this example.

Example:

Calculating Velocity and Change in Kinetic Energy: Inelastic Collision of a Puck and a Goalie

(a) Find the recoil velocity of a 70.0-kg ice hockey goalie, originally at rest, who catches a 0.150-kg hockey puck slapped at him at a velocity of 35.0 m/s. (b) How much kinetic energy is lost during the collision? Assume friction between the ice and the puck-goalie system is negligible. (See [\[link\]](#))



An ice hockey goalie catches a hockey puck and recoils backward. The initial kinetic energy of the puck is almost entirely converted to thermal energy and sound in this inelastic collision.

Strategy

Momentum is conserved because the net external force on the puck-goalie system is zero. We can thus use conservation of momentum to find the final velocity of the puck and goalie system. Note that the initial velocity of the goalie is zero and that the final velocity of the puck and goalie are the same. Once the final velocity is found, the kinetic energies can be calculated before and after the collision and compared as requested.

Solution for (a)

Momentum is conserved because the net external force on the puck-goalie system is zero.

Conservation of momentum is

Equation:

$$p_1 + p_2 = p'_1 + p'_2$$

or

Equation:

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2.$$

Because the goalie is initially at rest, we know $v_2 = 0$. Because the goalie catches the puck, the final velocities are equal, or $v'_1 = v'_2 = v'$. Thus, the

conservation of momentum equation simplifies to

Equation:

$$m_1 v_1 = (m_1 + m_2) v_f.$$

Solving for v_f yields

Equation:

$$v_f = \frac{m_1}{m_1 + m_2} v_1.$$

Entering known values in this equation, we get

Equation:

$$v_f = \left(\frac{0.150 \text{ kg}}{70.0 \text{ kg} + 0.150 \text{ kg}} \right) (35.0 \text{ m/s}) = 7.48 \times 10^{-2} \text{ m/s}.$$

Discussion for (a)

This recoil velocity is small and in the same direction as the puck's original velocity, as we might expect.

Solution for (b)

Before the collision, the internal kinetic energy KE_{int} of the system is that of the hockey puck, because the goalie is initially at rest. Therefore, KE_{int} is initially

Equation:

$$\begin{aligned} \text{KE}_{\text{int}} &= \frac{1}{2} m v^2 = \frac{1}{2} (0.150 \text{ kg}) (35.0 \text{ m/s})^2 \\ &= 91.9 \text{ J}. \end{aligned}$$

After the collision, the internal kinetic energy is

Equation:

$$\begin{aligned} \text{KE}'_{\text{int}} &= \frac{1}{2} (m + M) v^2 = \frac{1}{2} (70.15 \text{ kg}) (7.48 \times 10^{-2} \text{ m/s})^2 \\ &= 0.196 \text{ J}. \end{aligned}$$

The change in internal kinetic energy is thus

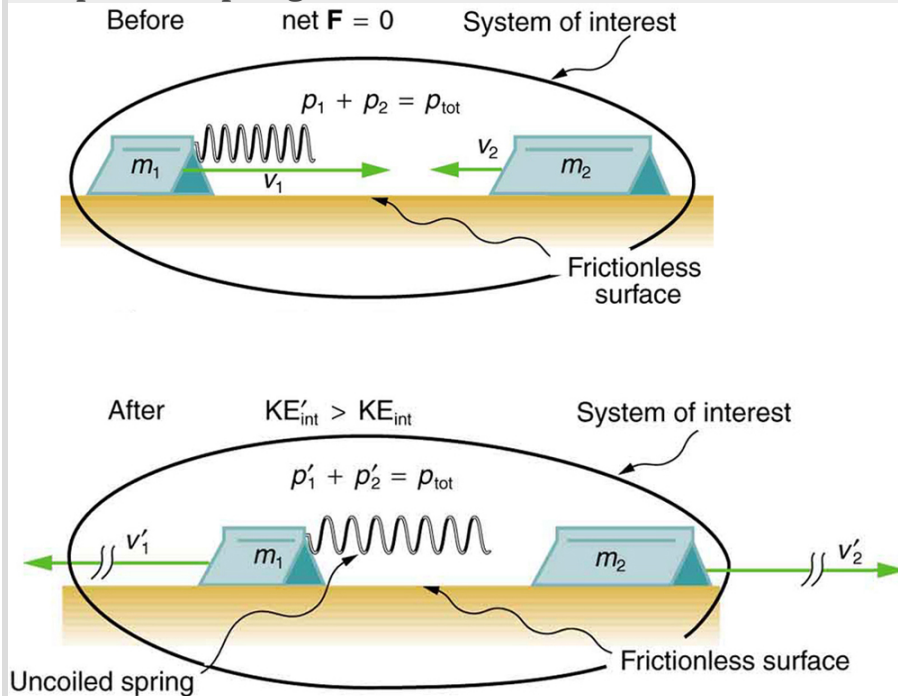
Equation:

$$\begin{aligned} KE_{\text{int}} - KE_{\text{int}} &= 0.196 \text{ J} - 91.9 \text{ J} \\ &= -91.7 \text{ J} \end{aligned}$$

where the minus sign indicates that the energy was lost.

Discussion for (b)

Nearly all of the initial internal kinetic energy is lost in this perfectly inelastic collision. KE_{int} is mostly converted to thermal energy and sound. During some collisions, the objects do not stick together and less of the internal kinetic energy is removed—such as happens in most automobile accidents. Alternatively, stored energy may be converted into internal kinetic energy during a collision. [\[link\]](#) shows a one-dimensional example in which two carts on an air track collide, releasing potential energy from a compressed spring. [\[link\]](#) deals with data from such a collision.



An air track is nearly frictionless, so that momentum is conserved. Motion is one-dimensional. In this collision, examined in [\[link\]](#), the potential energy of a compressed spring is released during the collision and is converted to internal kinetic energy.

Collisions are particularly important in sports and the sporting and leisure industry utilizes elastic and inelastic collisions. Let us look briefly at tennis. Recall that in a collision, it is momentum and not force that is important. So, a heavier tennis racquet will have the advantage over a lighter one. This conclusion also holds true for other sports—a lightweight bat (such as a softball bat) cannot hit a hardball very far.

The location of the impact of the tennis ball on the racquet is also important, as is the part of the stroke during which the impact occurs. A smooth motion results in the maximizing of the velocity of the ball after impact and reduces sports injuries such as tennis elbow. A tennis player tries to hit the ball on the “sweet spot” on the racquet, where the vibration and impact are minimized and the ball is able to be given more velocity. Sports science and technologies also use physics concepts such as momentum and rotational motion and vibrations.

Note:

Take-Home Experiment—Bouncing of Tennis Ball

1. Find a racquet (a tennis, badminton, or other racquet will do). Place the racquet on the floor and stand on the handle. Drop a tennis ball on the strings from a measured height. Measure how high the ball bounces. Now ask a friend to hold the racquet firmly by the handle and drop a tennis ball from the same measured height above the racquet. Measure how high the ball bounces and observe what happens to your friend’s hand during the collision. Explain your observations and measurements.
2. The coefficient of restitution (c) is a measure of the elasticity of a collision between a ball and an object, and is defined as the ratio of the speeds after and before the collision. A perfectly elastic collision has a c of 1. For a ball bouncing off the floor (or a racquet on the floor), c can be shown to be $c = (h/H)^{1/2}$ where h is the height to which the ball bounces and H is the height from which the ball is dropped. Determine c for the cases in Part 1 and for the case of a tennis ball bouncing off a concrete or wooden floor ($c = 0.85$ for new tennis balls used on a tennis court).

Example:**Calculating Final Velocity and Energy Release: Two Carts Collide**

In the collision pictured in [\[link\]](#), two carts collide inelastically. Cart 1 (denoted m_1) carries a spring which is initially compressed. During the collision, the spring releases its potential energy and converts it to internal kinetic energy. The mass of cart 1 and the spring is 0.350 kg, and the cart and the spring together have an initial velocity of 2.00 m/s. Cart 2 (denoted m_2 in [\[link\]](#)) has a mass of 0.500 kg and an initial velocity of -0.500 m/s. After the collision, cart 1 is observed to recoil with a velocity of -4.00 m/s. (a) What is the final velocity of cart 2? (b) How much energy was released by the spring (assuming all of it was converted into internal kinetic energy)?

Strategy

We can use conservation of momentum to find the final velocity of cart 2, because $F_{\text{net}} = 0$ (the track is frictionless and the force of the spring is internal). Once this velocity is determined, we can compare the internal kinetic energy before and after the collision to see how much energy was released by the spring.

Solution for (a)

As before, the equation for conservation of momentum in a two-object system is

Equation:

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2.$$

The only unknown in this equation is v'_2 . Solving for v'_2 and substituting known values into the previous equation yields

Equation:

$$\begin{aligned} v'_2 &= \frac{m_1 v_1 + m_2 v_2 - m_1 v'_1}{m_2} \\ &= \frac{(0.350 \text{ kg})(2.00 \text{ m/s}) + (0.500 \text{ kg})(-0.500 \text{ m/s})}{0.500 \text{ kg}} - \frac{(0.350 \text{ kg})(-4.00 \text{ m/s})}{0.500 \text{ kg}} \\ &= 3.70 \text{ m/s.} \end{aligned}$$

Solution for (b)

The internal kinetic energy before the collision is

Equation:

$$\begin{aligned}\text{KE}_{\text{int}} &= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \\ &= \frac{1}{2}(0.350 \text{ kg})(2.00 \text{ m/s})^2 + \frac{1}{2}(0.500 \text{ kg})(-0.500 \text{ m/s})^2 \\ &= 0.763 \text{ J.}\end{aligned}$$

After the collision, the internal kinetic energy is

Equation:

$$\begin{aligned}\text{KE}'_{\text{int}} &= \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2 \\ &= \frac{1}{2}(0.350 \text{ kg})(-4.00 \text{ m/s})^2 + \frac{1}{2}(0.500 \text{ kg})(3.70 \text{ m/s})^2 \\ &= 6.22 \text{ J.}\end{aligned}$$

The change in internal kinetic energy is thus

Equation:

$$\begin{aligned}\text{KE}'_{\text{int}} - \text{KE}_{\text{int}} &= 6.22 \text{ J} - 0.763 \text{ J} \\ &= 5.46 \text{ J.}\end{aligned}$$

Discussion

The final velocity of cart 2 is large and positive, meaning that it is moving to the right after the collision. The internal kinetic energy in this collision increases by 5.46 J. That energy was released by the spring.

Test Prep for AP Courses

Exercise:

Problem:

Mass A (2.0 kg) has an initial velocity of 4 m/s in the $+x$ -direction. Mass B (2.0 kg) has an initial velocity of 5 m/s in the $-x$ -direction. If the two masses have an elastic collision, what will be the final velocities of the masses after the collision?

- a. Both will move 0.5 m/s in the $-x$ -direction.
 - b. Mass A will stop; mass B will move 9 m/s in the $+x$ -direction.
 - c. Mass B will stop; mass A will move 9 m/s in the $-x$ -direction.
 - d. Mass A will move 5 m/s in the $-x$ -direction; mass B will move 4 m/s in the $+x$ -direction.
-

Solution:

(d)

Exercise:**Problem:**

Mass A has an initial velocity of 22 m/s in the $+x$ -direction. Mass B is three times more massive than mass A and has an initial velocity of 22 m/s in the $-x$ -direction. If the two masses have an elastic collision, what will be the final velocities of the masses after the collision?

Exercise:**Problem:**

Mass A (2.0 kg) is moving with an initial velocity of 15 m/s in the $+x$ -direction, and it collides with mass B (5.0 kg), initially at rest. After the collision, the two objects stick together and move as one. What is the change in kinetic energy of the system as a result of the collision?

- a. no change
- b. decrease by 225 J
- c. decrease by 161 J
- d. decrease by 64 J

Solution:

(c). Because of conservation of momentum, the final velocity of the combined mass must be 4.286 m/s. The initial kinetic energy is $(0.5)(2.0)(15)^2 = 225 \text{ J}$. The final kinetic energy is $(0.5)(7.0)(4.286)^2 = 64 \text{ J}$, so the difference is -161 J .

Exercise:**Problem:**

Mass A (2.0 kg) is moving with an initial velocity of 15 m/s in the +x-direction, and it collides with mass B (4.0 kg), initially moving 7.0 m/s in the +x-direction. After the collision, the two objects stick together and move as one. What is the change in kinetic energy of the system as a result of the collision?

Exercise:**Problem:**

Mass A slides across a rough table with an initial velocity of 12 m/s in the +x-direction. By the time mass A collides with mass B (a stationary object with equal mass), mass A has slowed to 10 m/s. After the collision, the two objects stick together and move as one. Immediately after the collision, the velocity of the system is measured to be 5 m/s in the +x-direction, and the system eventually slides to a stop. Which of the following statements is true about this motion?

- a. Momentum is conserved during the collision, but it is not conserved during the motion before and after the collision.
- b. Momentum is not conserved at any time during this analysis.
- c. Momentum is conserved at all times during this analysis.
- d. Momentum is not conserved during the collision, but it is conserved during the motion before and after the collision.

Solution:

(a)

Exercise:

Problem:

Mass A is initially moving with a velocity of 12 m/s in the $+x$ -direction. Mass B is twice as massive as mass A and is initially at rest. After the two objects collide, the two masses move together as one with a velocity of 4 m/s in the $+x$ -direction. Is momentum conserved in this collision?

Exercise:

Problem:

Mass A is initially moving with a velocity of 24 m/s in the $+x$ -direction. Mass B is twice as massive as mass A and is initially at rest. The two objects experience a totally inelastic collision. What is the final speed of both objects after the collision?

- a. A is not moving; B is moving 24 m/s in the $+x$ -direction.
- b. Neither A nor B is moving.
- c. A is moving 24 m/s in the $-x$ -direction. B is not moving.
- d. Both A and B are moving together 8 m/s in the $+x$ -direction.

Solution:

(d)

Exercise:

Problem:

Mass A is initially moving with some unknown velocity in the $+x$ -direction. Mass B is twice as massive as mass A and initially at rest. The two objects collide, and after the collision, they move together with a speed of 6 m/s in the $+x$ -direction. (a) Is this collision elastic or inelastic? Explain. (b) Determine the initial velocity of mass A.

Exercise:

Problem:

Mass A is initially moving with a velocity of 2 m/s in the $+x$ -direction. Mass B is initially moving with a velocity of 6 m/s in the $-x$ -direction. The two objects have equal masses. After they collide, mass A moves with a speed of 4 m/s in the $-x$ -direction. What is the final velocity of mass B after the collision?

- a. 6 m/s in the $+x$ -direction
 - b. 4 m/s in the $+x$ -direction
 - c. zero
 - d. 4 m/s in the $-x$ -direction
-

Solution:

(c)

Exercise:**Problem:**

Mass A is initially moving with a velocity of 15 m/s in the $+x$ -direction. Mass B is twice as massive and is initially moving with a velocity of 10 m/s in the $-x$ -direction. The two objects collide, and after the collision, mass A moves with a speed of 15 m/s in the $-x$ -direction. (a) What is the final velocity of mass B after the collision? (b) Calculate the change in kinetic energy as a result of the collision, assuming mass A is 5.0 kg.

Section Summary

- An inelastic collision is one in which the internal kinetic energy changes (it is not conserved).
- A collision in which the objects stick together is sometimes called perfectly inelastic because it reduces internal kinetic energy more than does any other type of inelastic collision.

- Sports science and technologies also use physics concepts such as momentum and rotational motion and vibrations.

Conceptual Questions

Exercise:

Problem:

What is an inelastic collision? What is a perfectly inelastic collision?

Exercise:

Problem:

Mixed-pair ice skaters performing in a show are standing motionless at arms length just before starting a routine. They reach out, clasp hands, and pull themselves together by only using their arms. Assuming there is no friction between the blades of their skates and the ice, what is their velocity after their bodies meet?

Exercise:

Problem:

A small pickup truck that has a camper shell slowly coasts toward a red light with negligible friction. Two dogs in the back of the truck are moving and making various inelastic collisions with each other and the walls. What is the effect of the dogs on the motion of the center of mass of the system (truck plus entire load)? What is their effect on the motion of the truck?

Problems & Exercises

Exercise:

Problem:

A 0.240-kg billiard ball that is moving at 3.00 m/s strikes the bumper of a pool table and bounces straight back at 2.40 m/s (80% of its original speed). The collision lasts 0.0150 s. (a) Calculate the average force exerted on the ball by the bumper. (b) How much kinetic energy in joules is lost during the collision? (c) What percent of the original energy is left?

Solution:

(a) 86.4 N perpendicularly away from the bumper

(b) 0.389 J

(c) 64.0%

Exercise:**Problem:**

During an ice show, a 60.0-kg skater leaps into the air and is caught by an initially stationary 75.0-kg skater. (a) What is their final velocity assuming negligible friction and that the 60.0-kg skater's original horizontal velocity is 4.00 m/s? (b) How much kinetic energy is lost?

Exercise:**Problem: Professional Application**

Using mass and speed data from [\[link\]](#) and assuming that the football player catches the ball with his feet off the ground with both of them moving horizontally, calculate: (a) the final velocity if the ball and player are going in the same direction and (b) the loss of kinetic energy in this case. (c) Repeat parts (a) and (b) for the situation in which the ball and the player are going in opposite directions. Might the loss of kinetic energy be related to how much it hurts to catch the pass?

Solution:

(a) 8.06 m/s

(b) -56.0 J

(c)(i) 7.88 m/s; (ii) -223 J

Exercise:

Problem:

A battleship that is 6.00×10^7 kg and is originally at rest fires a 1100-kg artillery shell horizontally with a velocity of 575 m/s. (a) If the shell is fired straight aft (toward the rear of the ship), there will be negligible friction opposing the ship's recoil. Calculate its recoil velocity. (b) Calculate the increase in internal kinetic energy (that is, for the ship and the shell). This energy is less than the energy released by the gun powder—significant heat transfer occurs.

Exercise:

Problem: Professional Application

Two manned satellites approaching one another, at a relative speed of 0.250 m/s, intending to dock. The first has a mass of 4.00×10^3 kg, and the second a mass of 7.50×10^3 kg. (a) Calculate the final velocity (after docking) by using the frame of reference in which the first satellite was originally at rest. (b) What is the loss of kinetic energy in this inelastic collision? (c) Repeat both parts by using the frame of reference in which the second satellite was originally at rest. Explain why the change in velocity is different in the two frames, whereas the change in kinetic energy is the same in both.

Solution:

(a) 0.163 m/s in the direction of motion of the more massive satellite

(b) 81.6 J

(c) 8.70×10^{-2} m/s in the direction of motion of the less massive satellite, 81.5 J. Because there are no external forces, the velocity of the center of mass of the two-satellite system is unchanged by the collision. The two velocities calculated above are the velocity of the center of mass in each of the two different individual reference frames. The loss in KE is the same in both reference frames because the KE lost to internal forces (heat, friction, etc.) is the same regardless of the coordinate system chosen.

Exercise:

Problem: Professional Application

A 30,000-kg freight car is coasting at 0.850 m/s with negligible friction under a hopper that dumps 110,000 kg of scrap metal into it. (a) What is the final velocity of the loaded freight car? (b) How much kinetic energy is lost?

Exercise:

Problem: Professional Application

Space probes may be separated from their launchers by exploding bolts. (They bolt away from one another.) Suppose a 4800-kg satellite uses this method to separate from the 1500-kg remains of its launcher, and that 5000 J of kinetic energy is supplied to the two parts. What are their subsequent velocities using the frame of reference in which they were at rest before separation?

Solution:

0.704 m/s

−2.25 m/s

Exercise:

Problem:

A 0.0250-kg bullet is accelerated from rest to a speed of 550 m/s in a 3.00-kg rifle. The pain of the rifle's kick is much worse if you hold the gun loosely a few centimeters from your shoulder rather than holding it tightly against your shoulder. (a) Calculate the recoil velocity of the rifle if it is held loosely away from the shoulder. (b) How much kinetic energy does the rifle gain? (c) What is the recoil velocity if the rifle is held tightly against the shoulder, making the effective mass 28.0 kg? (d) How much kinetic energy is transferred to the rifle-shoulder combination? The pain is related to the amount of kinetic energy, which is significantly less in this latter situation. (e) Calculate the momentum of a 110-kg football player running at 8.00 m/s. Compare the player's momentum with the momentum of a hard-thrown 0.410-kg football that has a speed of 25.0 m/s. Discuss its relationship to this problem.

Solution:

(a) 4.58 m/s away from the bullet

(b) 31.5 J

(c) -0.491 m/s

(d) 3.38 J

Exercise:**Problem: Professional Application**

One of the waste products of a nuclear reactor is plutonium-239 (^{239}Pu). This nucleus is radioactive and decays by splitting into a helium-4 nucleus and a uranium-235 nucleus ($^4\text{He} + ^{235}\text{U}$), the latter of which is also radioactive and will itself decay some time later. The energy emitted in the plutonium decay is 8.40×10^{-13} J and is entirely converted to kinetic energy of the helium and uranium nuclei.

The mass of the helium nucleus is 6.68×10^{-27} kg, while that of the uranium is 3.92×10^{-25} kg (note that the ratio of the masses is 4 to 235). (a) Calculate the velocities of the two nuclei, assuming the plutonium nucleus is originally at rest. (b) How much kinetic energy does each nucleus carry away? Note that the data given here are accurate to three digits only.

Exercise:

Problem: Professional Application

The Moon's craters are remnants of meteorite collisions. Suppose a fairly large asteroid that has a mass of 5.00×10^{12} kg (about a kilometer across) strikes the Moon at a speed of 15.0 km/s. (a) At what speed does the Moon recoil after the perfectly inelastic collision (the mass of the Moon is 7.36×10^{22} kg)? (b) How much kinetic energy is lost in the collision? Such an event may have been observed by medieval English monks who reported observing a red glow and subsequent haze about the Moon. (c) In October 2009, NASA crashed a rocket into the Moon, and analyzed the plume produced by the impact. (Significant amounts of water were detected.) Answer part (a) and (b) for this real-life experiment. The mass of the rocket was 2000 kg and its speed upon impact was 9000 km/h. How does the plume produced alter these results?

Solution:

(a) 1.02×10^{-6} m/s

(b) 5.63×10^{20} J (almost all KE lost)

(c) Recoil speed is 6.79×10^{-17} m/s, energy lost is 6.25×10^9 J. The plume will not affect the momentum result because the plume is still part of the Moon system. The plume may affect the kinetic energy result because a significant part of the initial kinetic energy may be transferred to the kinetic energy of the plume particles.

Exercise:

Problem: Professional Application

Two football players collide head-on in midair while trying to catch a thrown football. The first player is 95.0 kg and has an initial velocity of 6.00 m/s, while the second player is 115 kg and has an initial velocity of -3.50 m/s. What is their velocity just after impact if they cling together?

Exercise:**Problem:**

What is the speed of a garbage truck that is 1.20×10^4 kg and is initially moving at 25.0 m/s just after it hits and adheres to a trash can that is 80.0 kg and is initially at rest?

Solution:

24.8 m/s

Exercise:**Problem:**

During a circus act, an elderly performer thrills the crowd by catching a cannon ball shot at him. The cannon ball has a mass of 10.0 kg and the horizontal component of its velocity is 8.00 m/s when the 65.0-kg performer catches it. If the performer is on nearly frictionless roller skates, what is his recoil velocity?

Exercise:**Problem:**

(a) During an ice skating performance, an initially motionless 80.0-kg clown throws a fake barbell away. The clown's ice skates allow her to recoil frictionlessly. If the clown recoils with a velocity of 0.500 m/s and the barbell is thrown with a velocity of 10.0 m/s, what is the mass of the barbell? (b) How much kinetic energy is gained by this maneuver? (c) Where does the kinetic energy come from?

Solution:

(a) 4.00 kg

(b) 210 J

(c) The clown does work to throw the barbell, so the kinetic energy comes from the muscles of the clown. The muscles convert the chemical potential energy of ATP into kinetic energy.

Glossary

inelastic collision

a collision in which internal kinetic energy is not conserved

perfectly inelastic collision

a collision in which the colliding objects stick together

Collisions of Point Masses in Two Dimensions

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Discuss two-dimensional collisions as an extension of one-dimensional analysis.
- Define point masses.
- Derive an expression for conservation of momentum along the x -axis and y -axis.
- Describe elastic collisions of two objects with equal mass.
- Determine the magnitude and direction of the final velocity given initial velocity and scattering angle.

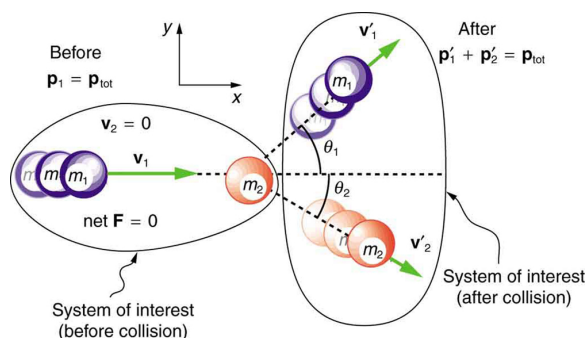
The information presented in this section supports the following AP® learning objectives and science practices:

- **5.D.1.2** The student is able to apply the principles of conservation of momentum and restoration of kinetic energy to reconcile a situation that appears to be isolated and elastic, but in which data indicate that linear momentum and kinetic energy are not the same after the interaction, by refining a scientific question to identify interactions that have not been considered. Students will be expected to solve qualitatively and/or quantitatively for one-dimensional situations and only qualitatively in two-dimensional situations.
- **5.D.3.3** The student is able to make predictions about the velocity of the center of mass for interactions within a defined two-dimensional system.

In the previous two sections, we considered only one-dimensional collisions; during such collisions, the incoming and outgoing velocities are all along the same line. But what about collisions, such as those between billiard balls, in which objects scatter to the side? These are two-dimensional collisions, and we shall see that their study is an extension of the one-dimensional analysis already presented. The approach taken (similar to the approach in discussing two-dimensional kinematics and dynamics) is to choose a convenient coordinate system and resolve the motion into components along perpendicular axes. Resolving the motion yields a pair of one-dimensional problems to be solved simultaneously.

One complication arising in two-dimensional collisions is that the objects might rotate before or after their collision. For example, if two ice skaters hook arms as they pass by one another, they will spin in circles. We will not consider such rotation until later, and so for now we arrange things so that no rotation is possible. To avoid rotation, we consider only the scattering of **point masses**—that is, structureless particles that cannot rotate or spin.

We start by assuming that $\mathbf{F}_{\text{net}} = 0$, so that momentum \mathbf{p} is conserved. The simplest collision is one in which one of the particles is initially at rest. (See [\[link\]](#).) The best choice for a coordinate system is one with an axis parallel to the velocity of the incoming particle, as shown in [\[link\]](#). Because momentum is conserved, the components of momentum along the x - and y -axes (p_x and p_y) will also be conserved, but with the chosen coordinate system, p_y is initially zero and p_x is the momentum of the incoming particle. Both facts simplify the analysis. (Even with the simplifying assumptions of point masses, one particle initially at rest, and a convenient coordinate system, we still gain new insights into nature from the analysis of two-dimensional collisions.)



A two-dimensional collision with the coordinate system chosen so that m_2 is initially at rest and v_1 is parallel to the x -axis. This coordinate system is sometimes called the laboratory coordinate system, because many scattering experiments have a target that is stationary in the laboratory, while particles are scattered from it to determine the particles that make-up the target and how they are bound together. The particles may not be observed directly, but their initial and final velocities are.

Along the x -axis, the equation for conservation of momentum is
Equation:

$$p_{1x} + p_{2x} = p'_{1x} + p'_{2x}.$$

Where the subscripts denote the particles and axes and the primes denote the situation after the collision. In terms of masses and velocities, this equation is

Equation:

$$m_1 v_{1x} + m_2 v_{2x} = m_1 v'_{1x} + m_2 v'_{2x}.$$

But because particle 2 is initially at rest, this equation becomes

Equation:

$$m_1 v_{1x} = m_1 v'_{1x} + m_2 v'_{2x}.$$

The components of the velocities along the x -axis have the form $v \cos \theta$. Because particle 1 initially moves along the x -axis, we find $v_{1x} = v_1$.

Conservation of momentum along the x -axis gives the following equation:

Equation:

$$m_1 v_1 = m_1 v'_1 \cos \theta_1 + m_2 v'_2 \cos \theta_2,$$

where θ_1 and θ_2 are as shown in [\[link\]](#).

Note:

Conservation of Momentum along the x -axis

Equation:

$$m_1 v_1 = m_1 v'_1 \cos \theta_1 + m_2 v'_2 \cos \theta_2$$

Along the y -axis, the equation for conservation of momentum is

Equation:

$$p_{1y} + p_{2y} = p'_{1y} + p'_{2y}$$

or

Equation:

$$m_1 v_{1y} + m_2 v_{2y} = m_1 v'_{1y} + m_2 v'_{2y}.$$

But v_{1y} is zero, because particle 1 initially moves along the x -axis. Because particle 2 is initially at rest, v_{2y} is also zero. The equation for conservation of momentum along the y -axis becomes

Equation:

$$0 = m_1 v'_{1y} + m_2 v'_{2y}.$$

The components of the velocities along the y -axis have the form $v \sin \theta$.

Thus, conservation of momentum along the y -axis gives the following equation:

Equation:

$$0 = m_1 v'_1 \sin \theta_1 + m_2 v'_2 \sin \theta_2.$$

Note:

Conservation of Momentum along the y -axis

Equation:

$$0 = m_1 v'_1 \sin \theta_1 + m_2 v'_2 \sin \theta_2$$

The equations of conservation of momentum along the x -axis and y -axis are very useful in analyzing two-dimensional collisions of particles, where one is originally stationary (a common laboratory situation). But two equations can only be used to find two unknowns, and so other data may be necessary when collision experiments are used to explore nature at the subatomic level.

Note:

Making Connections: Real World Connections

We have seen, in one-dimensional collisions when momentum is conserved, that the center-of-mass velocity of the system remains unchanged as a result of the collision. If you calculate the momentum and center-of-mass

velocity before the collision, you will get the same answer as if you calculate both quantities after the collision. This logic also works for two-dimensional collisions.

For example, consider two cars of equal mass. Car A is driving east (+ x -direction) with a speed of 40 m/s. Car B is driving north (+ y -direction) with a speed of 80 m/s. What is the velocity of the center-of-mass of this system before and after an inelastic collision, in which the cars move together as one mass after the collision? Since both cars have equal mass, the center-of-mass velocity components are just the average of the components of the individual velocities before the collision. The x -component of the center of mass velocity is 20 m/s, and the y -component is 40 m/s.

Using momentum conservation for the collision in both the x -component and y -component yields similar answers:

Equation:

$$m(40) + m(0) = (2m)v_{\text{final}}(x)$$

Equation:

$$v_{\text{final}}(x) = 20 \text{ m/s}$$

Equation:

$$m(0) + m(80) = (2m)v_{\text{final}}(y)$$

Equation:

$$v_{\text{final}}(y) = 40 \text{ m/s}$$

Since the two masses move together after the collision, the velocity of this combined object is equal to the center-of-mass velocity. Thus, the center-of-mass velocity before and after the collision is identical, even in two-dimensional collisions, when momentum is conserved.

Example:

Determining the Final Velocity of an Unseen Object from the Scattering of Another Object

Suppose the following experiment is performed. A 0.250-kg object (m_1) is slid on a frictionless surface into a dark room, where it strikes an initially stationary object with mass of 0.400 kg (m_2). The 0.250-kg object emerges from the room at an angle of 45.0° with its incoming direction.

The speed of the 0.250-kg object is originally 2.00 m/s and is 1.50 m/s after the collision. Calculate the magnitude and direction of the velocity (v_2 and θ_2) of the 0.400-kg object after the collision.

Strategy

Momentum is conserved because the surface is frictionless. The coordinate system shown in [\[link\]](#) is one in which m_2 is originally at rest and the initial velocity is parallel to the x -axis, so that conservation of momentum along the x - and y -axes is applicable.

Everything is known in these equations except v_2 and θ_2 , which are precisely the quantities we wish to find. We can find two unknowns because we have two independent equations: the equations describing the conservation of momentum in the x - and y -directions.

Solution

Solving $m_1 v_1 = m_1 v_1 \cos \theta_1 + m_2 v_2 \cos \theta_2$ for $v_2 \cos \theta_2$ and $0 = m_1 v_1 \sin \theta_1 + m_2 v_2 \sin \theta_2$ for $v_2 \sin \theta_2$ and taking the ratio yields an equation (in which θ_2 is the only unknown quantity. Applying the identity ($\tan \theta = \frac{\sin \theta}{\cos \theta}$), we obtain:

Equation:

$$\tan \theta_2 = \frac{v_1 \sin \theta_1}{v_1 \cos \theta_1 - v_1}$$

Entering known values into the previous equation gives

Equation:

$$\tan \theta_2 = \frac{(1.50 \text{ m/s})(0.7071)}{(1.50 \text{ m/s})(0.7071) - 2.00 \text{ m/s}} = -1.129.$$

Thus,

Equation:

$$\theta_2 = \tan^{-1}(-1.129) = 311.5^\circ \approx 312^\circ.$$

Angles are defined as positive in the counter clockwise direction, so this angle indicates that m_2 is scattered to the right in [\[link\]](#), as expected (this angle is in the fourth quadrant). Either equation for the x - or y -axis can now be used to solve for v'_2 , but the latter equation is easiest because it has fewer terms.

Equation:

$$v'_2 = -\frac{m_1}{m_2} v_1 \frac{\sin \theta_1}{\sin \theta_2}$$

Entering known values into this equation gives

Equation:

$$v'_2 = -\left(\frac{0.250 \text{ kg}}{0.400 \text{ kg}}\right)(1.50 \text{ m/s})\left(\frac{0.7071}{-0.7485}\right).$$

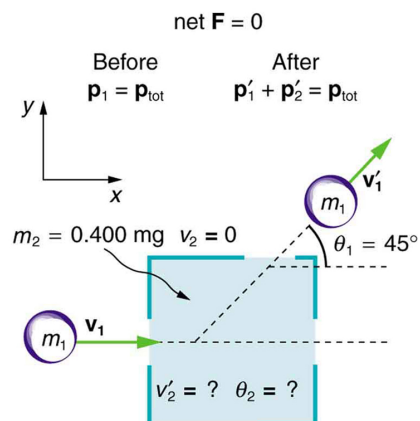
Thus,

Equation:

$$v'_2 = 0.886 \text{ m/s}.$$

Discussion

It is instructive to calculate the internal kinetic energy of this two-object system before and after the collision. (This calculation is left as an end-of-chapter problem.) If you do this calculation, you will find that the internal kinetic energy is less after the collision, and so the collision is inelastic. This type of result makes a physicist want to explore the system further.



A collision taking place in a dark room is explored in [\[link\]](#).

The incoming object m_1 is scattered by an initially

stationary object. Only the stationary object's mass m_2 is known. By measuring the angle and speed at which m_1 emerges from the room, it is possible to calculate the magnitude and direction of the initially stationary object's velocity after the collision.

Elastic Collisions of Two Objects with Equal Mass

Some interesting situations arise when the two colliding objects have equal mass and the collision is elastic. This situation is nearly the case with colliding billiard balls, and precisely the case with some subatomic particle collisions. We can thus get a mental image of a collision of subatomic particles by thinking about billiards (or pool). (Refer to [\[link\]](#) for masses and angles.) First, an elastic collision conserves internal kinetic energy. Again, let us assume object 2 (m_2) is initially at rest. Then, the internal kinetic energy before and after the collision of two objects that have equal masses is

Equation:

$$\frac{1}{2}mv_1^2 = \frac{1}{2}mv_1'^2 + \frac{1}{2}mv_2'^2.$$

Because the masses are equal, $m_1 = m_2 = m$. Algebraic manipulation (left to the reader) of conservation of momentum in the x - and y -directions can show that

Equation:

$$\frac{1}{2}mv_1^2 = \frac{1}{2}mv_1'^2 + \frac{1}{2}mv_2'^2 + mv_1v_2' \cos(\theta_1 - \theta_2).$$

(Remember that θ_2 is negative here.) The two preceding equations can both be true only if

Equation:

$$mv_1v_2' \cos(\theta_1 - \theta_2) = 0.$$

There are three ways that this term can be zero. They are

- $v_1 = 0$: head-on collision; incoming ball stops
- $v_2' = 0$: no collision; incoming ball continues unaffected
- $\cos(\theta_1 - \theta_2) = 0$: angle of separation ($\theta_1 - \theta_2$) is 90° after the collision

All three of these ways are familiar occurrences in billiards and pool, although most of us try to avoid the second. If you play enough pool, you will notice that the angle between the balls is very close to 90° after the collision, although it will vary from this value if a great deal of spin is placed on the ball. (Large spin carries in extra energy and a quantity called *angular momentum*, which must also be conserved.) The assumption that the scattering of billiard balls is elastic is reasonable based on the correctness of the three results it produces. This assumption also implies that, to a good approximation, momentum is conserved for the two-ball system in billiards and pool. The problems below explore these and other characteristics of two-dimensional collisions.

Note:**Connections to Nuclear and Particle Physics**

Two-dimensional collision experiments have revealed much of what we know about subatomic particles, as we shall see in [Medical Applications of Nuclear Physics](#) and [Particle Physics](#). Ernest Rutherford, for example, discovered the nature of the atomic nucleus from such experiments.

Test Prep for AP Courses**Exercise:****Problem:**

Two cars of equal mass approach an intersection. Car A is moving east at a speed of 45 m/s. Car B is moving south at a speed of 35 m/s. They collide inelastically and stick together after the collision, moving as one object. Which of the following statements is true about the center-of-mass velocity of this system?

- The center-of-mass velocity will decrease after the collision as a result of lost energy (but not drop to zero).
- The center-of-mass velocity will remain the same after the collision since momentum is conserved.
- The center-of-mass velocity will drop to zero since the two objects stick together.
- The magnitude of the center-of-mass velocity will remain the same, but the direction of the velocity will change.

Solution:

(b)

Exercise:**Problem:**

Car A has a mass of 2000 kg and approaches an intersection with a velocity of 38 m/s directed to the east. Car B has a mass of 3500 kg and approaches the intersection with a velocity of 53 m/s directed 63° north of east. The two cars collide and stick together after the collision. Will the center-of-mass velocity change as a result of the collision? Explain why or why not. Calculate the center-of-mass velocity before and after the collision.

Section Summary

- The approach to two-dimensional collisions is to choose a convenient coordinate system and break the motion into components along perpendicular axes. Choose a coordinate system with the x -axis parallel to the velocity of the incoming particle.
- Two-dimensional collisions of point masses where mass 2 is initially at rest conserve momentum along the initial direction of mass 1 (the x -axis), stated by $m_1 v_1 = m_1 v'_1 \cos \theta_1 + m_2 v'_2 \cos \theta_2$ and along the direction perpendicular to the initial direction (the y -axis) stated by $0 = m_1 v'_1 \sin \theta_1 + m_2 v'_2 \sin \theta_2$.
- The internal kinetic before and after the collision of two objects that have equal masses is

Equation:

$$\frac{1}{2} m v_1^2 = \frac{1}{2} m v_1'^2 + \frac{1}{2} m v_2'^2 + m v_1 v_2' \cos(\theta_1 - \theta_2).$$

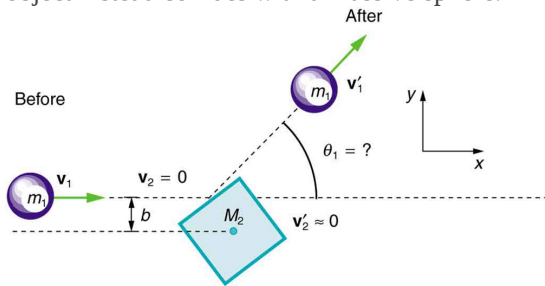
- Point masses are structureless particles that cannot spin.

Conceptual Questions

Exercise:

Problem:

[\[link\]](#) shows a cube at rest and a small object heading toward it. (a) Describe the directions (angle θ_1) at which the small object can emerge after colliding elastically with the cube. How does θ_1 depend on b , the so-called impact parameter? Ignore any effects that might be due to rotation after the collision, and assume that the cube is much more massive than the small object. (b) Answer the same questions if the small object instead collides with a massive sphere.



A small object approaches a collision with a much more massive cube, after which its velocity has the direction θ_1 . The angles at which the small object can be scattered are determined by the shape of the object it strikes and the impact parameter b .

Problems & Exercises

Exercise:

Problem:

Two identical pucks collide on an air hockey table. One puck was originally at rest. (a) If the incoming puck has a speed of 6.00 m/s and scatters to an angle of 30.0° , what is the velocity (magnitude and direction) of the second puck? (You may use the result that $\theta_1 - \theta_2 = 90^\circ$ for elastic collisions of objects that have identical masses.) (b) Confirm that the collision is elastic.

Solution:

(a) 3.00 m/s, 60° below x -axis

(b) Find speed of first puck after collision:

$$0 = mv_1 \sin 30^\circ - mv_2 \sin 60^\circ \Rightarrow v_1 = v_2 \frac{\sin 60^\circ}{\sin 30^\circ} = 5.196 \text{ m/s}$$

$$\left. \begin{array}{l} \text{Verify that ratio of initial to final KE equals one:} \\ \text{KE} = \frac{1}{2}mv_1^2 = 18m \text{ J} \\ \text{KE} = \frac{1}{2}mv_1'^2 + \frac{1}{2}mv_2'^2 = 18m \text{ J} \end{array} \right\} \frac{\text{KE}}{\text{KE}'} = 1.00$$

Exercise:

Problem:

Confirm that the results of the example [\[link\]](#) do conserve momentum in both the x - and y -directions.

Exercise:**Problem:**

A 3000-kg cannon is mounted so that it can recoil only in the horizontal direction. (a) Calculate its recoil velocity when it fires a 15.0-kg shell at 480 m/s at an angle of 20.0° above the horizontal. (b) What is the kinetic energy of the cannon? This energy is dissipated as heat transfer in shock absorbers that stop its recoil. (c) What happens to the vertical component of momentum that is imparted to the cannon when it is fired?

Solution:

(a) -2.26 m/s

(b) $7.63 \times 10^3 \text{ J}$

(c) The ground will exert a normal force to oppose recoil of the cannon in the vertical direction. The momentum in the vertical direction is transferred to the earth. The energy is transferred into the ground, making a dent where the cannon is. After long barrages, cannon have erratic aim because the ground is full of divots.

Exercise:**Problem: Professional Application**

A 5.50-kg bowling ball moving at 9.00 m/s collides with a 0.850-kg bowling pin, which is scattered at an angle of 85.0° to the initial direction of the bowling ball and with a speed of 15.0 m/s. (a) Calculate the final velocity (magnitude and direction) of the bowling ball. (b) Is the collision elastic? (c) Linear kinetic energy is greater after the collision. Discuss how spin on the ball might be converted to linear kinetic energy in the collision.

Exercise:**Problem: Professional Application**

Ernest Rutherford (the first New Zealander to be awarded the Nobel Prize in Chemistry) demonstrated that nuclei were very small and dense by scattering helium-4 nuclei (^4He) from gold-197 nuclei (^{197}Au). The energy of the incoming helium nucleus was $8.00 \times 10^{-13} \text{ J}$, and the masses of the helium and gold nuclei were $6.68 \times 10^{-27} \text{ kg}$ and $3.29 \times 10^{-25} \text{ kg}$, respectively (note that their mass ratio is 4 to 197). (a) If a helium nucleus scatters to an angle of 120° during an elastic collision with a gold nucleus, calculate the helium nucleus's final speed and the final velocity (magnitude and direction) of the gold nucleus. (b) What is the final kinetic energy of the helium nucleus?

Solution:

(a) $5.36 \times 10^5 \text{ m/s}$ at -29.5°

(b) $7.52 \times 10^{-13} \text{ J}$

Exercise:**Problem: Professional Application**

Two cars collide at an icy intersection and stick together afterward. The first car has a mass of 1200 kg and is approaching at 8.00 m/s due south. The second car has a mass of 850 kg and is approaching at 17.0 m/s due west. (a) Calculate the final velocity (magnitude and direction) of the cars. (b) How much kinetic energy is lost in the collision? (This energy goes into deformation of the cars.) Note that because both cars have an initial velocity, you cannot use the equations for conservation of momentum along the x -axis and y -axis; instead, you must look for other simplifying aspects.

Exercise:

Problem:

Starting with equations $m_1 v_1 = m_1 v'_1 \cos \theta_1 + m_2 v'_2 \cos \theta_2$ and $0 = m_1 v'_1 \sin \theta_1 + m_2 v'_2 \sin \theta_2$ for conservation of momentum in the x - and y -directions and assuming that one object is originally stationary, prove that for an elastic collision of two objects of equal masses,

Equation:

$$\frac{1}{2} m v_1^2 = \frac{1}{2} m v_1'^2 + \frac{1}{2} m v_2'^2 + m v'_1 v'_2 \cos (\theta_1 - \theta_2)$$

as discussed in the text.

Solution:

We are given that $m_1 = m_2 \equiv m$. The given equations then become:

$$v_1 = v'_1 \cos \theta_1 + v'_2 \cos \theta_2$$

and

$$0 = v'_1 \sin \theta_1 + v'_2 \sin \theta_2.$$

Square each equation to get

$$v_1^2 = v_1'^2 \cos^2 \theta_1 + v_2'^2 \cos^2 \theta_2 + 2v'_1 v'_2 \cos \theta_1 \cos \theta_2$$

$$0 = v_1'^2 \sin^2 \theta_1 + v_2'^2 \sin^2 \theta_2 + 2v'_1 v'_2 \sin \theta_1 \sin \theta_2.$$

Add these two equations and simplify:

$$\begin{aligned} v_1^2 &= v_1'^2 + v_2'^2 + 2v'_1 v'_2 (\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) \\ &= v_1'^2 + v_2'^2 + 2v'_1 v'_2 \left[\frac{1}{2} \cos (\theta_1 - \theta_2) + \frac{1}{2} \cos (\theta_1 + \theta_2) + \frac{1}{2} \cos (\theta_1 - \theta_2) - \frac{1}{2} \cos (\theta_1 + \theta_2) \right] \\ &= v_1'^2 + v_2'^2 + 2v'_1 v'_2 \cos (\theta_1 - \theta_2). \end{aligned}$$

Multiply the entire equation by $\frac{1}{2}m$ to recover the kinetic energy:

$$\frac{1}{2} m v_1^2 = \frac{1}{2} m v_1'^2 + \frac{1}{2} m v_2'^2 + m v'_1 v'_2 \cos (\theta_1 - \theta_2)$$

Exercise:

Problem: Integrated Concepts

A 90.0-kg ice hockey player hits a 0.150-kg puck, giving the puck a velocity of 45.0 m/s. If both are initially at rest and if the ice is frictionless, how far does the player recoil in the time it takes the puck to reach the goal 15.0 m away?

Glossary

point masses

structureless particles with no rotation or spin

Introduction to Rocket Propulsion

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- State Newton's third law of motion.
- Explain the principle involved in propulsion of rockets and jet engines.
- Derive an expression for the acceleration of the rocket.
- Discuss the factors that affect the rocket's acceleration.
- Describe the function of a space shuttle.

Rockets range in size from fireworks so small that ordinary people use them to immense Saturn Vs that once propelled massive payloads toward the Moon. The propulsion of all rockets, jet engines, deflating balloons, and even squids and octopuses is explained by the same physical principle—Newton's third law of motion. Matter is forcefully ejected from a system, producing an equal and opposite reaction on what remains. Another common example is the recoil of a gun. The gun exerts a force on a bullet to accelerate it and consequently experiences an equal and opposite force, causing the gun's recoil or kick.

Note:

Making Connections: Take-Home Experiment—Propulsion of a Balloon

Hold a balloon and fill it with air. Then, let the balloon go. In which direction does the air come out of the balloon and in which direction does

the balloon get propelled? If you fill the balloon with water and then let the balloon go, does the balloon's direction change? Explain your answer.

[\[link\]](#) shows a rocket accelerating straight up. In part (a), the rocket has a mass m and a velocity v relative to Earth, and hence a momentum mv . In part (b), a time Δt has elapsed in which the rocket has ejected a mass Δm of hot gas at a velocity v_e relative to the rocket. The remainder of the mass ($m - \Delta m$) now has a greater velocity ($v + \Delta v$). The momentum of the entire system (rocket plus expelled gas) has actually decreased because the force of gravity has acted for a time Δt , producing a negative impulse $\Delta p = -mg\Delta t$. (Remember that impulse is the net external force on a system multiplied by the time it acts, and it equals the change in momentum of the system.) So, the center of mass of the system is in free fall but, by rapidly expelling mass, part of the system can accelerate upward. It is a commonly held misconception that the rocket exhaust pushes on the ground. If we consider thrust; that is, the force exerted on the rocket by the exhaust gases, then a rocket's thrust is greater in outer space than in the atmosphere or on the launch pad. In fact, gases are easier to expel into a vacuum.

By calculating the change in momentum for the entire system over Δt , and equating this change to the impulse, the following expression can be shown to be a good approximation for the acceleration of the rocket.

Equation:

$$a = \frac{v_e}{m} \frac{\Delta m}{\Delta t} - g$$

“The rocket” is that part of the system remaining after the gas is ejected, and g is the acceleration due to gravity.

Note:

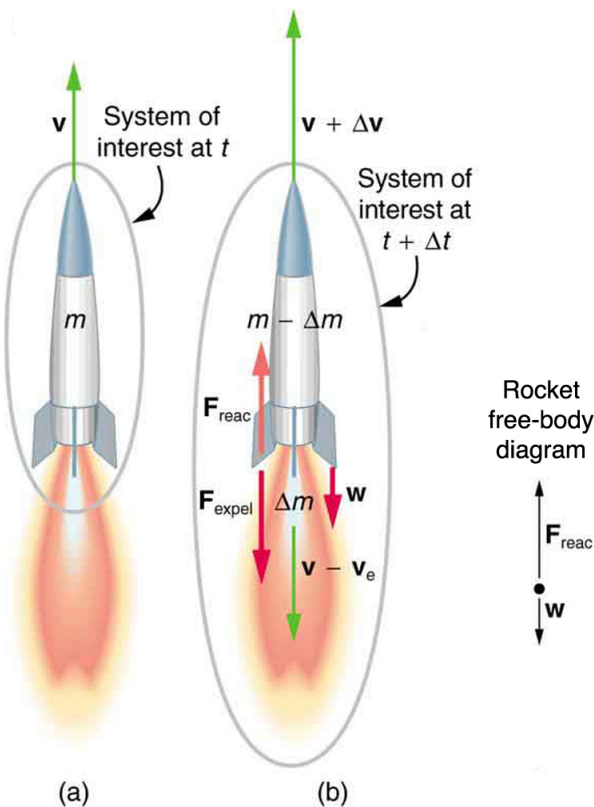
Acceleration of a Rocket

Acceleration of a rocket is

Equation:

$$a = \frac{v_e}{m} \frac{\Delta m}{\Delta t} - g,$$

where a is the acceleration of the rocket, v_e is the escape velocity, m is the mass of the rocket, Δm is the mass of the ejected gas, and Δt is the time in which the gas is ejected.



(a) This rocket has a mass m and an upward velocity v . The net external force on the system is $-mg$, if air resistance is neglected. (b) A time Δt later the system has two main parts, the ejected gas and the

remainder of the rocket. The reaction force on the rocket is what overcomes the gravitational force and accelerates it upward.

A rocket's acceleration depends on three major factors, consistent with the equation for acceleration of a rocket. First, the greater the exhaust velocity of the gases relative to the rocket, v_e , the greater the acceleration is. The practical limit for v_e is about 2.5×10^3 m/s for conventional (non-nuclear) hot-gas propulsion systems. The second factor is the rate at which mass is ejected from the rocket. This is the factor $\Delta m / \Delta t$ in the equation. The quantity $(\Delta m / \Delta t)v_e$, with units of newtons, is called "thrust." The faster the rocket burns its fuel, the greater its thrust, and the greater its acceleration. The third factor is the mass m of the rocket. The smaller the mass is (all other factors being the same), the greater the acceleration. The rocket mass m decreases dramatically during flight because most of the rocket is fuel to begin with, so that acceleration increases continuously, reaching a maximum just before the fuel is exhausted.

Note:**Factors Affecting a Rocket's Acceleration**

- The greater the exhaust velocity v_e of the gases relative to the rocket, the greater the acceleration.
- The faster the rocket burns its fuel, the greater its acceleration.
- The smaller the rocket's mass (all other factors being the same), the greater the acceleration.

Example:

Calculating Acceleration: Initial Acceleration of a Moon Launch

A Saturn V's mass at liftoff was 2.80×10^6 kg, its fuel-burn rate was 1.40×10^4 kg/s, and the exhaust velocity was 2.40×10^3 m/s. Calculate its initial acceleration.

Strategy

This problem is a straightforward application of the expression for acceleration because a is the unknown and all of the terms on the right side of the equation are given.

Solution

Substituting the given values into the equation for acceleration yields

Equation:

$$\begin{aligned} a &= \frac{v_e}{m} \frac{\Delta m}{\Delta t} - g \\ &= \frac{2.40 \times 10^3 \text{ m/s}}{2.80 \times 10^6 \text{ kg}} (1.40 \times 10^4 \text{ kg/s}) - 9.80 \text{ m/s}^2 \\ &= 2.20 \text{ m/s}^2. \end{aligned}$$

Discussion

This value is fairly small, even for an initial acceleration. The acceleration does increase steadily as the rocket burns fuel, because m decreases while v_e and $\frac{\Delta m}{\Delta t}$ remain constant. Knowing this acceleration and the mass of the rocket, you can show that the thrust of the engines was 3.36×10^7 N.

To achieve the high speeds needed to hop continents, obtain orbit, or escape Earth's gravity altogether, the mass of the rocket other than fuel must be as small as possible. It can be shown that, in the absence of air resistance and neglecting gravity, the final velocity of a one-stage rocket initially at rest is

Equation:

$$v = v_e \ln \frac{m_0}{m_r},$$

where $\ln(m_0/m_r)$ is the natural logarithm of the ratio of the initial mass of the rocket (m_0) to what is left (m_r) after all of the fuel is exhausted. (Note

that v is actually the change in velocity, so the equation can be used for any segment of the flight. If we start from rest, the change in velocity equals the final velocity.) For example, let us calculate the mass ratio needed to escape Earth's gravity starting from rest, given that the escape velocity from Earth is about 11.2×10^3 m/s, and assuming an exhaust velocity $v_e = 2.5 \times 10^3$ m/s.

Equation:

$$\ln \frac{m_0}{m_r} = \frac{v}{v_e} = \frac{11.2 \times 10^3 \text{ m/s}}{2.5 \times 10^3 \text{ m/s}} = 4.48$$

Solving for m_0/m_r gives

Equation:

$$\frac{m_0}{m_r} = e^{4.48} = 88.$$

Thus, the mass of the rocket is

Equation:

$$m_r = \frac{m_0}{88}.$$

This result means that only 1/88 of the mass is left when the fuel is burnt, and 87/88 of the initial mass was fuel. Expressed as percentages, 98.9% of the rocket is fuel, while payload, engines, fuel tanks, and other components make up only 1.10%. Taking air resistance and gravitational force into account, the mass m_r remaining can only be about $m_0/180$. It is difficult to build a rocket in which the fuel has a mass 180 times everything else. The solution is multistage rockets. Each stage only needs to achieve part of the final velocity and is discarded after it burns its fuel. The result is that each successive stage can have smaller engines and more payload relative to its fuel. Once out of the atmosphere, the ratio of payload to fuel becomes more favorable, too.

The space shuttle was an attempt at an economical vehicle with some reusable parts, such as the solid fuel boosters and the craft itself. (See [\[link\]](#)) The shuttle's need to be operated by humans, however, made it at least as costly for launching satellites as expendable, unmanned rockets. Ideally, the shuttle would only have been used when human activities were required for the success of a mission, such as the repair of the Hubble space telescope. Rockets with satellites can also be launched from airplanes. Using airplanes has the double advantage that the initial velocity is significantly above zero and a rocket can avoid most of the atmosphere's resistance.



The space shuttle had a number of reusable parts.

Solid fuel boosters on either side were recovered and refueled after each flight, and the entire orbiter returned to Earth for use in subsequent flights. The large liquid fuel tank was expended. The space

shuttle was a complex assemblage of technologies, employing both solid and liquid fuel and pioneering ceramic tiles as reentry heat shields. As a result, it permitted multiple launches as opposed to single-use rockets.
(credit: NASA)

Note:

PhET Explorations: Lunar Lander

Can you avoid the boulder field and land safely, just before your fuel runs out, as Neil Armstrong did in 1969? Our version of this classic video game accurately simulates the real motion of the lunar lander with the correct mass, thrust, fuel consumption rate, and lunar gravity. The real lunar lander is very hard to control.

[Lunar](#)
[Lander](#)
[r](#)

Section Summary

- Newton's third law of motion states that to every action, there is an equal and opposite reaction.

- Acceleration of a rocket is $a = \frac{v_e}{m} \frac{\Delta m}{\Delta t} - g$.
- A rocket's acceleration depends on three main factors. They are
 1. The greater the exhaust velocity of the gases, the greater the acceleration.
 2. The faster the rocket burns its fuel, the greater its acceleration.
 3. The smaller the rocket's mass, the greater the acceleration.

Conceptual Questions

Exercise:

Problem: Professional Application

Suppose a fireworks shell explodes, breaking into three large pieces for which air resistance is negligible. How is the motion of the center of mass affected by the explosion? How would it be affected if the pieces experienced significantly more air resistance than the intact shell?

Exercise:

Problem: Professional Application

During a visit to the International Space Station, an astronaut was positioned motionless in the center of the station, out of reach of any solid object on which he could exert a force. Suggest a method by which he could move himself away from this position, and explain the physics involved.

Exercise:

Problem: Professional Application

It is possible for the velocity of a rocket to be greater than the exhaust velocity of the gases it ejects. When that is the case, the gas velocity

and gas momentum are in the same direction as that of the rocket. How is the rocket still able to obtain thrust by ejecting the gases?

Problems & Exercises

Exercise:

Problem: Professional Application

Antiballistic missiles (ABMs) are designed to have very large accelerations so that they may intercept fast-moving incoming missiles in the short time available. What is the takeoff acceleration of a 10,000-kg ABM that expels 196 kg of gas per second at an exhaust velocity of 2.50×10^3 m/s?

Solution:

$$39.2 \text{ m/s}^2$$

Exercise:

Problem: Professional Application

What is the acceleration of a 5000-kg rocket taking off from the Moon, where the acceleration due to gravity is only 1.6 m/s^2 , if the rocket expels 8.00 kg of gas per second at an exhaust velocity of 2.20×10^3 m/s?

Exercise:

Problem: Professional Application

Calculate the increase in velocity of a 4000-kg space probe that expels 3500 kg of its mass at an exhaust velocity of 2.00×10^3 m/s. You may assume the gravitational force is negligible at the probe's location.

Solution:

$$4.16 \times 10^3 \text{ m/s}$$

Exercise:**Problem: Professional Application**

Ion-propulsion rockets have been proposed for use in space. They employ atomic ionization techniques and nuclear energy sources to produce extremely high exhaust velocities, perhaps as great as $8.00 \times 10^6 \text{ m/s}$. These techniques allow a much more favorable payload-to-fuel ratio. To illustrate this fact: (a) Calculate the increase in velocity of a 20,000-kg space probe that expels only 40.0-kg of its mass at the given exhaust velocity. (b) These engines are usually designed to produce a very small thrust for a very long time—the type of engine that might be useful on a trip to the outer planets, for example. Calculate the acceleration of such an engine if it expels $4.50 \times 10^{-6} \text{ kg/s}$ at the given velocity, assuming the acceleration due to gravity is negligible.

Exercise:

Problem: Derive the equation for the vertical acceleration of a rocket.

Solution:

The force needed to give a small mass Δm an acceleration $a_{\Delta m}$ is $F = \Delta m a_{\Delta m}$. To accelerate this mass in the small time interval Δt at a speed v_e requires $v_e = a_{\Delta m} \Delta t$, so $F = v_e \frac{\Delta m}{\Delta t}$. By Newton's third law, this force is equal in magnitude to the thrust force acting on the rocket, so $F_{\text{thrust}} = v_e \frac{\Delta m}{\Delta t}$, where all quantities are positive. Applying Newton's second law to the rocket gives $F_{\text{thrust}} - mg = ma \Rightarrow a = \frac{v_e}{m} \frac{\Delta m}{\Delta t} - g$, where m is the mass of the rocket and unburnt fuel.

Exercise:

Problem: Professional Application

(a) Calculate the maximum rate at which a rocket can expel gases if its acceleration cannot exceed seven times that of gravity. The mass of the rocket just as it runs out of fuel is 75,000-kg, and its exhaust velocity is 2.40×10^3 m/s. Assume that the acceleration of gravity is the same as on Earth's surface (9.80 m/s^2). (b) Why might it be necessary to limit the acceleration of a rocket?

Exercise:**Problem:**

Given the following data for a fire extinguisher-toy wagon rocket experiment, calculate the average exhaust velocity of the gases expelled from the extinguisher. Starting from rest, the final velocity is 10.0 m/s. The total mass is initially 75.0 kg and is 70.0 kg after the extinguisher is fired.

Exercise:**Problem:**

How much of a single-stage rocket that is 100,000 kg can be anything but fuel if the rocket is to have a final speed of 8.00 km/s, given that it expels gases at an exhaust velocity of 2.20×10^3 m/s?

Solution:

$$2.63 \times 10^3 \text{ kg}$$

Exercise:**Problem: Professional Application**

(a) A 5.00-kg squid initially at rest ejects 0.250-kg of fluid with a velocity of 10.0 m/s. What is the recoil velocity of the squid if the ejection is done in 0.100 s and there is a 5.00-N frictional force

opposing the squid's movement. (b) How much energy is lost to work done against friction?

Solution:

(a) 0.421 m/s away from the ejected fluid.

(b) 0.237 J.

Exercise:

Problem: Unreasonable Results

Squids have been reported to jump from the ocean and travel 30.0 m (measured horizontally) before re-entering the water. (a) Calculate the initial speed of the squid if it leaves the water at an angle of 20.0° , assuming negligible lift from the air and negligible air resistance. (b) The squid propels itself by squirting water. What fraction of its mass would it have to eject in order to achieve the speed found in the previous part? The water is ejected at 12.0 m/s; gravitational force and friction are neglected. (c) What is unreasonable about the results? (d) Which premise is unreasonable, or which premises are inconsistent?

Exercise:

Problem: Construct Your Own Problem

Consider an astronaut in deep space cut free from her space ship and needing to get back to it. The astronaut has a few packages that she can throw away to move herself toward the ship. Construct a problem in which you calculate the time it takes her to get back by throwing all the packages at one time compared to throwing them one at a time. Among the things to be considered are the masses involved, the force she can exert on the packages through some distance, and the distance to the ship.

Exercise:

Problem: Construct Your Own Problem

Consider an artillery projectile striking armor plating. Construct a problem in which you find the force exerted by the projectile on the plate. Among the things to be considered are the mass and speed of the projectile and the distance over which its speed is reduced. Your instructor may also wish for you to consider the relative merits of depleted uranium versus lead projectiles based on the greater density of uranium.

Connection for AP® Courses
class="introduction"

On a short time
scale, rocks like
these in Australia's
Kings Canyon are
static, or motionless
relative to the Earth.

(credit:
freeaussiestock.com
)



What might desks, bridges, buildings, trees, and mountains have in common? What do these objects have in common with a car moving at a constant velocity? While it may be apparent that the objects in the first group are all motionless relative to Earth, they also share something with the moving car and all objects moving at a constant velocity. All of these

objects, stationary and moving, share an acceleration of zero. How can this be? Consider Newton's second law, $F = ma$. When acceleration is zero, as is the case for both stationary objects and objects moving at a constant velocity, the net external force must also be zero (Big Idea 3). Forces are acting on both stationary objects and on objects moving at a constant velocity, but the forces are balanced. That is, they are in *equilibrium*. In equilibrium, the net force is zero.

The first two sections of this chapter will focus on the two conditions necessary for equilibrium. They will not only help you to distinguish between stationary bridges and cars moving at constant velocity, but will introduce a second equilibrium condition, this time involving rotation. As you explore the second equilibrium condition, you will learn about torque, in support of both Enduring Understanding 3.F and Essential Knowledge 3.F.1. Much like a force, torque provides the capability for acceleration; however, with careful attention, torques may also be balanced and equilibrium can be reached.

The remainder of this chapter will discuss a variety of interesting equilibrium applications. From the art of balancing, to simple machines, to the muscles in your body, the world around you relies upon the principles of equilibrium to remain stable. This chapter will help you to see just how closely related these events truly are.

The content in this chapter supports:

Big Idea 3 The interactions of an object with other objects can be described by forces.

Enduring Understanding 3.F A force exerted on an object can cause a torque on that object.

Essential Knowledge 3.F.1 Only the force component perpendicular to the line connecting the axis of rotation and the point of application of the force results in a torque about that axis.

The First Condition for Equilibrium

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- State the first condition of equilibrium.
- Explain static equilibrium.
- Explain dynamic equilibrium.

The first condition necessary to achieve equilibrium is the one already mentioned: the net external force on the system must be zero. Expressed as an equation, this is simply

Equation:

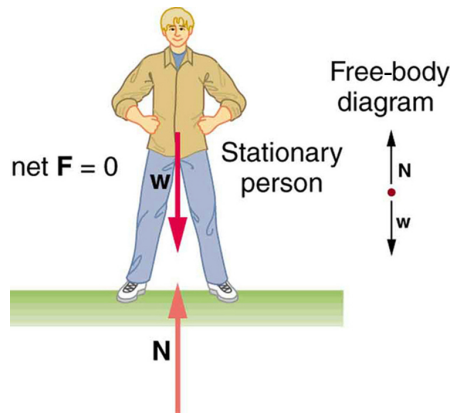
$$\text{net } \mathbf{F} = 0$$

Note that if net F is zero, then the net external force in *any* direction is zero. For example, the net external forces along the typical x- and y-axes are zero. This is written as

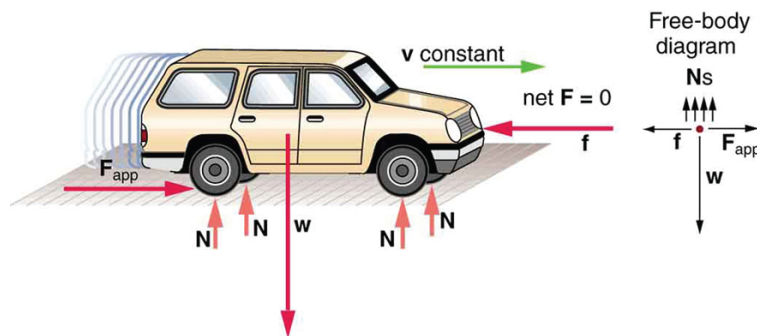
Equation:

$$\text{net } F_x = 0 \text{ and } F_y = 0$$

[\[link\]](#) and [\[link\]](#) illustrate situations where net $F = 0$ for both **static equilibrium** (motionless), and **dynamic equilibrium** (constant velocity).



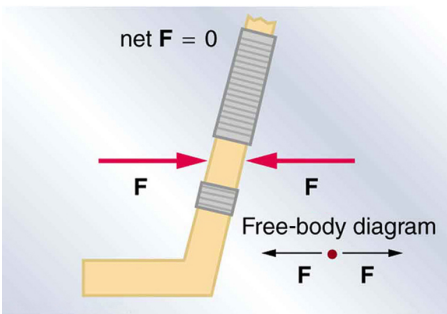
This motionless person is in static equilibrium. The forces acting on him add up to zero. Both forces are vertical in this case.



This car is in dynamic equilibrium because it is moving at constant velocity. There are horizontal and vertical forces, but the net external force in any direction is zero. The applied force F_{app} between the tires and the road is balanced by air friction, and the weight of the car is supported by the normal forces, here shown to be equal for all four tires.

However, it is not sufficient for the net external force of a system to be zero for a system to be in equilibrium. Consider the two situations illustrated in [\[link\]](#) and [\[link\]](#) where forces are applied to an ice hockey stick lying flat on ice. The net external force is zero in both situations shown in the figure; but in one case, equilibrium is achieved, whereas in the other, it is not. In [\[link\]](#), the ice hockey stick remains motionless. But in [\[link\]](#), with the same forces applied in different places, the stick experiences accelerated rotation. Therefore, we know that the point at which a force is applied is another factor in determining whether or not equilibrium is achieved. This will be explored further in the next section.

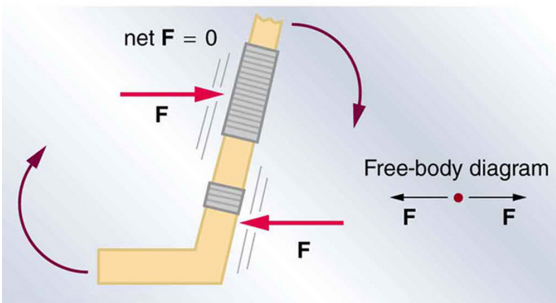
Equilibrium: remains stationary



An ice hockey stick lying flat on ice with two equal and opposite horizontal forces applied to it. Friction is negligible, and the gravitational force is balanced by the support of the ice (a normal force). Thus, net $F = 0$.

Equilibrium is achieved, which is static equilibrium in this case.

Nonequilibrium: rotation accelerates



The same forces are applied at other points and the stick rotates—in fact, it experiences an accelerated rotation. Here net $F = 0$ but the system is *not* at equilibrium. Hence, the net $F = 0$ is a necessary—but not sufficient—condition for achieving equilibrium.

Note:

PhET Explorations: Torque

Investigate how torque causes an object to rotate. Discover the relationships between angular acceleration, moment of inertia, angular momentum and torque.

[Torqu](#)
[e](#)

Section Summary

- Statics is the study of forces in equilibrium.
- Two conditions must be met to achieve equilibrium, which is defined to be motion without linear or rotational acceleration.
- The first condition necessary to achieve equilibrium is that the net external force on the system must be zero, so that net $\mathbf{F} = 0$.

Conceptual Questions

Exercise:

Problem:

What can you say about the velocity of a moving body that is in dynamic equilibrium? Draw a sketch of such a body using clearly labeled arrows to represent all external forces on the body.

Exercise:

Problem:

Under what conditions can a rotating body be in equilibrium? Give an example.

Glossary

static equilibrium

a state of equilibrium in which the net external force and torque acting on a system is zero

dynamic equilibrium

a state of equilibrium in which the net external force and torque on a system moving with constant velocity are zero

The Second Condition for Equilibrium

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- State the second condition that is necessary to achieve equilibrium.
- Explain torque and the factors on which it depends.
- Describe the role of torque in rotational mechanics.

The information presented in this section supports the following AP[®] learning objectives and science practices:

- **3.F.1.1** The student is able to use representations of the relationship between force and torque. **(S.P. 1.4)**
- **3.F.1.2** The student is able to compare the torques on an object caused by various forces. **(S.P. 1.4)**
- **3.F.1.3** The student is able to estimate the torque on an object caused by various forces in comparison to other situations. **(S.P. 2.3)**

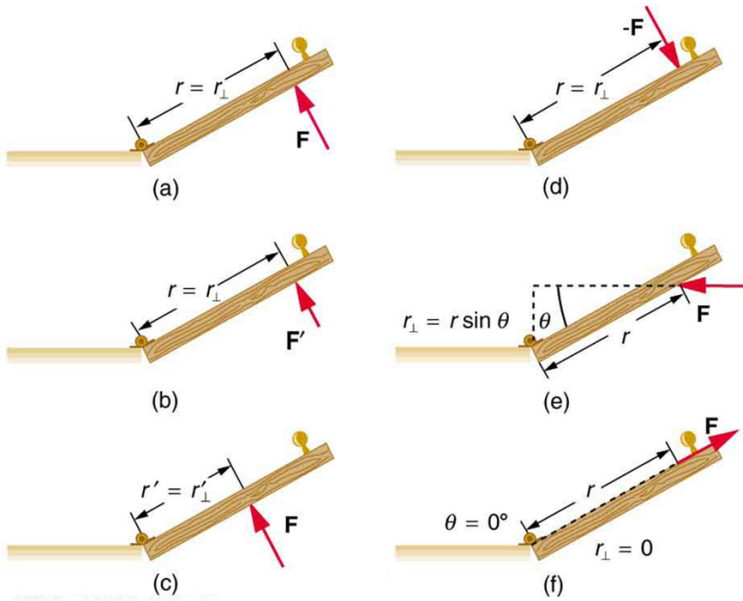
Note:

Torque

The second condition necessary to achieve equilibrium involves avoiding accelerated rotation (maintaining a constant angular velocity). A rotating body or system can be in equilibrium if its rate of rotation is constant and remains unchanged by the forces acting on it. To understand what factors

affect rotation, let us think about what happens when you open an ordinary door by rotating it on its hinges.

Several familiar factors determine how effective you are in opening the door. See [\[link\]](#). First of all, the larger the force, the more effective it is in opening the door—obviously, the harder you push, the more rapidly the door opens. Also, the point at which you push is crucial. If you apply your force too close to the hinges, the door will open slowly, if at all. Most people have been embarrassed by making this mistake and bumping up against a door when it did not open as quickly as expected. Finally, the direction in which you push is also important. The most effective direction is perpendicular to the door—we push in this direction almost instinctively.



Torque is the turning or twisting effectiveness of a force, illustrated here for door rotation on its hinges (as viewed from overhead). Torque has both magnitude and direction. (a) Counterclockwise torque is produced by this force, which means that the door will rotate in a counterclockwise due to \mathbf{F} . Note that r_{\perp} is the perpendicular distance

of the pivot from the line of action of the force. (b) A smaller counterclockwise torque is produced by a smaller force \mathbf{F}' acting at the same distance from the hinges (the pivot point). (c) The same force as in (a) produces a smaller counterclockwise torque when applied at a smaller distance from the hinges. (d) The same force as in (a), but acting in the opposite direction, produces a clockwise torque. (e) A smaller counterclockwise torque is produced by the same magnitude force acting at the same point but in a different direction. Here, θ is less than 90° . (f) Torque is zero here since the force just pulls on the hinges, producing no rotation. In this case, $\theta = 0^\circ$.

The magnitude, direction, and point of application of the force are incorporated into the definition of the physical quantity called torque.

Torque is the rotational equivalent of a force. It is a measure of the effectiveness of a force in changing or accelerating a rotation (changing the angular velocity over a period of time). In equation form, the magnitude of torque is defined to be

Equation:

$$\tau = rF \sin \theta$$

where τ (the Greek letter tau) is the symbol for torque, r is the distance from the pivot point to the point where the force is applied, F is the magnitude of the force, and θ is the angle between the force and the vector directed from the point of application to the pivot point, as seen in [\[link\]](#) and [\[link\]](#). An alternative expression for torque is given in terms of the **perpendicular lever arm** r_\perp as shown in [\[link\]](#) and [\[link\]](#), which is defined as

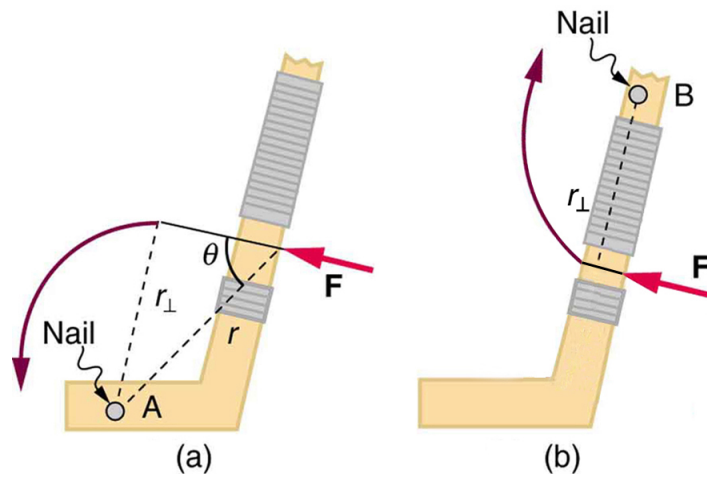
Equation:

$$r_{\perp} = r \sin \theta$$

so that

Equation:

$$\tau = r_{\perp} F.$$



A force applied to an object can produce a torque, which depends on the location of the pivot point. (a) The three factors r , F , and θ for pivot point A on a body are shown here— r is the distance from the chosen pivot point to the point where the force F is applied, and θ is the angle between F and the vector directed from the point of application to the pivot point. If the object can rotate around point A, it will rotate counterclockwise. This means that torque is counterclockwise relative to pivot A. (b) In this case, point B is the pivot point. The torque from the applied force will cause a clockwise rotation around point B,

and so it is a clockwise torque relative to B.

The perpendicular lever arm r_{\perp} is the shortest distance from the pivot point to the line along which \mathbf{F} acts; it is shown as a dashed line in [\[link\]](#) and [\[link\]](#). Note that the line segment that defines the distance r_{\perp} is perpendicular to \mathbf{F} , as its name implies. It is sometimes easier to find or visualize r_{\perp} than to find both r and θ . In such cases, it may be more convenient to use $\tau = r_{\perp}F$ rather than $\tau = rF \sin \theta$ for torque, but both are equally valid.

The **SI unit of torque** is newtons times meters, usually written as $\text{N} \cdot \text{m}$. For example, if you push perpendicular to the door with a force of 40 N at a distance of 0.800 m from the hinges, you exert a torque of $32 \text{ N} \cdot \text{m}$ ($0.800 \text{ m} \times 40 \text{ N} \times \sin 90^\circ$) relative to the hinges. If you reduce the force to 20 N, the torque is reduced to $16 \text{ N} \cdot \text{m}$, and so on.

The torque is always calculated with reference to some chosen pivot point. For the same applied force, a different choice for the location of the pivot will give you a different value for the torque, since both r and θ depend on the location of the pivot. Any point in any object can be chosen to calculate the torque about that point. The object may not actually pivot about the chosen “pivot point.”

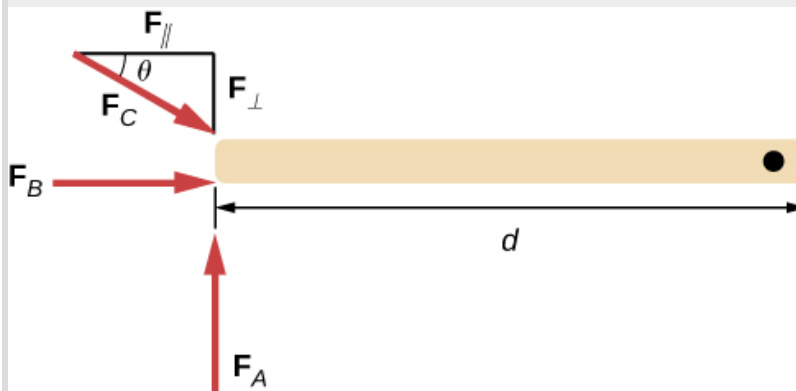
Note that for rotation in a plane, torque has two possible directions. Torque is either clockwise or counterclockwise relative to the chosen pivot point, as illustrated for points B and A, respectively, in [\[link\]](#). If the object can rotate about point A, it will rotate counterclockwise, which means that the torque for the force is shown as counterclockwise relative to A. But if the object can rotate about point B, it will rotate clockwise, which means the torque for the force shown is clockwise relative to B. Also, the magnitude of the torque is greater when the lever arm is longer.

Note:

Making Connections: Pivoting Block

A solid block of length d is pinned to a wall on its right end. Three forces act on the block as shown below: \mathbf{F}_A , \mathbf{F}_B , and \mathbf{F}_C . While all three forces are of equal magnitude, and all three are equal distances away from the pivot point, all three forces will create a different torque upon the object.

\mathbf{F}_A is vectored perpendicular to its distance from the pivot point; as a result, the magnitude of its torque can be found by the equation $\tau = F_A * d$. Vector \mathbf{F}_B is parallel to the line connecting the point of application of force and the pivot point. As a result, it does not provide an ability to rotate the object and, subsequently, its torque is zero. \mathbf{F}_C , however, is directed at an angle θ to the line connecting the point of application of force and the pivot point. In this instance, only the component perpendicular to this line is exerting a torque. This component, labeled F_{\perp} , can be found using the equation $F_{\perp} = F_C \sin \theta$. The component of the force parallel to this line, labeled F_{\parallel} , does not provide an ability to rotate the object and, as a result, does not provide a torque. Therefore, the resulting torque created by \mathbf{F}_C is $\tau = F_{\perp} * d$.



Forces on a block pinned to a wall. A solid block of length d is pinned to a wall on its right end. Three forces act on the block: \mathbf{F}_A , \mathbf{F}_B , and \mathbf{F}_C .

Now, the second condition necessary to achieve equilibrium is that the net external torque on a system must be zero. An external torque is one that is

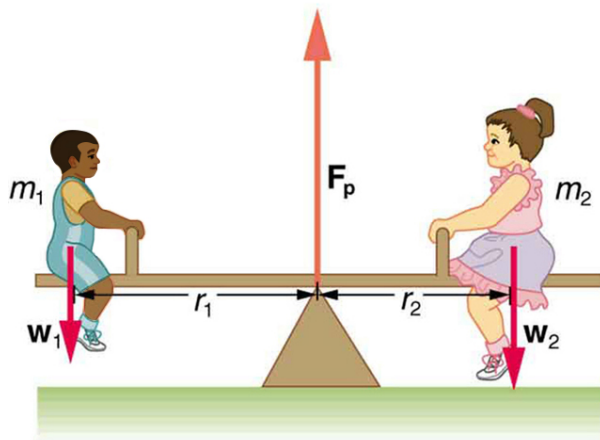
created by an external force. You can choose the point around which the torque is calculated. The point can be the physical pivot point of a system or any other point in space—but it must be the same point for all torques. If the second condition (net external torque on a system is zero) is satisfied for one choice of pivot point, it will also hold true for any other choice of pivot point in or out of the system of interest. (This is true only in an inertial frame of reference.) The second condition necessary to achieve equilibrium is stated in equation form as

Equation:

$$\text{net } \tau = 0$$

where net means total. Torques, which are in opposite directions are assigned opposite signs. A common convention is to call counterclockwise (ccw) torques positive and clockwise (cw) torques negative.

When two children balance a seesaw as shown in [\[link\]](#), they satisfy the two conditions for equilibrium. Most people have perfect intuition about seesaws, knowing that the lighter child must sit farther from the pivot and that a heavier child can keep a lighter one off the ground indefinitely.



Two children balancing a seesaw satisfy both conditions for equilibrium. The lighter child sits farther from the pivot to create a torque equal in

magnitude to that of the heavier child.

Example:

She Saw Torques On A Seesaw

The two children shown in [\[link\]](#) are balanced on a seesaw of negligible mass. (This assumption is made to keep the example simple—more involved examples will follow.) The first child has a mass of 26.0 kg and sits 1.60 m from the pivot. (a) If the second child has a mass of 32.0 kg, how far is she from the pivot? (b) What is F_p , the supporting force exerted by the pivot?

Strategy

Both conditions for equilibrium must be satisfied. In part (a), we are asked for a distance; thus, the second condition (regarding torques) must be used, since the first (regarding only forces) has no distances in it. To apply the second condition for equilibrium, we first identify the system of interest to be the seesaw plus the two children. We take the supporting pivot to be the point about which the torques are calculated. We then identify all external forces acting on the system.

Solution (a)

The three external forces acting on the system are the weights of the two children and the supporting force of the pivot. Let us examine the torque produced by each. Torque is defined to be

Equation:

$$\tau = rF \sin \theta.$$

Here $\theta = 90^\circ$, so that $\sin \theta = 1$ for all three forces. That means $r_\perp = r$ for all three. The torques exerted by the three forces are first,

Equation:

$$\tau_1 = r_1 w_1$$

second,

Equation:

$$\tau_2 = -r_2 w_2$$

and third,

Equation:

$$\begin{aligned}\tau_p &= r_p F_p \\ &= 0 \cdot F_p \\ &= 0.\end{aligned}$$

Note that a minus sign has been inserted into the second equation because this torque is clockwise and is therefore negative by convention. Since F_p acts directly on the pivot point, the distance r_p is zero. A force acting on the pivot cannot cause a rotation, just as pushing directly on the hinges of a door will not cause it to rotate. Now, the second condition for equilibrium is that the sum of the torques on both children is zero. Therefore

Equation:

$$\tau_2 = -\tau_1,$$

or

Equation:

$$r_2 w_2 = r_1 w_1.$$

Weight is mass times the acceleration due to gravity. Entering mg for w , we get

Equation:

$$r_2 m_2 g = r_1 m_1 g.$$

Solve this for the unknown r_2 :

Equation:

$$r_2 = r_1 \frac{m_1}{m_2}.$$

The quantities on the right side of the equation are known; thus, r_2 is

Equation:

$$r_2 = (1.60 \text{ m}) \frac{26.0 \text{ kg}}{32.0 \text{ kg}} = 1.30 \text{ m}.$$

As expected, the heavier child must sit closer to the pivot (1.30 m versus 1.60 m) to balance the seesaw.

Solution (b)

This part asks for a force F_p . The easiest way to find it is to use the first condition for equilibrium, which is

Equation:

$$\text{net } \mathbf{F} = 0.$$

The forces are all vertical, so that we are dealing with a one-dimensional problem along the vertical axis; hence, the condition can be written as

Equation:

$$\text{net } F_y = 0$$

where we again call the vertical axis the y -axis. Choosing upward to be the positive direction, and using plus and minus signs to indicate the directions of the forces, we see that

Equation:

$$F_p - w_1 - w_2 = 0.$$

This equation yields what might have been guessed at the beginning:

Equation:

$$F_p = w_1 + w_2.$$

So, the pivot supplies a supporting force equal to the total weight of the system:

Equation:

$$F_p = m_1 g + m_2 g.$$

Entering known values gives

Equation:

$$\begin{aligned} F_p &= (26.0 \text{ kg})(9.80 \text{ m/s}^2) + (32.0 \text{ kg})(9.80 \text{ m/s}^2) \\ &= 568 \text{ N.} \end{aligned}$$

Discussion

The two results make intuitive sense. The heavier child sits closer to the pivot. The pivot supports the weight of the two children. Part (b) can also be solved using the second condition for equilibrium, since both distances are known, but only if the pivot point is chosen to be somewhere other than the location of the seesaw's actual pivot!

Several aspects of the preceding example have broad implications. First, the choice of the pivot as the point around which torques are calculated simplified the problem. Since F_p is exerted on the pivot point, its lever arm is zero. Hence, the torque exerted by the supporting force F_p is zero relative to that pivot point. The second condition for equilibrium holds for any choice of pivot point, and so we choose the pivot point to simplify the solution of the problem.

Second, the acceleration due to gravity canceled in this problem, and we were left with a ratio of masses. *This will not always be the case.* Always enter the correct forces—do not jump ahead to enter some ratio of masses.

Third, the weight of each child is distributed over an area of the seesaw, yet we treated the weights as if each force were exerted at a single point. This is not an approximation—the distances r_1 and r_2 are the distances to points directly below the **center of gravity** of each child. As we shall see in the next section, the mass and weight of a system can act as if they are located at a single point.

Finally, note that the concept of torque has an importance beyond static equilibrium. *Torque plays the same role in rotational motion that force plays in linear motion.* We will examine this in the next chapter.

Note:**Take-Home Experiment**

Take a piece of modeling clay and put it on a table, then mash a cylinder down into it so that a ruler can balance on the round side of the cylinder while everything remains still. Put a penny 8 cm away from the pivot. Where would you need to put two pennies to balance? Three pennies?

Section Summary

- The second condition assures those torques are also balanced. Torque is the rotational equivalent of a force in producing a rotation and is defined to be

Equation:

$$\tau = rF \sin \theta$$

where τ is torque, r is the distance from the pivot point to the point where the force is applied, F is the magnitude of the force, and θ is the angle between \mathbf{F} and the vector directed from the point where the force acts to the pivot point. The perpendicular lever arm r_{\perp} is defined to be

Equation:

$$r_{\perp} = r \sin \theta$$

so that

Equation:

$$\tau = r_{\perp} F.$$

- The perpendicular lever arm r_{\perp} is the shortest distance from the pivot point to the line along which F acts. The SI unit for torque is newton-meter (N·m). The second condition necessary to achieve equilibrium is that the net external torque on a system must be zero:

Equation:

$$\text{net } \tau = 0$$

By convention, counterclockwise torques are positive, and clockwise torques are negative.

Conceptual Questions

Exercise:

Problem:

What three factors affect the torque created by a force relative to a specific pivot point?

Exercise:

Problem:

A wrecking ball is being used to knock down a building. One tall unsupported concrete wall remains standing. If the wrecking ball hits the wall near the top, is the wall more likely to fall over by rotating at its base or by falling straight down? Explain your answer. How is it most likely to fall if it is struck with the same force at its base? Note that this depends on how firmly the wall is attached at its base.

Exercise:

Problem:

Mechanics sometimes put a length of pipe over the handle of a wrench when trying to remove a very tight bolt. How does this help? (It is also hazardous since it can break the bolt.)

Problems & Exercises

Exercise:

Problem:

(a) When opening a door, you push on it perpendicularly with a force of 55.0 N at a distance of 0.850 m from the hinges. What torque are you exerting relative to the hinges? (b) Does it matter if you push at the same height as the hinges?

Solution:

a) 46.8 N·m

b) It does not matter at what height you push. The torque depends on only the magnitude of the force applied and the perpendicular distance of the force's application from the hinges. (Children don't have a tougher time opening a door because they push lower than adults, they have a tougher time because they don't push far enough from the hinges.)

Exercise:**Problem:**

When tightening a bolt, you push perpendicularly on a wrench with a force of 165 N at a distance of 0.140 m from the center of the bolt. (a) How much torque are you exerting in newton \times meters (relative to the center of the bolt)? (b) Convert this torque to footpounds.

Exercise:**Problem:**

Two children push on opposite sides of a door during play. Both push horizontally and perpendicular to the door. One child pushes with a force of 17.5 N at a distance of 0.600 m from the hinges, and the second child pushes at a distance of 0.450 m. What force must the second child exert to keep the door from moving? Assume friction is negligible.

Solution:

23.3 N

Exercise:

Problem:

Use the second condition for equilibrium (net $\tau = 0$) to calculate F_p in [\[link\]](#), employing any data given or solved for in part (a) of the example.

Exercise:

Problem:

Repeat the seesaw problem in [\[link\]](#) with the center of mass of the seesaw 0.160 m to the left of the pivot (on the side of the lighter child) and assuming a mass of 12.0 kg for the seesaw. The other data given in the example remain unchanged. Explicitly show how you follow the steps in the Problem-Solving Strategy for static equilibrium.

Solution:

Given:

$$\begin{aligned} m_1 &= 26.0 \text{ kg}, m_2 = 32.0 \text{ kg}, m_s = 12.0 \text{ kg}, \\ r_1 &= 1.60 \text{ m}, r_s = 0.160 \text{ m}, \text{ find (a) } r_2, \text{ (b) } F_p \end{aligned}$$

a) Since children are balancing:

$$\begin{aligned} \text{net } \tau_{\text{cw}} &= -\text{net } \tau_{\text{ccw}} \\ \Rightarrow w_1 r_1 + m_s g r_s &= w_2 r_2 \end{aligned}$$

So, solving for r_2 gives:

$$\begin{aligned} r_2 &= \frac{w_1 r_1 + m_s g r_s}{w_2} = \frac{m_1 g r_1 + m_s g r_s}{m_2 g} = \frac{m_1 r_1 + m_s r_s}{m_2} \\ &= \frac{(26.0 \text{ kg})(1.60 \text{ m}) + (12.0 \text{ kg})(0.160 \text{ m})}{32.0 \text{ kg}} \\ &= 1.36 \text{ m} \end{aligned}$$

b) Since the children are not moving:

$$\text{net } F = 0 = F_p - w_1 - w_2 - w_s$$

$$\Rightarrow F_p = w_1 + w_2 + w_s$$

So that

$$\begin{aligned} F_p &= (26.0 \text{ kg} + 32.0 \text{ kg} + 12.0 \text{ kg})(9.80 \text{ m/s}^2) \\ &= 686 \text{ N} \end{aligned}$$

Test Prep for AP Courses

Exercise:

Problem:

Which of the following is not an example of an object undergoing a torque?

- a. A car is rounding a bend at a constant speed.
- b. A merry-go-round increases from rest to a constant rotational speed.
- c. A pendulum swings back and forth.
- d. A bowling ball rolls down a bowling alley.

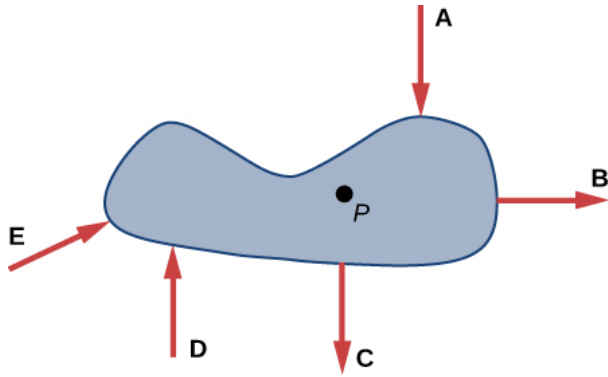
Solution:

(a)

Exercise:

Problem:

Five forces of equal magnitude, labeled **A–E**, are applied to the object shown below. If the object is anchored at point **P**, which force will provide the greatest torque?



Five forces acting on an object.

- a. Force **A**
- b. Force **B**
- c. Force **C**
- d. Force **D**
- e. Force **E**

Glossary

torque

turning or twisting effectiveness of a force

perpendicular lever arm

the shortest distance from the pivot point to the line along which **F** lies

SI units of torque

newton times meters, usually written as $\text{N}\cdot\text{m}$

center of gravity

the point where the total weight of the body is assumed to be concentrated

Stability

LEARNING OBJECTIVES

By the end of this section, you will be able to:

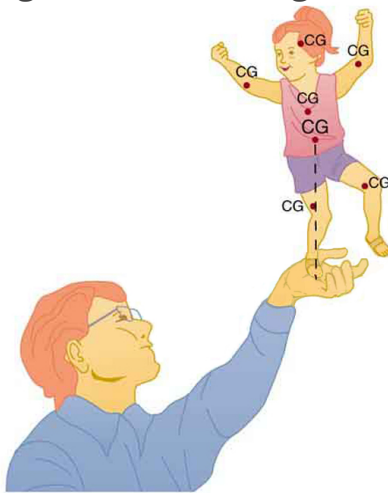
- State the types of equilibrium.
- Describe stable and unstable equilibriums.
- Describe neutral equilibrium.

The information presented in this section supports the following AP[®] learning objectives and science practices:

- **3.F.1.1** The student is able to use representations of the relationship between force and torque. **(S.P. 1.4)**
- **3.F.1.2** The student is able to compare the torques on an object caused by various forces. **(S.P. 1.4)**
- **3.F.1.3** The student is able to estimate the torque on an object caused by various forces in comparison to other situations. **(S.P. 2.3)**
- **3.F.1.4** The student is able to design an experiment and analyze data testing a question about torques in a balanced rigid system. **(S.P. 4.1, 4.2, 5.1)**
- **3.F.1.5** The student is able to calculate torques on a two-dimensional system in static equilibrium, by examining a representation or model (such as a diagram or physical construction). **(S.P. 1.4, 2.2)**

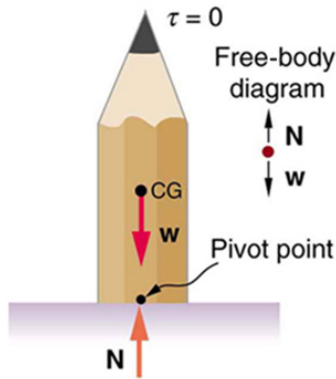
It is one thing to have a system in equilibrium; it is quite another for it to be stable. The toy doll perched on the man's hand in [\[link\]](#), for example, is not in stable equilibrium. There are *three types of equilibrium: stable, unstable, and neutral*. Figures throughout this module illustrate various examples.

[\[link\]](#) presents a balanced system, such as the toy doll on the man's hand, which has its center of gravity (cg) directly over the pivot, so that the torque of the total weight is zero. This is equivalent to having the torques of the individual parts balanced about the pivot point, in this case the hand. The cgs of the arms, legs, head, and torso are labeled with smaller type.



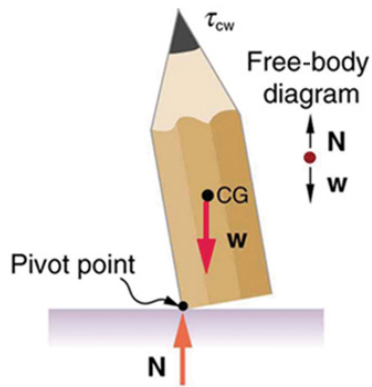
A man balances a
toy doll on one
hand.

A system is said to be in **stable equilibrium** if, when displaced from equilibrium, it experiences a net force or torque in a direction opposite to the direction of the displacement. For example, a marble at the bottom of a bowl will experience a *restoring* force when displaced from its equilibrium position. This force moves it back toward the equilibrium position. Most systems are in stable equilibrium, especially for small displacements. For another example of stable equilibrium, see the pencil in [\[link\]](#).

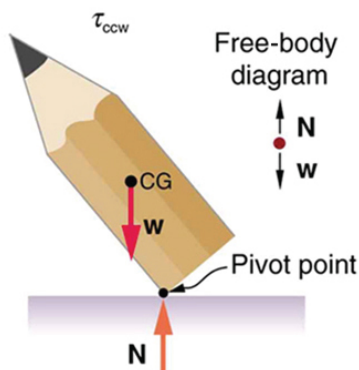


This pencil is in the condition of equilibrium. The net force on the pencil is zero and the total torque about any pivot is zero.

A system is in **unstable equilibrium** if, when displaced, it experiences a net force or torque in the *same* direction as the displacement from equilibrium. A system in unstable equilibrium accelerates away from its equilibrium position if displaced even slightly. An obvious example is a ball resting on top of a hill. Once displaced, it accelerates away from the crest. See the next several figures for examples of unstable equilibrium.

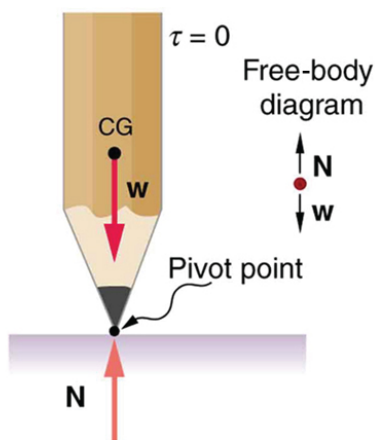


If the pencil is displaced slightly to the side (counterclockwise), it is no longer in equilibrium. Its weight produces a clockwise torque that returns the pencil to its equilibrium position.

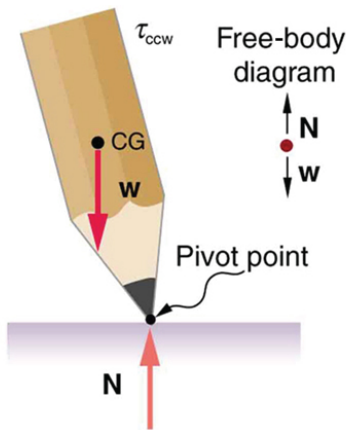


If the pencil is

displaced too far,
the torque caused
by its weight
changes direction
to
counterclockwise
and causes the
displacement to
increase.

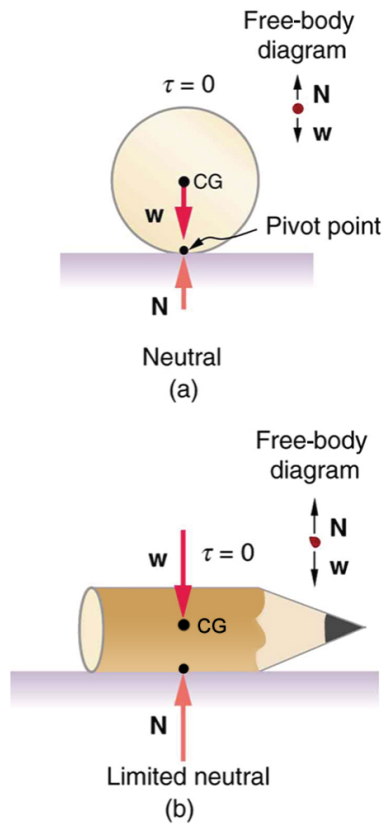


This figure shows
unstable
equilibrium,
although both
conditions for
equilibrium are
satisfied.



If the pencil is displaced even slightly, a torque is created by its weight that is in the same direction as the displacement, causing the displacement to increase.

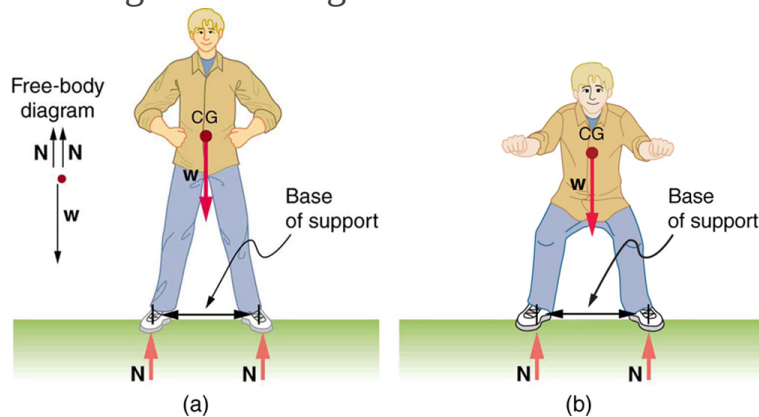
A system is in **neutral equilibrium** if its equilibrium is independent of displacements from its original position. A marble on a flat horizontal surface is an example. Combinations of these situations are possible. For example, a marble on a saddle is stable for displacements toward the front or back of the saddle and unstable for displacements to the side. [\[link\]](#) shows another example of neutral equilibrium.



(a) Here we see neutral equilibrium. The cg of a sphere on a flat surface lies directly above the point of support, independent of the position on the surface. The sphere is therefore in equilibrium in any location, and if displaced, it will remain put. (b) Because it has a circular cross section, the pencil

is in neutral equilibrium for displacements perpendicular to its length.

When we consider how far a system in stable equilibrium can be displaced before it becomes unstable, we find that some systems in stable equilibrium are more stable than others. The pencil in [\[link\]](#) and the person in [\[link\]](#)(a) are in stable equilibrium, but become unstable for relatively small displacements to the side. The critical point is reached when the cg is no longer *above* the base of support. Additionally, since the cg of a person's body is above the pivots in the hips, displacements must be quickly controlled. This control is a central nervous system function that is developed when we learn to hold our bodies erect as infants. For increased stability while standing, the feet should be spread apart, giving a larger base of support. Stability is also increased by lowering one's center of gravity by bending the knees, as when a football player prepares to receive a ball or braces themselves for a tackle. A cane, a crutch, or a walker increases the stability of the user, even more as the base of support widens. Usually, the cg of a female is lower (closer to the ground) than a male. Young children have their center of gravity between their shoulders, which increases the challenge of learning to walk.



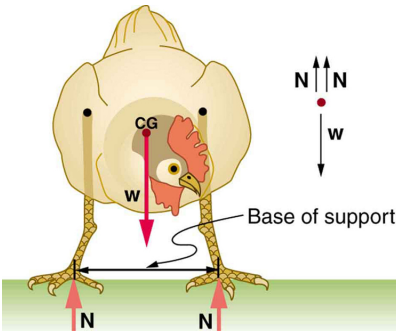
(a) The center of gravity of an adult is above the hip joints (one of the main

pivots in the body) and lies between two narrowly-separated feet. Like a pencil standing on its eraser, this person is in stable equilibrium in relation to sideways displacements, but relatively small displacements take his cg outside the base of support and make him unstable.

Humans are less stable relative to forward and backward displacements because the feet are not very long. Muscles are used extensively to balance the body in the front-to-back direction. (b) While bending in the manner shown, stability is increased by lowering the center of gravity. Stability is also increased if the base is expanded by placing the feet farther apart.

Animals such as chickens have easier systems to control. [\[link\]](#) shows that the cg of a chicken lies below its hip joints and between its widely separated and broad feet. Even relatively large displacements of the chicken's cg are stable and result in restoring forces and torques that return the cg to its equilibrium position with little effort on the chicken's part. Not all birds are like chickens, of course. Some birds, such as the flamingo, have balance systems that are almost as sophisticated as that of humans.

[\[link\]](#) shows that the cg of a chicken is below the hip joints and lies above a broad base of support formed by widely-separated and large feet. Hence, the chicken is in very stable equilibrium, since a relatively large displacement is needed to render it unstable. The body of the chicken is supported from above by the hips and acts as a pendulum between the hips. Therefore, the chicken is stable for front-to-back displacements as well as for side-to-side displacements.



The center of gravity of a chicken is below the hip joints. The chicken is in stable equilibrium. The body of the chicken is supported from above by the hips and acts as a pendulum between them.

Engineers and architects strive to achieve extremely stable equilibriums for buildings and other systems that must withstand wind, earthquakes, and other forces that displace them from equilibrium. Although the examples in this section emphasize gravitational forces, the basic conditions for equilibrium are the same for all types of forces. The net external force must be zero, and the net torque must also be zero.

Note:

Take-Home Experiment

Stand straight with your heels, back, and head against a wall. Bend forward from your waist, keeping your heels and bottom against the wall, to touch your toes. Can you do this without toppling over? Explain why and what

you need to do to be able to touch your toes without losing your balance. Is it easier for a woman to do this?

Section Summary

- A system is said to be in stable equilibrium if, when displaced from equilibrium, it experiences a net force or torque in a direction opposite the direction of the displacement.
- A system is in unstable equilibrium if, when displaced from equilibrium, it experiences a net force or torque in the same direction as the displacement from equilibrium.
- A system is in neutral equilibrium if its equilibrium is independent of displacements from its original position.

Conceptual Questions

Exercise:

Problem:

A round pencil lying on its side as in [\[link\]](#) is in neutral equilibrium relative to displacements perpendicular to its length. What is its stability relative to displacements parallel to its length?

Exercise:

Problem:

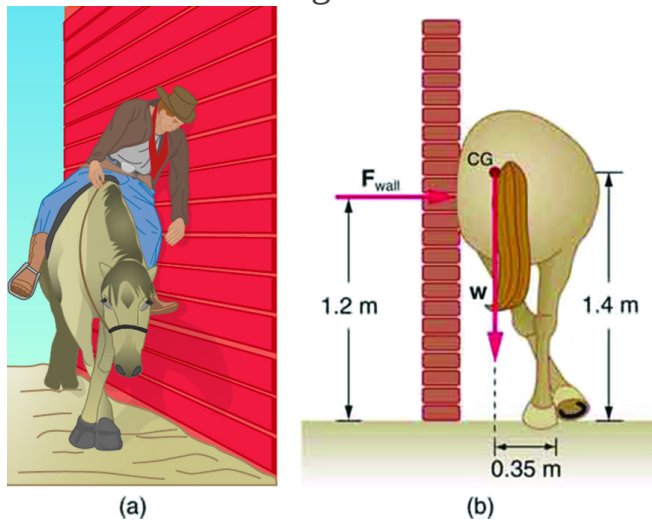
Explain the need for tall towers on a suspension bridge to ensure stable equilibrium.

Problems & Exercises

Exercise:

Problem:

Suppose a horse leans against a wall as in [\[link\]](#). Calculate the force exerted on the wall assuming that force is horizontal while using the data in the schematic representation of the situation. Note that the force exerted on the wall is equal in magnitude and opposite in direction to the force exerted on the horse, keeping it in equilibrium. The total mass of the horse and rider is 500 kg. Take the data to be accurate to three digits.

**Solution:**

$$F_{\text{wall}} = 1.43 \times 10^3 \text{ N}$$

Exercise:**Problem:**

Two children of mass 20 kg and 30 kg sit balanced on a seesaw with the pivot point located at the center of the seesaw. If the children are separated by a distance of 3 m, at what distance from the pivot point is the small child sitting in order to maintain the balance?

Exercise:

Problem:

(a) Calculate the magnitude and direction of the force on each foot of the horse in [\[link\]](#) (two are on the ground), assuming the center of mass of the horse is midway between the feet. The total mass of the horse and rider is 500kg. (b) What is the minimum coefficient of friction between the hooves and ground? Note that the force exerted by the wall is horizontal.

Solution:

a) 2.55×10^3 N, 16.3° to the left of vertical (i.e., toward the wall)

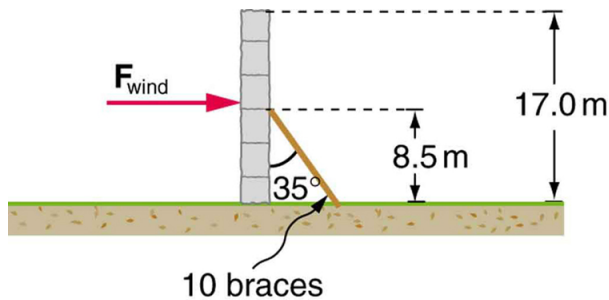
b) 0.292

Exercise:**Problem:**

A person carries a plank of wood 2 m long with one hand pushing down on it at one end with a force F_1 and the other hand holding it up at 50 cm from the end of the plank with force F_2 . If the plank has a mass of 20 kg and its center of gravity is at the middle of the plank, what are the magnitudes of the forces F_1 and F_2 ?

Exercise:**Problem:**

A 17.0-m-high and 11.0-m-long wall under construction and its bracing are shown in [\[link\]](#). The wall is in stable equilibrium without the bracing but can pivot at its base. Calculate the force exerted by each of the 10 braces if a strong wind exerts a horizontal force of 650 N on each square meter of the wall. Assume that the net force from the wind acts at a height halfway up the wall and that all braces exert equal forces parallel to their lengths. Neglect the thickness of the wall.



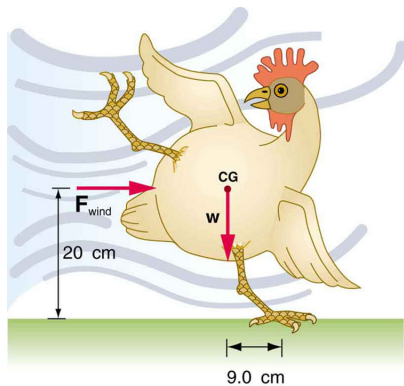
Solution:

$$F_B = 2.12 \times 10^4 \text{ N}$$

Exercise:

Problem:

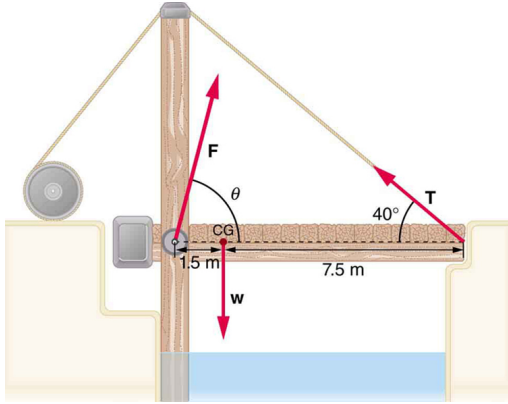
(a) What force must be exerted by the wind to support a 2.50-kg chicken in the position shown in [\[link\]](#)? (b) What is the ratio of this force to the chicken's weight? (c) Does this support the contention that the chicken has a relatively stable construction?



Exercise:

Problem:

Suppose the weight of the drawbridge in [\[link\]](#) is supported entirely by its hinges and the opposite shore, so that its cables are slack. (a) What fraction of the weight is supported by the opposite shore if the point of support is directly beneath the cable attachments? (b) What is the direction and magnitude of the force the hinges exert on the bridge under these circumstances? The mass of the bridge is 2500 kg.



A small drawbridge,
showing the forces on the
hinges (F), its weight (w),
and the tension in its
wires (T).

Solution:

a) 0.167, or about one-sixth of the weight is supported by the opposite shore.

b) $F = 2.0 \times 10^4$ N, straight up.

Exercise:

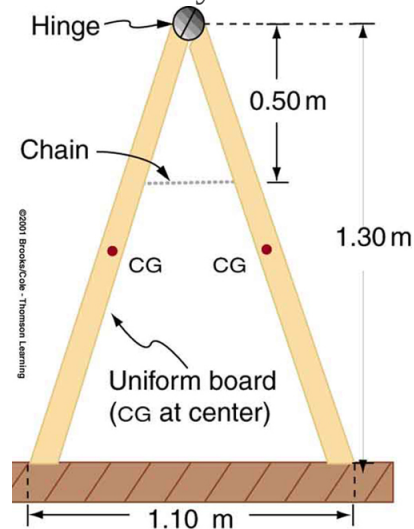
Problem:

Suppose a 900-kg car is on the bridge in [\[link\]](#) with its center of mass halfway between the hinges and the cable attachments. (The bridge is supported by the cables and hinges only.) (a) Find the force in the cables. (b) Find the direction and magnitude of the force exerted by the hinges on the bridge.

Exercise:

Problem:

A sandwich board advertising sign is constructed as shown in [\[link\]](#). The sign's mass is 8.00 kg. (a) Calculate the tension in the chain assuming no friction between the legs and the sidewalk. (b) What force is exerted by each side on the hinge?



A sandwich board
advertising sign
demonstrates
tension.

Solution:

a) 21.6 N

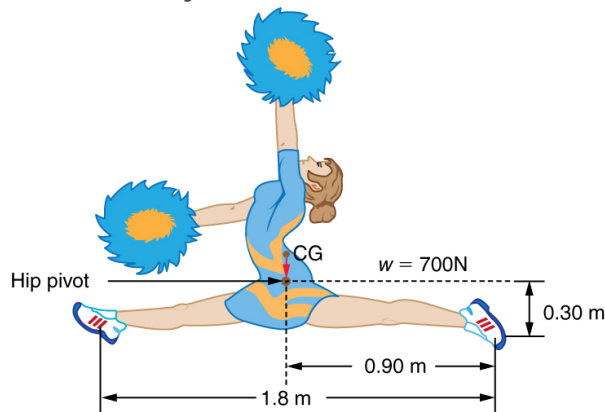
b) 21.6 N

Exercise:**Problem:**

(a) What minimum coefficient of friction is needed between the legs and the ground to keep the sign in [\[link\]](#) in the position shown if the chain breaks? (b) What force is exerted by each side on the hinge?

Exercise:**Problem:**

A gymnast is attempting to perform splits. From the information given in [\[link\]](#), calculate the magnitude and direction of the force exerted on each foot by the floor.



A gymnast performs full split.
The center of gravity and the
various distances from it are
shown.

Solution:

350 N directly upwards

Test Prep for AP Courses**Exercise:****Problem:**

Using the concept of torque, explain why a traffic cone placed on its base is in stable equilibrium, while a traffic cone placed on its tip is in unstable equilibrium.

Solution:

Both objects are in equilibrium. However, they will respond differently if a force is applied to their sides. If the cone placed on its base is displaced to the side, its center of gravity will remain over its base and it will return to its original position. When the traffic cone placed on its tip is displaced to the side, its center of gravity will drift from its base, causing a torque that will accelerate it to the ground.

Glossary**neutral equilibrium**

a state of equilibrium that is independent of a system's displacements from its original position

stable equilibrium

a system, when displaced, experiences a net force or torque in a direction opposite to the direction of the displacement

unstable equilibrium

a system, when displaced, experiences a net force or torque in the same direction as the displacement from equilibrium

Applications of Statics, Including Problem-Solving Strategies

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Discuss the applications of statics in real life.
- State and discuss various problem-solving strategies in statics.

The information presented in this section supports the following AP[®] learning objectives and science practices:

- **3.F.1.1** The student is able to use representations of the relationship between force and torque. **(S.P. 1.4)**
- **3.F.1.2** The student is able to compare the torques on an object caused by various forces. **(S.P. 1.4)**
- **3.F.1.3** The student is able to estimate the torque on an object caused by various forces in comparison to other situations. **(S.P. 2.3)**
- **3.F.1.4** The student is able to design an experiment and analyze data testing a question about torques in a balanced rigid system. **(S.P. 4.1, 4.2, 5.1)**
- **3.F.1.5** The student is able to calculate torques on a two-dimensional system in static equilibrium, by examining a representation or model (such as a diagram or physical construction). **(S.P. 1.4, 2.2)**

Statics can be applied to a variety of situations, ranging from raising a drawbridge to bad posture and back strain. We begin with a discussion of

problem-solving strategies specifically used for statics. Since statics is a special case of Newton's laws, both the general problem-solving strategies and the special strategies for Newton's laws, discussed in [Problem-Solving Strategies](#), still apply.

Note:

Problem-Solving Strategy: Static Equilibrium Situations

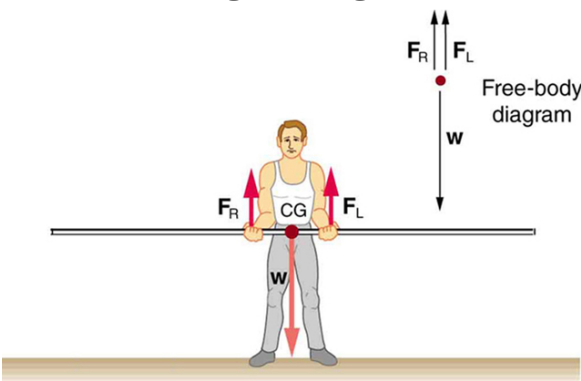
1. The first step is to determine whether or not the system is in **static equilibrium**. This condition is always the case when the *acceleration of the system is zero and accelerated rotation does not occur*.
2. It is particularly important to *draw a free body diagram for the system of interest*. Carefully label all forces, and note their relative magnitudes, directions, and points of application whenever these are known.
3. Solve the problem by applying either or both of the conditions for equilibrium (represented by the equations $\text{net } F = 0$ and $\text{net } \tau = 0$, depending on the list of known and unknown factors. If the second condition is involved, *choose the pivot point to simplify the solution*. Any pivot point can be chosen, but the most useful ones cause torques by unknown forces to be zero. (Torque is zero if the force is applied at the pivot (then $r = 0$), or along a line through the pivot point (then $\theta = 0$)). Always choose a convenient coordinate system for projecting forces.
4. *Check the solution to see if it is reasonable* by examining the magnitude, direction, and units of the answer. The importance of this last step never diminishes, although in unfamiliar applications, it is usually more difficult to judge reasonableness. These judgments become progressively easier with experience.

Now let us apply this problem-solving strategy for the pole vaulter shown in the three figures below. The pole is uniform and has a mass of 5.00 kg. In [\[link\]](#), the pole's cg lies halfway between the vaulter's hands. It seems

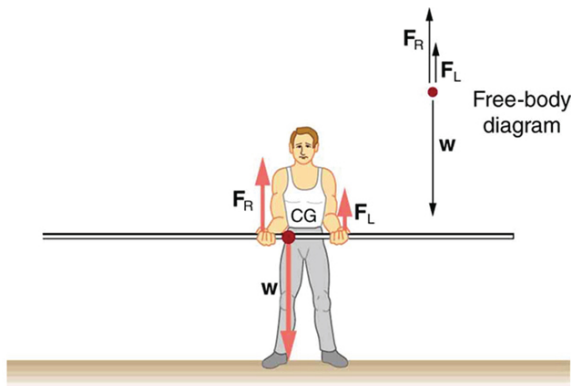
reasonable that the force exerted by each hand is equal to half the weight of the pole, or 24.5 N. This obviously satisfies the first condition for equilibrium (net $F = 0$). The second condition (net $\tau = 0$) is also satisfied, as we can see by choosing the cg to be the pivot point. The weight exerts no torque about a pivot point located at the cg, since it is applied at that point and its lever arm is zero. The equal forces exerted by the hands are equidistant from the chosen pivot, and so they exert equal and opposite torques. Similar arguments hold for other systems where supporting forces are exerted symmetrically about the cg. For example, the four legs of a uniform table each support one-fourth of its weight.

In [\[link\]](#), a pole vaulter holding a pole with its cg halfway between his hands is shown. Each hand exerts a force equal to half the weight of the pole, $F_R = F_L = w/2$. (b) The pole vaulter moves the pole to his left, and the forces that the hands exert are no longer equal. See [\[link\]](#). If the pole is held with its cg to the left of the person, then he must push down with his right hand and up with his left. The forces he exerts are larger here because they are in opposite directions and the cg is at a long distance from either hand.

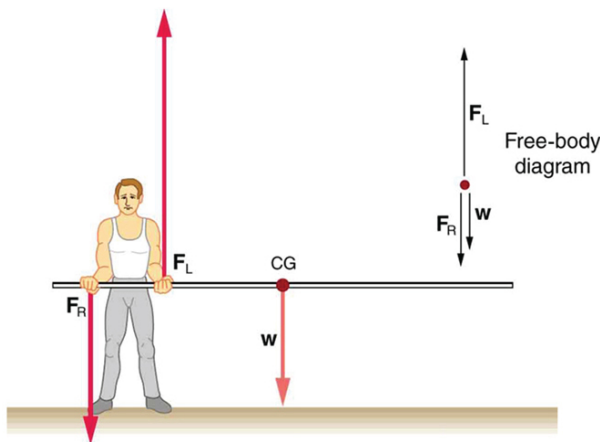
Similar observations can be made using a meter stick held at different locations along its length.



A pole vaulter holds a pole horizontally with both hands.



A pole vaulter is holding a pole horizontally with both hands. The center of gravity is near his right hand.



A pole vaulter is holding a pole horizontally with both hands. The center of gravity is to the left side of the vaulter.

If the pole vaulter holds the pole as shown in [\[link\]](#), the situation is not as simple. The total force he exerts is still equal to the weight of the pole, but it is not evenly divided between his hands. (If $F_L = F_R$, then the torques about the cg would not be equal since the lever arms are different.)

Logically, the right hand should support more weight, since it is closer to the cg. In fact, if the right hand is moved directly under the cg, it will support all the weight. This situation is exactly analogous to two people carrying a load; the one closer to the cg carries more of its weight. Finding the forces F_L and F_R is straightforward, as the next example shows.

If the pole vaulter holds the pole from near the end of the pole ([\[link\]](#)), the direction of the force applied by the right hand of the vaulter reverses its direction.

Example:

What Force Is Needed to Support a Weight Held Near Its CG?

For the situation shown in [\[link\]](#), calculate: (a) F_R , the force exerted by the right hand, and (b) F_L , the force exerted by the left hand. The hands are 0.900 m apart, and the cg of the pole is 0.600 m from the left hand.

Strategy

[\[link\]](#) includes a free body diagram for the pole, the system of interest. There is not enough information to use the first condition for equilibrium (net $F = 0$), since two of the three forces are unknown and the hand forces cannot be assumed to be equal in this case. There is enough information to use the second condition for equilibrium (net $\tau = 0$) if the pivot point is chosen to be at either hand, thereby making the torque from that hand zero. We choose to locate the pivot at the left hand in this part of the problem, to eliminate the torque from the left hand.

Solution for (a)

There are now only two nonzero torques, those from the gravitational force (τ_w) and from the push or pull of the right hand (τ_R). Stating the second condition in terms of clockwise and counterclockwise torques,

Equation:

$$\text{net } \tau_{cw} = -\text{net } \tau_{ccw}.$$

or the algebraic sum of the torques is zero.

Here this is

Equation:

$$\tau_R = -\tau_w$$

since the weight of the pole creates a counterclockwise torque and the right hand counters with a clockwise torque. Using the definition of torque, $\tau = rF \sin \theta$, noting that $\theta = 90^\circ$, and substituting known values, we obtain

Equation:

$$(0.900 \text{ m})(F_R) = (0.600 \text{ m})(mg).$$

Thus,

Equation:

$$\begin{aligned} F_R &= (0.667)(5.00 \text{ kg})(9.80 \text{ m/s}^2) \\ &= 32.7 \text{ N.} \end{aligned}$$

Solution for (b)

The first condition for equilibrium is based on the free body diagram in the figure. This implies that by Newton's second law:

Equation:

$$F_L + F_R - mg = 0$$

From this we can conclude:

Equation:

$$F_L + F_R = w = mg$$

Solving for F_L , we obtain

Equation:

$$\begin{aligned} F_L &= mg - F_R \\ &= mg - 32.7 \text{ N} \\ &= (5.00 \text{ kg})(9.80 \text{ m/s}^2) - 32.7 \text{ N} \\ &= 16.3 \text{ N} \end{aligned}$$

Discussion

F_L is seen to be exactly half of F_R , as we might have guessed, since F_L is applied twice as far from the cg as F_R .

If the pole vaulter holds the pole as he might at the start of a run, shown in [\[link\]](#), the forces change again. Both are considerably greater, and one force reverses direction.

Note:

Take-Home Experiment

This is an experiment to perform while standing in a bus or a train. Stand facing sideways. How do you move your body to readjust the distribution of your mass as the bus accelerates and decelerates? Now stand facing forward. How do you move your body to readjust the distribution of your mass as the bus accelerates and decelerates? Why is it easier and safer to stand facing sideways rather than forward? Note: For your safety (and those around you), make sure you are holding onto something while you carry out this activity!

Note:

PhET Explorations: Balancing Act

Play with objects on a teeter totter to learn about balance. Test what you've learned by trying the Balance Challenge game.

[Balancing Act](#)

Summary

- Statics can be applied to a variety of situations, ranging from raising a drawbridge to bad posture and back strain. We have discussed the problem-solving strategies specifically useful for statics. Statics is a special case of Newton's laws, both the general problem-solving strategies and the special strategies for Newton's laws, discussed in [Problem-Solving Strategies](#), still apply.

Conceptual Questions

Exercise:

Problem:

When visiting some countries, you may see a person balancing a load on the head. Explain why the center of mass of the load needs to be directly above the person's neck vertebrae.

Problems & Exercises

Exercise:

Problem:

To get up on the roof, a person (mass 70.0 kg) places a 6.00-m aluminum ladder (mass 10.0 kg) against the house on a concrete pad with the base of the ladder 2.00 m from the house. The ladder rests against a plastic rain gutter, which we can assume to be frictionless. The center of mass of the ladder is 2 m from the bottom. The person is standing 3 m from the bottom. What are the magnitudes of the forces on the ladder at the top and bottom?

Exercise:

Problem:

In [\[link\]](#), the cg of the pole held by the pole vaulter is 2.00 m from the left hand, and the hands are 0.700 m apart. Calculate the force exerted by (a) his right hand and (b) his left hand. (c) If each hand supports half the weight of the pole in [\[link\]](#), show that the second condition for equilibrium (net $\tau = 0$) is satisfied for a pivot other than the one located at the center of gravity of the pole. Explicitly show how you follow the steps in the Problem-Solving Strategy for static equilibrium described above.

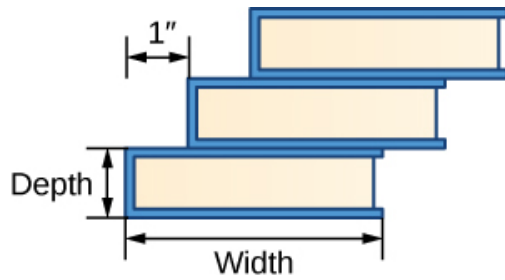
Test Prep for AP Courses**Exercise:****Problem:**

A child sits on the end of a playground see-saw. Which of the following values is the most appropriate estimate of the torque created by the child?

- a. 6 N•m
- b. 60 N•m
- c. 600 N•m
- d. 6000 N•m

Exercise:**Problem:**

A group of students is stacking a set of identical books, each one overhanging the one below it by 1 inch. They would like to estimate how many books they could place on top of each other before the stack tipped. What information below would they need to know to make this calculation?



3 overlapping stacked books.

- I. The mass of each book
 - II. The width of each book
 - III. The depth of each book
-
- a. I only
 - b. I and II only
 - c. I and III only
 - d. II only
 - e. I, II, and III

Solution:

(d)

Exercise:

Problem:

A 10 N board of uniform density is 5 meters long. It is supported on the left by a string bearing a 3 N upward force. In order to prevent the string from breaking, a person must place an upward force of 7 N at a position along the bottom surface of the board. At what distance from its left edge would they need to place this force in order for the board to be in static equilibrium?

- a. $\frac{3}{7}$ m

- b. $\frac{5}{2}$ m
- c. $\frac{25}{7}$ m
- d. $\frac{30}{7}$ m
- e. 5 m

Exercise:

Problem:

A bridge is supported by two piers located 20 meters apart. Both the left and right piers provide an upward force on the bridge, labeled F_L and F_R respectively.

- a. If a 1000 kg car comes to rest at a point 5 meters from the left pier, how much force will the bridge provide to the left and right piers?
- b. How will F_L and F_R change as the car drives to the right side of the bridge?

Solution:

- a. $F_L = 7350$ N, $F_R = 2450$ N
- b. As the car moves to the right side of the bridge, F_L will decrease and F_R will increase. (At exactly halfway across the bridge, F_L and F_R will both be 4900 N.)

Exercise:

Problem:

An object of unknown mass is provided to a student. Without using a scale, design an experimental procedure detailing how the magnitude of this mass could be experimentally found. Your explanation must include the concept of torque and all steps should be provided in an orderly sequence. You may include a labeled diagram of your setup to help in your description. Include enough detail so that another student could carry out your procedure.

Glossary

static equilibrium

equilibrium in which the acceleration of the system is zero and accelerated rotation does not occur

Simple Machines

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Describe different simple machines.
- Calculate the mechanical advantage.

The information presented in this section supports the following AP[®] learning objectives and science practices:

- **3.F.1.1** The student is able to use representations of the relationship between force and torque. **(S.P. 1.4)**
- **3.F.1.2** The student is able to compare the torques on an object caused by various forces. **(S.P. 1.4)**
- **3.F.1.3** The student is able to estimate the torque on an object caused by various forces in comparison to other situations. **(S.P. 2.3)**
- **3.F.1.5** The student is able to calculate torques on a two-dimensional system in static equilibrium, by examining a representation or model (such as a diagram or physical construction). **(S.P. 1.4, 2.2)**

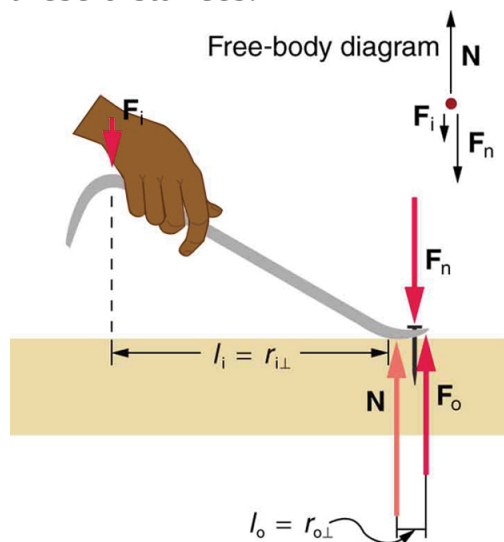
Simple machines are devices that can be used to multiply or augment a force that we apply – often at the expense of a distance through which we apply the force. The word for “machine” comes from the Greek word meaning “to help make things easier.” Levers, gears, pulleys, wedges, and screws are some examples of machines. Energy is still conserved for these

devices because a machine cannot do more work than the energy put into it. However, machines can reduce the input force that is needed to perform the job. The ratio of output to input force magnitudes for any simple machine is called its **mechanical advantage** (MA).

Equation:

$$MA = \frac{F_o}{F_i}$$

One of the simplest machines is the lever, which is a rigid bar pivoted at a fixed place called the fulcrum. Torques are involved in levers, since there is rotation about a pivot point. Distances from the physical pivot of the lever are crucial, and we can obtain a useful expression for the MA in terms of these distances.



A nail puller is a lever with a large mechanical advantage. The external forces on the nail puller are represented by solid arrows. The force that the nail puller applies to the nail (F_o) is not a force on the nail puller. The reaction force the nail

exerts back on the puller (\mathbf{F}_n) is an external force and is equal and opposite to \mathbf{F}_o . The perpendicular lever arms of the input and output forces are l_i and l_o .

[\[link\]](#) shows a lever type that is used as a nail puller. Crowbars, seesaws, and other such levers are all analogous to this one. \mathbf{F}_i is the input force and \mathbf{F}_o is the output force. There are three vertical forces acting on the nail puller (the system of interest) – these are \mathbf{F}_i , \mathbf{F}_n , and \mathbf{N} . \mathbf{F}_n is the reaction force back on the system, equal and opposite to \mathbf{F}_o . (Note that \mathbf{F}_o is not a force on the system.) \mathbf{N} is the normal force upon the lever, and its torque is zero since it is exerted at the pivot. The torques due to \mathbf{F}_i and \mathbf{F}_n must be equal to each other if the nail is not moving, to satisfy the second condition for equilibrium (net $\tau = 0$). (In order for the nail to actually move, the torque due to \mathbf{F}_i must be ever-so-slightly greater than torque due to \mathbf{F}_n .) Hence,

Equation:

$$l_i F_i = l_o F_o$$

where l_i and l_o are the distances from where the input and output forces are applied to the pivot, as shown in the figure. Rearranging the last equation gives

Equation:

$$\frac{F_o}{F_i} = \frac{l_i}{l_o}.$$

What interests us most here is that the magnitude of the force exerted by the nail puller, F_o , is much greater than the magnitude of the input force applied to the puller at the other end, F_i . For the nail puller,

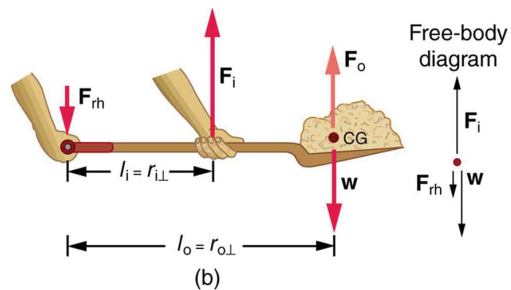
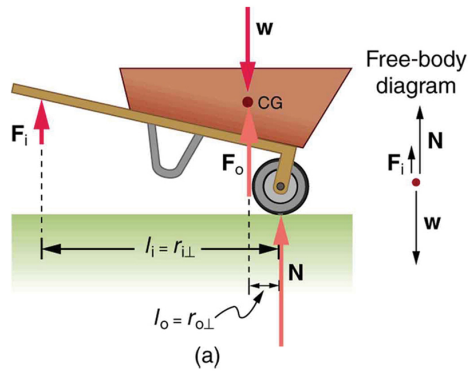
Equation:

$$\text{MA} = \frac{F_o}{F_i} = \frac{l_i}{l_o}.$$

This equation is true for levers in general. For the nail puller, the MA is certainly greater than one. The longer the handle on the nail puller, the greater the force you can exert with it.

Two other types of levers that differ slightly from the nail puller are a wheelbarrow and a shovel, shown in [\[link\]](#). All these lever types are similar in that only three forces are involved – the input force, the output force, and the force on the pivot – and thus their MAs are given by $\text{MA} = \frac{F_o}{F_i}$ and $\text{MA} = \frac{d_1}{d_2}$, with distances being measured relative to the physical pivot. The wheelbarrow and shovel differ from the nail puller because both the input and output forces are on the same side of the pivot.

In the case of the wheelbarrow, the output force or load is between the pivot (the wheel's axle) and the input or applied force. In the case of the shovel, the input force is between the pivot (at the end of the handle) and the load, but the input lever arm is shorter than the output lever arm. In this case, the MA is less than one.



(a) In the case of the wheelbarrow, the output force or load is between the pivot and the input force. The pivot is the wheel's axle. Here, the output force is greater than the input force.

Thus, a wheelbarrow enables you to lift much heavier loads than you could with your body alone. (b) In the case of the shovel, the input force is between the pivot and the load, but the input lever arm is shorter than the output lever arm. The pivot is at the handle held by the right hand. Here, the output force

(supporting the shovel's load) is less than the input force (from the hand nearest the load), because the input is exerted closer to the pivot than is the output.

Example:**What is the Advantage for the Wheelbarrow?**

In the wheelbarrow of [\[link\]](#), the load has a perpendicular lever arm of 7.50 cm, while the hands have a perpendicular lever arm of 1.02 m. (a) What upward force must you exert to support the wheelbarrow and its load if their combined mass is 45.0 kg? (b) What force does the wheelbarrow exert on the ground?

Strategy

Here, we use the concept of mechanical advantage.

Solution

(a) In this case, $\frac{F_o}{F_i} = \frac{l_i}{l_o}$ becomes

Equation:

$$F_i = F_o \frac{l_o}{l_i}.$$

Adding values into this equation yields

Equation:

$$F_i = (45.0 \text{ kg}) \left(9.80 \text{ m/s}^2 \right) \frac{0.075 \text{ m}}{1.02 \text{ m}} = 32.4 \text{ N}.$$

The free-body diagram (see [\[link\]](#)) gives the following normal force:

$F_i + N = W$. Therefore, $N = (45.0 \text{ kg}) \left(9.80 \text{ m/s}^2 \right) - 32.4 \text{ N} = 409 \text{ N}$

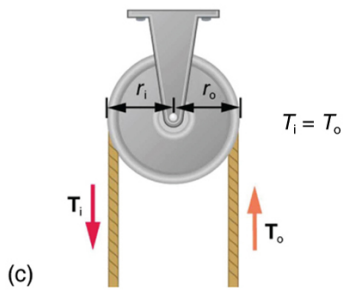
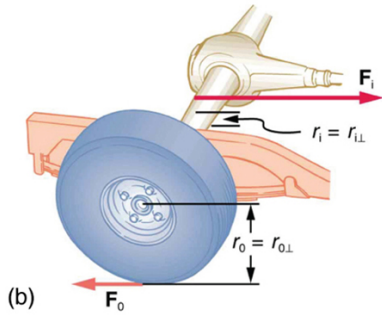
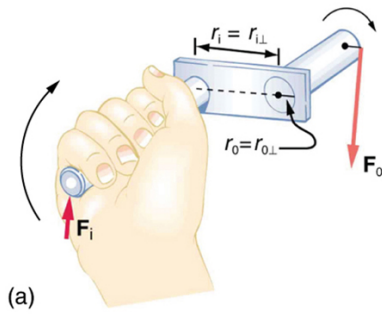
. N is the normal force acting on the wheel; by Newton's third law, the force the wheel exerts on the ground is 409 N.

Discussion

An even longer handle would reduce the force needed to lift the load. The MA here is $MA = 1.02/0.0750 = 13.6$.

Another very simple machine is the inclined plane. Pushing a cart up a plane is easier than lifting the same cart straight up to the top using a ladder, because the applied force is less. However, the work done in both cases (assuming the work done by friction is negligible) is the same. Inclined lanes or ramps were probably used during the construction of the Egyptian pyramids to move large blocks of stone to the top.

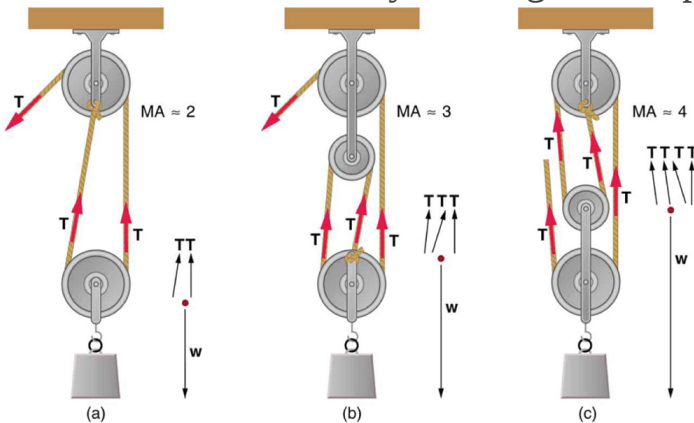
A crank is a lever that can be rotated 360° about its pivot, as shown in [\[link\]](#). Such a machine may not look like a lever, but the physics of its actions remain the same. The MA for a crank is simply the ratio of the radii r_i/r_0 . Wheels and gears have this simple expression for their MAs too. The MA can be greater than 1, as it is for the crank, or less than 1, as it is for the simplified car axle driving the wheels, as shown. If the axle's radius is 2.0 cm and the wheel's radius is 24.0 cm, then $MA = 2.0/24.0 = 0.083$ and the axle would have to exert a force of 12,000 N on the wheel to enable it to exert a force of 1000 N on the ground.



(a) A crank is a type of lever that can be rotated 360° about its pivot. Cranks are usually designed to have a large MA. (b) A simplified automobile axle drives a wheel, which has a much larger diameter than the axle. The MA is less than 1.

(c) An ordinary pulley is used to lift a heavy load. The pulley changes the direction of the force T exerted by the cord without changing its magnitude. Hence, this machine has an MA of 1.

An ordinary pulley has an MA of 1; it only changes the direction of the force and not its magnitude. Combinations of pulleys, such as those illustrated in [\[link\]](#), are used to multiply force. If the pulleys are friction-free, then the force output is approximately an integral multiple of the tension in the cable. The number of cables pulling directly upward on the system of interest, as illustrated in the figures given below, is approximately the MA of the pulley system. Since each attachment applies an external force in approximately the same direction as the others, they add, producing a total force that is nearly an integral multiple of the input force T .



(a) The combination of pulleys is used to multiply force. The force is an integral multiple of tension if the pulleys are frictionless. This pulley

system has two cables attached to its load, thus applying a force of approximately $2T$. This machine has $MA \approx 2$. (b) Three pulleys are used to lift a load in such a way that the mechanical advantage is about 3. Effectively, there are three cables attached to the load. (c) This pulley system applies a force of $4T$, so that it has $MA \approx 4$. Effectively, four cables are pulling on the system of interest.

Section Summary

- Simple machines are devices that can be used to multiply or augment a force that we apply – often at the expense of a distance through which we have to apply the force.
- The ratio of output to input forces for any simple machine is called its mechanical advantage
- A few simple machines are the lever, nail puller, wheelbarrow, crank, etc.

Conceptual Questions

Exercise:

Problem:

Scissors are like a double-lever system. Which of the simple machines in [\[link\]](#) and [\[link\]](#) is most analogous to scissors?

Exercise:

Problem:

Suppose you pull a nail at a constant rate using a nail puller as shown in [\[link\]](#). Is the nail puller in equilibrium? What if you pull the nail with some acceleration – is the nail puller in equilibrium then? In which case is the force applied to the nail puller larger and why?

Exercise:**Problem:**

Why are the forces exerted on the outside world by the limbs of our bodies usually much smaller than the forces exerted by muscles inside the body?

Exercise:**Problem:**

Explain why the forces in our joints are several times larger than the forces we exert on the outside world with our limbs. Can these forces be even greater than muscle forces (see previous Question)?

Problems & Exercises**Exercise:****Problem:**

What is the mechanical advantage of a nail puller—similar to the one shown in [\[link\]](#)—where you exert a force 45 cm from the pivot and the nail is 1.8 cm on the other side? What minimum force must you exert to apply a force of 1250 N to the nail?

Solution:

25

50 N

Exercise:**Problem:**

Suppose you needed to raise a 250-kg mower a distance of 6.0 cm above the ground to change a tire. If you had a 2.0-m long lever, where would you place the fulcrum if your force was limited to 300 N?

Exercise:**Problem:**

a) What is the mechanical advantage of a wheelbarrow, such as the one in [\[link\]](#), if the center of gravity of the wheelbarrow and its load has a perpendicular lever arm of 5.50 cm, while the hands have a perpendicular lever arm of 1.02 m? (b) What upward force should you exert to support the wheelbarrow and its load if their combined mass is 55.0 kg? (c) What force does the wheel exert on the ground?

Solution:

- a) $MA = 18.5$
- b) $F_i = 29.1 \text{ N}$
- c) 510 N downward

Exercise:**Problem:**

A typical car has an axle with 1.10 cm radius driving a tire with a radius of 27.5 cm. What is its mechanical advantage assuming the very simplified model in [\[link\]](#)(b)?

Exercise:**Problem:**

What force does the nail puller in [\[link\]](#) exert on the supporting surface? The nail puller has a mass of 2.10 kg.

Solution:

$$1.3 \times 10^3 \text{ N}$$

Exercise:**Problem:**

If you used an ideal pulley of the type shown in [\[link\]](#)(a) to support a car engine of mass 115 kg, (a) What would be the tension in the rope? (b) What force must the ceiling supply, assuming you pull straight down on the rope? Neglect the pulley system's mass.

Exercise:**Problem:**

Repeat [\[link\]](#) for the pulley shown in [\[link\]](#)(c), assuming you pull straight up on the rope. The pulley system's mass is 7.00 kg.

Solution:

a) $T = 299 \text{ N}$

b) 897 N upward

Test Prep for AP Courses

Exercise:**Problem:**

As a young student, you likely learned that simple machines are capable of increasing the ability to lift and move objects. Now, as an educated AP Physics student, you are aware that this capability is governed by the relationship between force and torque.

In the space below, explain why torque is integral to the increase in force created by a simple machine. You may use an example or

diagram to assist in your explanation. Be sure to cite the mechanical advantage in your explanation as well.

Solution:

The student should mention that the guiding principle behind simple machines is the second condition of equilibrium. Though the torque leaving a machine must be equivalent to torque entering a machine, the same requirement does not exist for forces. As a result, by decreasing the lever arm to the existing force, the size of the existing force will be increased. The mechanical advantage will be equivalent to the ratio of the forces exiting and entering the machine.

Exercise:

Problem:

[Figure 9.24\(a\)](#) shows a wheelbarrow being lifted by an applied force F_i . If the wheelbarrow is filled with twenty bricks massing 3 kg each, estimate the value of the applied force F_i . Provide an explanation behind the total weight w and any reasoning toward your final answer. Additionally, provide a range of values over which you feel the force could exist.

Glossary

mechanical advantage

the ratio of output to input forces for any simple machine

Forces and Torques in Muscles and Joints

LEARNING OBJECTIVES

By the end of this section, you will be able to:

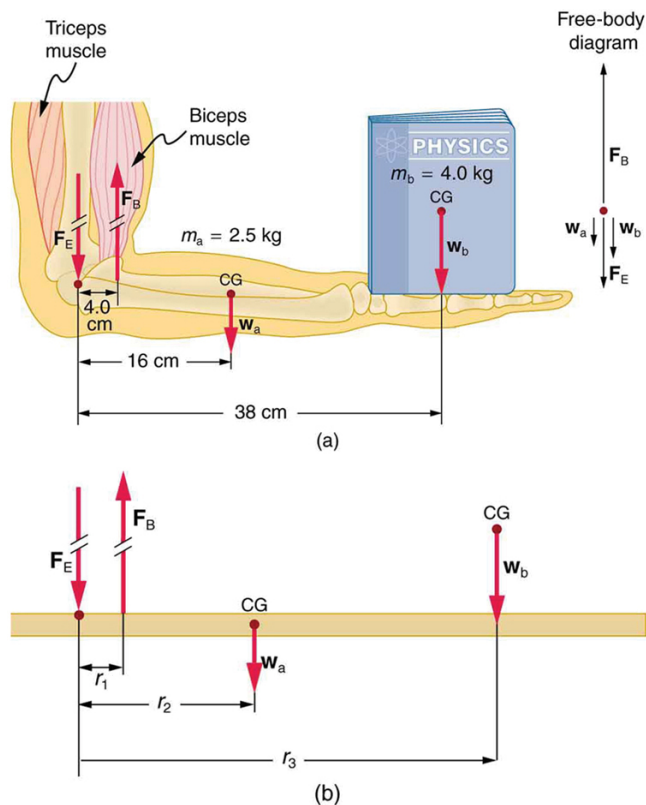
- Explain the forces exerted by muscles.
- State how a bad posture causes back strain.
- Discuss the benefits of skeletal muscles attached close to joints.
- Discuss various complexities in the real system of muscles, bones, and joints.

The information presented in this section supports the following AP[®] learning objectives and science practices:

- **3.F.1.1** The student is able to use representations of the relationship between force and torque. (S.P. 1.4)
- **3.F.1.2** The student is able to compare the torques on an object caused by various forces. (S.P. 1.4)
- **3.F.1.3** The student is able to estimate the torque on an object caused by various forces in comparison to other situations. (S.P. 4.1, 4.2, 5.1)
- **3.F.1.5** The student is able to calculate torques on a two-dimensional system in static equilibrium, by examining a representation or model (such as a diagram or physical construction). (S.P. 1.4, 2.2)

Muscles, bones, and joints are some of the most interesting applications of statics. There are some surprises. Muscles, for example, exert far greater forces than we might think. [\[link\]](#) shows a forearm holding a book and a schematic diagram of an analogous lever system. The schematic is a good approximation for the forearm, which looks more complicated than it is, and we can get some insight into the way typical muscle systems function by analyzing it.

Muscles can only contract, so they occur in pairs. In the arm, the biceps muscle is a flexor—that is, it closes the limb. The triceps muscle is an extensor that opens the limb. This configuration is typical of skeletal muscles, bones, and joints in humans and other vertebrates. Most skeletal muscles exert much larger forces within the body than the limbs apply to the outside world. The reason is clear once we realize that most muscles are attached to bones via tendons close to joints, causing these systems to have mechanical advantages much less than one. Viewing them as simple machines, the input force is much greater than the output force, as seen in [\[link\]](#).



(a) The figure shows the forearm of a person holding a book. The biceps exert a force F_B to support the weight of the forearm and the book. The triceps are assumed to be relaxed.

(b) Here, you can view an approximately equivalent mechanical system with the pivot at the elbow joint as seen in [\[link\]](#).

Example:

Muscles Exert Bigger Forces Than You Might Think

Calculate the force the biceps muscle must exert to hold the forearm and its load as shown in [\[link\]](#), and compare this force with the weight of the forearm plus its load. You may take the data in the figure to be accurate to three significant figures.

Strategy

There are four forces acting on the forearm and its load (the system of interest). The magnitude of the force of the biceps is F_B ; that of the elbow joint is F_E ; that of the weights of the forearm is w_a , and its load is w_b . Two of these are unknown (F_B and F_E), so that the first condition for equilibrium cannot by itself yield F_B . But if we use the second condition and choose the pivot to be at the elbow, then the torque due to F_E is zero, and the only unknown becomes F_B .

Solution

The torques created by the weights are clockwise relative to the pivot, while the torque created by the biceps is counterclockwise; thus, the second condition for equilibrium (net $\tau = 0$) becomes

Equation:

$$r_2 w_a + r_3 w_b = r_1 F_B.$$

Note that $\sin \theta = 1$ for all forces, since $\theta = 90^\circ$ for all forces. This equation can easily be solved for F_B in terms of known quantities, yielding

Equation:

$$F_B = \frac{r_2 w_a + r_3 w_b}{r_1}.$$

Entering the known values gives

Equation:

$$F_B = \frac{(0.160 \text{ m})(2.50 \text{ kg})(9.80 \text{ m/s}^2) + (0.380 \text{ m})(4.00 \text{ kg})(9.80 \text{ m/s}^2)}{0.0400 \text{ m}}$$

which yields

Equation:

$$F_B = 470 \text{ N}.$$

Now, the combined weight of the arm and its load is $(6.50 \text{ kg})(9.80 \text{ m/s}^2) = 63.7 \text{ N}$, so that the ratio of the force exerted by the biceps to the total weight is

Equation:

$$\frac{F_B}{w_a + w_b} = \frac{470}{63.7} = 7.38.$$

Discussion

This means that the biceps muscle is exerting a force 7.38 times the weight supported.

In the above example of the biceps muscle, the angle between the forearm and upper arm is 90° . If this angle changes, the force exerted by the biceps muscle also changes. In addition, the length of the biceps muscle changes. The force the biceps muscle can exert depends upon its length; it is smaller when it is shorter than when it is stretched.

Very large forces are also created in the joints. In the previous example, the downward force F_E exerted by the humerus at the elbow joint equals 407 N, or 6.38 times the total weight supported. (The calculation of F_E is straightforward and is left as an end-of-chapter problem.) Because muscles can contract, but not expand beyond their resting length, joints and muscles

often exert forces that act in opposite directions and thus subtract. (In the above example, the upward force of the muscle minus the downward force of the joint equals the weight supported—that is, $470\text{ N} - 407\text{ N} = 63\text{ N}$, approximately equal to the weight supported.) Forces in muscles and joints are largest when their load is a long distance from the joint, as the book is in the previous example.

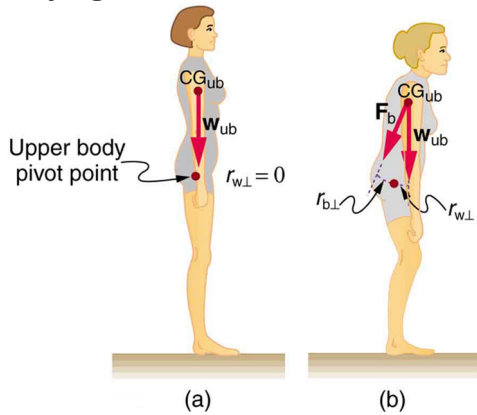
In racquet sports such as tennis the constant extension of the arm during game play creates large forces in this way. The mass times the lever arm of a tennis racquet is an important factor, and many players use the heaviest racquet they can handle. It is no wonder that joint deterioration and damage to the tendons in the elbow, such as “tennis elbow,” can result from repetitive motion, undue torques, and possibly poor racquet selection in such sports. Various tried techniques for holding and using a racquet or bat or stick not only increases sporting prowess but can minimize fatigue and long-term damage to the body. For example, tennis balls correctly hit at the “sweet spot” on the racquet will result in little vibration or impact force being felt in the racquet and the body—less torque as explained in [Collisions of Extended Bodies in Two Dimensions](#). Twisting the hand to provide top spin on the ball or using an extended rigid elbow in a backhand stroke can also aggravate the tendons in the elbow.

Training coaches and physical therapists use the knowledge of relationships between forces and torques in the treatment of muscles and joints. In physical therapy, an exercise routine can apply a particular force and torque which can, over a period of time, revive muscles and joints. Some exercises are designed to be carried out under water, because this requires greater forces to be exerted, further strengthening muscles. However, connecting tissues in the limbs, such as tendons and cartilage as well as joints are sometimes damaged by the large forces they carry. Often, this is due to accidents, but heavily muscled athletes, such as weightlifters, can tear muscles and connecting tissue through effort alone.

The back is considerably more complicated than the arm or leg, with various muscles and joints between vertebrae, all having mechanical advantages less than 1. Back muscles must, therefore, exert very large forces, which are borne by the spinal column. Discs crushed by mere exertion are very common. The jaw is somewhat exceptional—the masseter muscles that close the jaw have a mechanical advantage greater than 1 for the back teeth, allowing us to exert very large forces with them. A cause of stress headaches is persistent clenching of teeth where the sustained large force translates into fatigue in muscles around the skull.

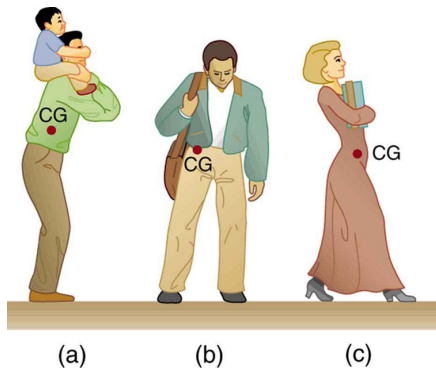
[\[link\]](#) shows how bad posture causes back strain. In part (a), we see a person with good posture. Note that her upper body's cg is directly above the pivot point in the hips, which in turn is directly above the base of support at her feet. Because of this, her upper body's weight exerts no torque about the hips. The only force needed is a vertical force at the hips equal to the weight supported. No muscle action is required, since the bones are rigid and transmit this force from the floor. This is a position of unstable equilibrium, but only small forces are needed to bring the upper body back to vertical if it is slightly displaced. Bad posture is shown in part (b); we see that the upper body's cg is in front of the pivot in the hips. This creates a clockwise torque around the hips that is counteracted by muscles in the lower back. These muscles must exert large forces, since they have typically small mechanical advantages. (In other words, the perpendicular lever arm for the muscles is much smaller than for the cg.) Poor posture can also cause muscle strain for people sitting at their desks using computers. Special

chairs are available that allow the body's CG to be more easily situated above the seat, to reduce back pain. Prolonged muscle action produces muscle strain. Note that the cg of the entire body is still directly above the base of support in part (b) of [\[link\]](#). This is compulsory; otherwise the person would not be in equilibrium. We lean forward for the same reason when carrying a load on our backs, to the side when carrying a load in one arm, and backward when carrying a load in front of us, as seen in [\[link\]](#).



(a) Good posture places the upper body's cg over the pivots in the hips, eliminating the need for muscle action to balance the body. (b) Poor posture requires exertion by the back muscles to counteract the clockwise torque produced around the pivot by the upper body's weight. The back muscles have a small effective perpendicular lever arm, $r_{b\perp}$, and must therefore exert a large force F_b . Note that the legs lean backward to keep the cg of the entire body above the base of support in the feet.

You have probably been warned against lifting objects with your back. This action, even more than bad posture, can cause muscle strain and damage discs and vertebrae, since abnormally large forces are created in the back muscles and spine.



People adjust their stance to maintain balance. (a) A father carrying his son piggyback leans forward to position their overall cg above the base of support at his feet. (b) A student carrying a shoulder bag leans to the side to keep the overall cg over his feet. (c) Another student carrying a load of books in her arms leans backward for the same reason.

Example:

Do Not Lift with Your Back

Consider the person lifting a heavy box with his back, shown in [\[link\]](#). (a) Calculate the magnitude of the force F_B in the back muscles that is needed to support the upper body plus the box and compare this with his weight. The mass of the upper body is 55.0 kg and the mass of the box is 30.0 kg. (b) Calculate the magnitude and direction of the force F_V exerted by the vertebrae on the spine at the indicated pivot point. Again, data in the figure may be taken to be accurate to three significant figures.

Strategy

By now, we sense that the second condition for equilibrium is a good place to start, and inspection of the known values confirms that it can be used to solve for F_B if the pivot is chosen to be at the hips. The torques created by \mathbf{w}_{ub} and \mathbf{w}_{box} are clockwise, while that created by \mathbf{F}_B is counterclockwise.

Solution for (a)

Using the perpendicular lever arms given in the figure, the second condition for equilibrium (net $\tau = 0$) becomes

Equation:

$$(0.350 \text{ m})(55.0 \text{ kg})(9.80 \text{ m/s}^2) + (0.500 \text{ m})(30.0 \text{ kg})(9.80 \text{ m/s}^2) = (0.0800 \text{ m})F_B.$$

Solving for F_B yields

Equation:

$$F_B = 4.20 \times 10^3 \text{ N}.$$

The ratio of the force the back muscles exert to the weight of the upper body plus its load is

Equation:

$$\frac{F_B}{w_{\text{ub}} + w_{\text{box}}} = \frac{4200 \text{ N}}{833 \text{ N}} = 5.04.$$

This force is considerably larger than it would be if the load were not present.

Solution for (b)

More important in terms of its damage potential is the force on the vertebrae \mathbf{F}_V . The first condition for equilibrium (net $\mathbf{F} = 0$) can be used to find its magnitude and direction. Using y for vertical and x for horizontal, the condition for the net external forces along those axes to be zero

Equation:

$$\text{net } F_y = 0 \text{ and net } F_x = 0.$$

Starting with the vertical (y) components, this yields

Equation:

$$F_{Vy} - w_{\text{ub}} - w_{\text{box}} - F_B \sin 29.0^\circ = 0.$$

Thus,

Equation:

$$\begin{aligned} F_{Vy} &= w_{\text{ub}} + w_{\text{box}} + F_B \sin 29.0^\circ \\ &= 833 \text{ N} + (4200 \text{ N}) \sin 29.0^\circ \end{aligned}$$

yielding

Equation:

$$F_{Vy} = 2.87 \times 10^3 \text{ N}.$$

Similarly, for the horizontal (x) components,

Equation:

$$F_{Vx} - F_B \cos 29.0^\circ = 0$$

yielding

Equation:

$$F_{Vx} = 3.67 \times 10^3 \text{ N.}$$

The magnitude of \mathbf{F}_V is given by the Pythagorean theorem:

Equation:

$$F_V = \sqrt{F_{Vx}^2 + F_{Vy}^2} = 4.66 \times 10^3 \text{ N.}$$

The direction of \mathbf{F}_V is

Equation:

$$\theta = \tan^{-1} \left(\frac{F_{Vy}}{F_{Vx}} \right) = 38.0^\circ.$$

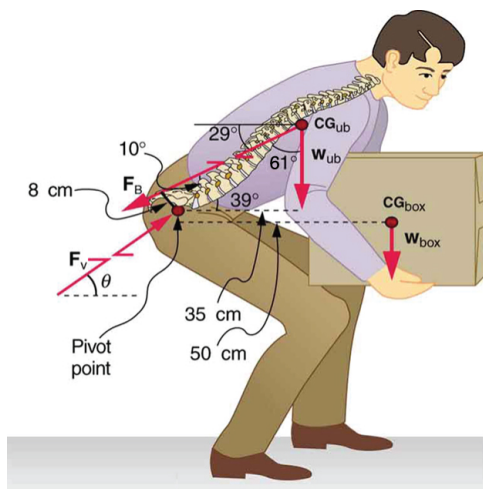
Note that the ratio of F_V to the weight supported is

Equation:

$$\frac{F_V}{w_{ub} + w_{box}} = \frac{4660 \text{ N}}{833 \text{ N}} = 5.59.$$

Discussion

This force is about 5.6 times greater than it would be if the person were standing erect. The trouble with the back is not so much that the forces are large—because similar forces are created in our hips, knees, and ankles—but that our spines are relatively weak. Proper lifting, performed with the back erect and using the legs to raise the body and load, creates much smaller forces in the back—in this case, about 5.6 times smaller.



This figure shows that large forces are exerted by the back muscles and experienced in the

vertebrae when a person lifts with their back, since these muscles have small effective perpendicular lever arms. The data shown here are analyzed in the preceding example, [\[link\]](#).

What are the benefits of having most skeletal muscles attached so close to joints? One advantage is speed because small muscle contractions can produce large movements of limbs in a short period of time. Other advantages are flexibility and agility, made possible by the large numbers of joints and the ranges over which they function. For example, it is difficult to imagine a system with biceps muscles attached at the wrist that would be capable of the broad range of movement we vertebrates possess.

There are some interesting complexities in real systems of muscles, bones, and joints. For instance, the pivot point in many joints changes location as the joint is flexed, so that the perpendicular lever arms and the mechanical advantage of the system change, too. Thus the force the biceps muscle must exert to hold up a book varies as the forearm is flexed. Similar mechanisms operate in the legs, which explain, for example, why there is less leg strain when a bicycle seat is set at the proper height. The methods employed in this section give a reasonable description of real systems provided enough is known about the dimensions of the system. There are many other interesting examples of force and torque in the body—a few of these are the subject of end-of-chapter problems.

Section Summary

- Statics plays an important part in understanding everyday strains in our muscles and bones.
- Many lever systems in the body have a mechanical advantage of significantly less than one, as many of our muscles are attached close to joints.
- Someone with good posture stands or sits in such a way that the person's center of gravity lies directly above the pivot point in the hips, thereby avoiding back strain and damage to disks.

Conceptual Questions

Exercise:

Problem:

Why are the forces exerted on the outside world by the limbs of our bodies usually much smaller than the forces exerted by muscles inside the body?

Exercise:

Problem:

Explain why the forces in our joints are several times larger than the forces we exert on the outside world with our limbs. Can these forces be even greater than muscle forces?

Exercise:**Problem:**

Certain types of dinosaurs were bipedal (walked on two legs). What is a good reason that these creatures invariably had long tails if they had long necks?

Exercise:**Problem:**

Swimmers and athletes during competition need to go through certain postures at the beginning of the race. Consider the balance of the person and why start-offs are so important for races.

Exercise:**Problem:**

If the maximum force the biceps muscle can exert is 1000 N, can we pick up an object that weighs 1000 N? Explain your answer.

Exercise:**Problem:**

Suppose the biceps muscle was attached through tendons to the upper arm close to the elbow and the forearm near the wrist. What would be the advantages and disadvantages of this type of construction for the motion of the arm?

Exercise:**Problem:**

Explain one of the reasons why pregnant women often suffer from back strain late in their pregnancy.

Problems & Exercises**Exercise:**

Problem: Verify that the force in the elbow joint in [\[link\]](#) is 407 N, as stated in the text.

Solution:

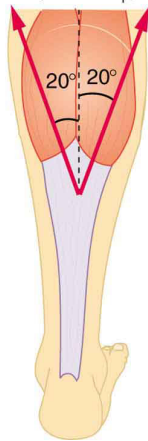
$$\begin{aligned}
 F_B &= 470 \text{ N}; r_1 = 4.00 \text{ cm}; w_a = 2.50 \text{ kg}; r_2 = 16.0 \text{ cm}; w_b = 4.00 \text{ kg}; r_3 = 38.0 \text{ cm} \\
 F_E &= w_a \left(\frac{r_2}{r_1} - 1 \right) + w_b \left(\frac{r_3}{r_1} - 1 \right) \\
 &= (2.50 \text{ kg})(9.80 \text{ m/s}^2) \left(\frac{16.0 \text{ cm}}{4.0 \text{ cm}} - 1 \right) \\
 &\quad + (4.00 \text{ kg})(9.80 \text{ m/s}^2) \left(\frac{38.0 \text{ cm}}{4.00 \text{ cm}} - 1 \right) \\
 &= 407 \text{ N}
 \end{aligned}$$

Exercise:

Problem:

Two muscles in the back of the leg pull on the Achilles tendon as shown in [\[link\]](#). What total force do they exert?

$F_2(200 \text{ N})$ $F_1(200 \text{ N})$

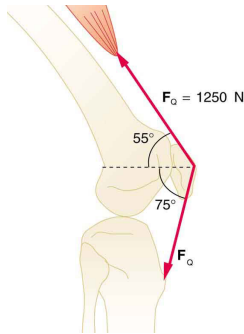


The Achilles tendon of the posterior leg serves to attach plantaris, gastrocnemius, and soleus muscles to calcaneus bone.

Exercise:

Problem:

The upper leg muscle (quadriceps) exerts a force of 1250 N, which is carried by a tendon over the kneecap (the patella) at the angles shown in [\[link\]](#). Find the direction and magnitude of the force exerted by the kneecap on the upper leg bone (the femur).



The knee joint works like a hinge to bend and straighten the lower leg. It permits a person to sit, stand, and pivot.

Solution:

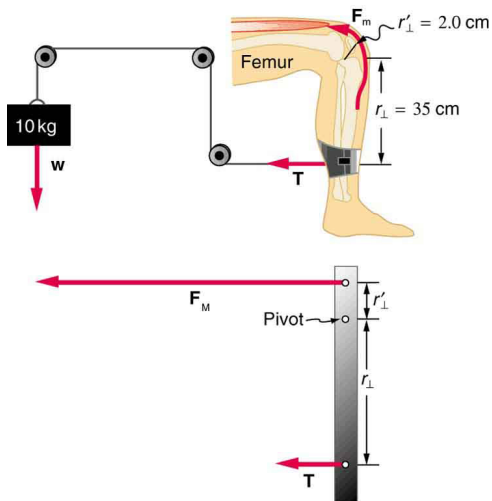
$$1.1 \times 10^3 \text{ N}$$

$$\theta = 190^\circ \text{ ccw from positive } x \text{ axis}$$

Exercise:

Problem:

A device for exercising the upper leg muscle is shown in [\[link\]](#), together with a schematic representation of an equivalent lever system. Calculate the force exerted by the upper leg muscle to lift the mass at a constant speed. Explicitly show how you follow the steps in the Problem-Solving Strategy for static equilibrium in [Applications of Statistics, Including Problem-Solving Strategies](#).

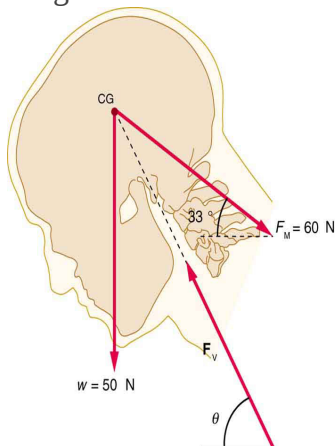


A mass is connected by pulleys and wires to the ankle in this exercise device.

Exercise:

Problem:

A person working at a drafting board may hold her head as shown in [\[link\]](#), requiring muscle action to support the head. The three major acting forces are shown. Calculate the direction and magnitude of the force supplied by the upper vertebrae \mathbf{F}_V to hold the head stationary, assuming that this force acts along a line through the center of mass as do the weight and muscle force.



Solution:

$$F_V = 97 \text{ N}, \theta = 59^\circ$$

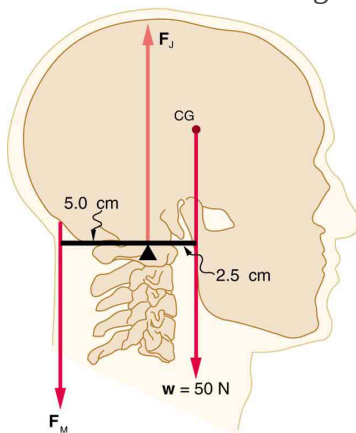
Exercise:

Problem:

We analyzed the biceps muscle example with the angle between forearm and upper arm set at 90° . Using the same numbers as in [\[link\]](#), find the force exerted by the biceps muscle when the angle is 120° and the forearm is in a downward position.

Exercise:**Problem:**

Even when the head is held erect, as in [\[link\]](#), its center of mass is not directly over the principal point of support (the atlanto-occipital joint). The muscles at the back of the neck should therefore exert a force to keep the head erect. That is why your head falls forward when you fall asleep in the class. (a) Calculate the force exerted by these muscles using the information in the figure. (b) What is the force exerted by the pivot on the head?



The center of mass of the head lies in front of its major point of support, requiring muscle action to hold the head erect. A simplified lever system is shown.

Solution:

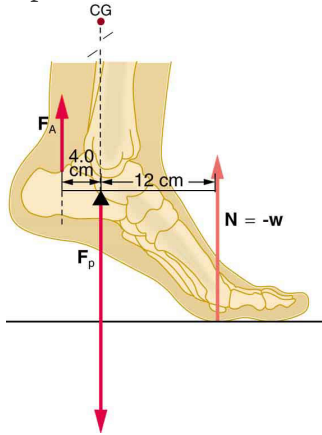
(a) 25 N downward

(b) 75 N upward

Exercise:

Problem:

A 75-kg man stands on his toes by exerting an upward force through the Achilles tendon, as in [\[link\]](#). (a) What is the force in the Achilles tendon if he stands on one foot? (b) Calculate the force at the pivot of the simplified lever system shown—that force is representative of forces in the ankle joint.



The muscles in the back of the leg pull the Achilles tendon when one stands on one's toes. A simplified lever system is shown.

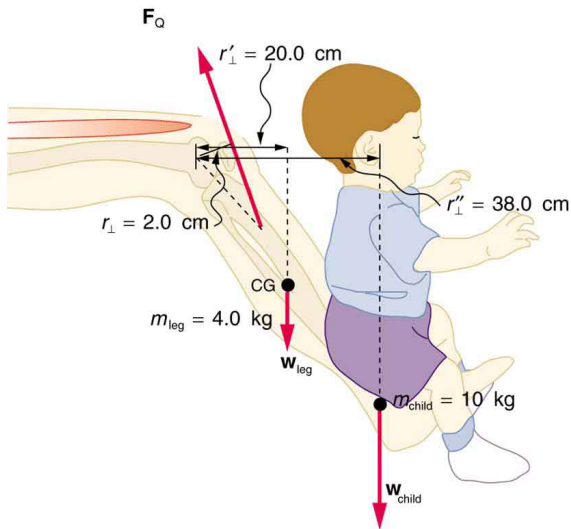
Solution:

(a) $F_A = 2.21 \times 10^3 \text{ N}$ upward

(b) $F_B = 2.94 \times 10^3 \text{ N}$ downward

Exercise:**Problem:**

A father lifts his child as shown in [\[link\]](#). What force should the upper leg muscle exert to lift the child at a constant speed?

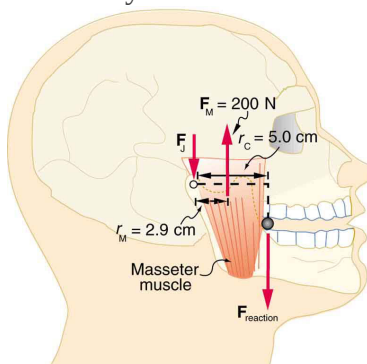


A child being lifted by a father's lower leg.

Exercise:

Problem:

Unlike most of the other muscles in our bodies, the masseter muscle in the jaw, as illustrated in [\[link\]](#), is attached relatively far from the joint, enabling large forces to be exerted by the back teeth. (a) Using the information in the figure, calculate the force exerted by the lower teeth on the bullet. (b) Calculate the force on the joint.



A person clenching a bullet between his teeth.

Solution:

(a) $F_{\text{teeth on bullet}} = 1.2 \times 10^2 \text{ N}$ upward

(b) $F_J = 84 \text{ N}$ downward

Exercise:

Problem: Integrated Concepts

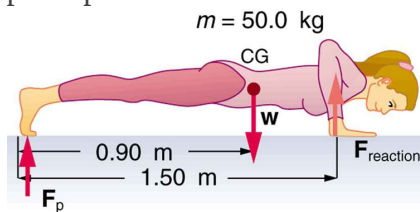
Suppose we replace the 4.0-kg book in [\[link\]](#) of the biceps muscle with an elastic exercise rope that obeys Hooke's Law. Assume its force constant $k = 600 \text{ N/m}$. (a) How much is the rope stretched (past equilibrium) to provide the same force F_B as in this example?

Assume the rope is held in the hand at the same location as the book. (b) What force is on the biceps muscle if the exercise rope is pulled straight up so that the forearm makes an angle of 25° with the horizontal? Assume the biceps muscle is still perpendicular to the forearm.

Exercise:

Problem:

(a) What force should the woman in [\[link\]](#) exert on the floor with each hand to do a push-up? Assume that she moves up at a constant speed. (b) The triceps muscle at the back of her upper arm has an effective lever arm of 1.75 cm, and she exerts force on the floor at a horizontal distance of 20.0 cm from the elbow joint. Calculate the magnitude of the force in each triceps muscle, and compare it to her weight. (c) How much work does she do if her center of mass rises 0.240 m? (d) What is her useful power output if she does 25 pushups in one minute?



A woman doing pushups.

Solution:

(a) 147 N downward

(b) 1680 N, 3.4 times her weight

(c) 118 J

(d) 49.0 W

Exercise:

Problem:

You have just planted a sturdy 2-m-tall palm tree in your front lawn for your mother's birthday. Your brother kicks a 500 g ball, which hits the top of the tree at a speed of 5 m/s and stays in contact with it for 10 ms. The ball falls to the ground near the base of the tree and the recoil of the tree is minimal. (a) What is the force on the tree? (b) The length of the sturdy section of the root is only 20 cm. Furthermore, the soil around the roots is loose and we can assume that an effective force is applied at the tip of the 20 cm length. What is the effective force exerted by the end of the tip of the root to keep the tree from toppling? Assume the tree will be uprooted rather than bend. (c) What could you have done to ensure that the tree does not uproot easily?

Exercise:**Problem: Unreasonable Results**

Suppose two children are using a uniform seesaw that is 3.00 m long and has its center of mass over the pivot. The first child has a mass of 30.0 kg and sits 1.40 m from the pivot. (a) Calculate where the second 18.0 kg child must sit to balance the seesaw. (b) What is unreasonable about the result? (c) Which premise is unreasonable, or which premises are inconsistent?

Solution:

a) $\bar{x}_2 = 2.33 \text{ m}$

b) The seesaw is 3.0 m long, and hence, there is only 1.50 m of board on the other side of the pivot. The second child is off the board.

c) The position of the first child must be shortened, i.e. brought closer to the pivot.

Exercise:**Problem: Construct Your Own Problem**

Consider a method for measuring the mass of a person's arm in anatomical studies. The subject lies on her back, extends her relaxed arm to the side and two scales are placed below the arm. One is placed under the elbow and the other under the back of her hand. Construct a problem in which you calculate the mass of the arm and find its center of mass based on the scale readings and the distances of the scales from the shoulder joint. You must include a free body diagram of the arm to direct the analysis. Consider changing the position of the scale under the hand to provide more information, if needed. You may wish to consult references to obtain reasonable mass values.

Test Prep for AP Courses

Exercise:**Problem:**

When you use your hand to raise a 20 lb dumbbell in a curling motion, the force on your bicep muscle is not equal to 20 lb.

- a. Compare the size of the force placed on your bicep muscle to the force of the 20 lb dumbbell lifted by your hand. Using the concept of torque, which force is greater and explain why the two forces are not identical.
- b. Does the force placed on your bicep muscle change as you curl the weight closer toward your body? (In other words, is the force on your muscle different when your forearm is 90° to your upper arm than when it is 45° to your upper arm?) Explain your answer using torque.

Solution:

- a. The force placed on your bicep muscle will be greater than the force placed on the dumbbell. The bicep muscle is closer to your elbow than the downward force placed on your hand from the dumbbell. Because the elbow is the pivot point of the system, this results in a decreased lever arm for the bicep. As a result, the force on the bicep must be greater than that placed on the dumbbell. (How much greater? The ratio between the bicep and dumbbell forces is equal to the inverted ratio of their distances from the elbow. If the dumbbell is ten times further from the elbow than the bicep, the force on the bicep will be 200 pounds!)
- b. The force placed on your bicep muscle will decrease. As the forearm lifts the dumbbell, it will get closer to the elbow. As a result, the torque placed on the arm from the weight will decrease and the countering torque created by the bicep muscle will do so as well.

Connection for AP® Courses
class="introduction"

The mention of
a tornado
conjures up
images of raw
destructive
power.

Tornadoes
blow houses
away as if they
were made of
paper and have
been known to
pierce tree
trunks with
pieces of straw.

They descend
from clouds in
funnel-like
shapes that spin
violently,
particularly at
the bottom
where they are
most narrow,
producing
winds as high
as 500 km/h.

(credit: Daphne
Zaras, U.S.

National
Oceanic and
Atmospheric
Administration

)



Why do tornados spin? And why do tornados spin so rapidly? The answer is that the air masses that produce tornados are themselves rotating, and when the radii of the air masses decrease, their rate of rotation increases. An ice skater increases her spin in an exactly analogous manner, as seen in [Figure 10.2](#). The skater starts her rotation with outstretched limbs and increases her rate of spin by pulling them in toward her body. The same physics describes the exhilarating spin of a skater and the wrenching force of a tornado. We will find that this is another example of the importance of conservation laws and their role in determining how changes happen in a system, supporting Big Idea 5. The idea that a change of a conserved quantity is always equal to the transfer of that quantity between interacting systems (Enduring Understanding 5.A) is presented for both energy and angular momentum (Enduring Understanding 5.E). The conservation of **angular momentum** in relation to the external net **torque** (Essential Knowledge 5.E.1) parallels that of linear momentum conservation in relation to the external net force. The concept of **rotational inertia** is introduced, a concept that takes into account not only the mass of an object or a system, but also the distribution of mass within the object or system. Therefore, changes in the rotational inertia of a system could lead to changes in the motion (Essential Knowledge 5.E.2) of the system. We shall see that all important aspects of

rotational motion either have already been defined for linear motion or have exact analogues in linear motion.

Clearly, therefore, force, energy, and power are associated with rotational motion. This supports Big Idea 3, that interactions are described by forces. The ability of forces to cause torques (Enduring Understanding 3.F) is extended to the interactions between objects that result in nonzero net torque. This nonzero net torque in turn causes changes in the rotational motion of an object (Essential Knowledge 3.F.2) and results in changes of the angular momentum of an object (Essential Knowledge 3.F.3).

Similarly, Big Idea 4, that interactions between systems cause changes in those systems, is supported by the empirical observation that when torques are exerted on rigid bodies these torques cause changes in the angular momentum of the system (Enduring Understanding 4.D).

Again, there is a clear analogy between linear and rotational motion in this interaction. Both the angular kinematics variables (angular displacement, angular velocity, and **angular acceleration**) and the dynamics variables (torque and angular momentum) are vectors with direction depending on whether the rotation is clockwise or counterclockwise with respect to an axis of rotation (Essential Knowledge 4.D.1). The angular momentum of the system can change due to interactions (Essential Knowledge 4.D.2). This change is defined as the product of the average torque and the time interval during which torque is exerted (Essential Knowledge 4.D.3), analogous to the impulse-momentum theorem for linear motion.

The concepts in this chapter support:

Big Idea 3. The interactions of an object with other objects can be described by forces.

Enduring Understanding 3.F. A force exerted on an object can cause a torque on that object.

Extended Knowledge 3.F.2. The presence of a net torque along any axis will cause a rigid system to change its rotational motion or an object to change its rotational motion about that axis.

Extended Knowledge 3.F.3. A torque exerted on an object can change the angular momentum of an object.

Big Idea 4. Interactions between systems can result in changes in those systems.

Enduring Understanding 4.D. A net torque exerted on a system by other objects or systems will change the angular momentum of the system.

Extended Knowledge 4.D.1. Torque, angular velocity, angular acceleration, and angular momentum are vectors and can be characterized as positive or negative depending upon whether they give rise to or correspond to counterclockwise or clockwise rotation with respect to an axis.

Extended Knowledge 4.D.2. The angular momentum of a system may change due to interactions with other objects or systems.

Extended Knowledge 4.D.3. The change in angular momentum is given by the product of the average torque and the time interval during which the torque is exerted.

Big Idea 5. Changes that occur as a result of interactions are constrained by conservation laws.

Enduring Understanding 5.A. Certain quantities are conserved, in the sense that the changes of those quantities in a given system are always equal to the transfer of that quantity to or from the system by all possible interactions with other systems.

Extended Knowledge 5.A.2. For all systems under all circumstances, energy, charge, linear momentum, and angular momentum are conserved.

Enduring Understanding 5.E. The angular momentum of a system is conserved.

Extended Knowledge 5.E.1. If the net external torque exerted on the system is zero, the angular momentum of the system does not change.

Extended Knowledge 5.E.2. The angular momentum of a system is determined by the locations and velocities of the objects that make up the system. The rotational inertia of an object or system depends upon the distribution of mass within the object or system. Changes in the radius of a system or in the distribution of mass within the system result in changes in the system's rotational inertia, and hence in its angular velocity and linear speed for a given angular momentum. Examples should include elliptical orbits in an Earth-satellite system. Mathematical expressions for the moments of inertia will be provided where needed. Students will not be expected to know the parallel axis theorem.



This figure skater increases her rate of spin by pulling her arms and her extended leg closer to her axis of rotation.

(credit: Luu, Wikimedia Commons)

Angular Acceleration

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Describe uniform circular motion.
- Explain nonuniform circular motion.
- Calculate angular acceleration of an object.
- Observe the link between linear and angular acceleration.

[Uniform Circular Motion and Gravitation](#) discussed only uniform circular motion, which is motion in a circle at constant speed and, hence, constant angular velocity. Recall that angular velocity ω was defined as the time rate of change of angle θ :

Equation:

$$\omega = \frac{\Delta\theta}{\Delta t},$$

where θ is the angle of rotation as seen in [\[link\]](#). The relationship between angular velocity ω and linear velocity v was also defined in [Rotation Angle and Angular Velocity](#) as

Equation:

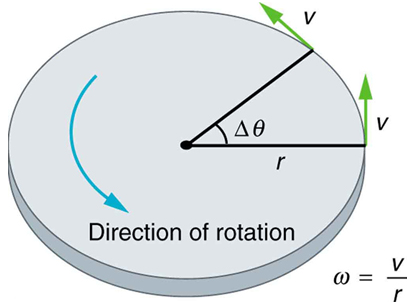
$$v = r\omega$$

or

Equation:

$$\omega = \frac{v}{r},$$

where r is the radius of curvature, also seen in [\[link\]](#). According to the sign convention, the counter clockwise direction is considered as positive direction and clockwise direction as negative



This figure shows uniform circular motion and some of its defined quantities.

Angular velocity is not constant when a skater pulls in her arms, when a child starts up a merry-go-round from rest, or when a computer's hard disk slows to a halt when switched off. In all these cases, there is an **angular acceleration**, in which ω changes. The faster the change occurs, the greater the angular acceleration. Angular acceleration α is defined as the rate of change of angular velocity. In equation form, angular acceleration is expressed as follows:

Equation:

$$\alpha = \frac{\Delta\omega}{\Delta t},$$

where $\Delta\omega$ is the **change in angular velocity** and Δt is the change in time. The units of angular acceleration are (rad/s)/s, or rad/s^2 . If ω increases, then α is positive. If ω decreases, then α is negative.

Example:

Calculating the Angular Acceleration and Deceleration of a Bike Wheel

Suppose a teenager puts her bicycle on its back and starts the rear wheel spinning from rest to a final frequency of 250 rpm in 5.00 s. (a) Calculate the angular acceleration in rad/s^2 . (b) If she now slams on the brakes, causing an angular acceleration of -87.3 rad/s^2 , how long does it take the wheel to stop?

Strategy for (a)

The angular acceleration can be found directly from its definition in $\alpha = \frac{\Delta\omega}{\Delta t}$ because the final frequency and time are given. We see that $\Delta\omega$ is the product of $2\pi \text{ rad/rev}$ and is 250 rpm, and Δt is 5.00 s.

Solution for (a)

First, calculate the change in angular velocity:

Equation:

$$\begin{aligned}\Delta\omega &= 250 \frac{\text{rev}}{\text{min}} \cdot \frac{2\pi \text{ rad}}{\text{rev}} \cdot \frac{1 \text{ min}}{60 \text{ sec}} \\ &= 26.2 \frac{\text{rad}}{\text{s}}.\end{aligned}$$

Entering this quantity into the expression for α , we get

Equation:

$$\begin{aligned}\alpha &= \frac{\Delta\omega}{\Delta t} \\ &= \frac{26.2 \text{ rad/s}}{5.00 \text{ s}} \\ &= 5.24 \text{ rad/s}^2.\end{aligned}$$

Strategy for (b)

In this part, we know the angular acceleration and the initial angular velocity. We can find the stoppage time by using the definition of angular

acceleration and solving for Δt , yielding

Equation:

$$\Delta t = \frac{\Delta \omega}{\alpha}.$$

Solution for (b)

Here the angular velocity decreases from 26.2 rad/s to zero, so that $\Delta \omega$ is -26.2 rad/s, and α is given to be -87.3 rad/s². Thus,

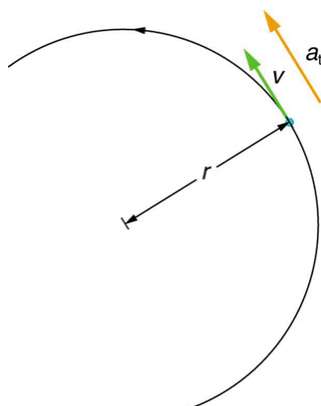
Equation:

$$\begin{aligned}\Delta t &= \frac{-26.2 \text{ rad/s}}{-87.3 \text{ rad/s}^2} \\ &= 0.300 \text{ s}.\end{aligned}$$

Discussion

Note that the angular acceleration as the girl spins the wheel is small and positive; it takes 5 s to produce an appreciable angular velocity. When she hits the brake, the angular acceleration is large and negative. The angular velocity quickly goes to zero. In both cases, the relationships are analogous to what happens with linear motion. For example, there is a large deceleration when you crash into a brick wall—the velocity change is large in a short time interval.

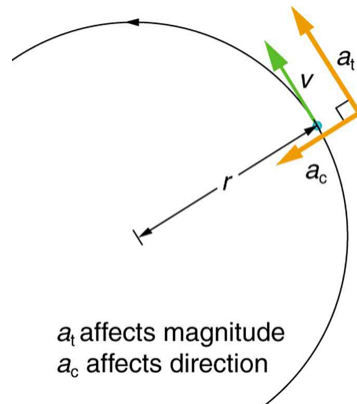
If the bicycle in the preceding example had been on its wheels instead of upside-down, it would first have accelerated along the ground and then come to a stop. This connection between circular motion and linear motion needs to be explored. For example, it would be useful to know how linear and angular acceleration are related. In circular motion, linear acceleration is *tangent* to the circle at the point of interest, as seen in [\[link\]](#). Thus, linear acceleration is called **tangential acceleration** a_t .



In circular motion, linear acceleration a , occurs as the magnitude of the velocity changes: a is tangent to the motion. In the context of circular motion, linear acceleration is also called tangential acceleration a_t .

Linear or tangential acceleration refers to changes in the magnitude of velocity but not its direction. We know from [Uniform Circular Motion and Gravitation](#) that in circular motion centripetal acceleration, a_c , refers to changes in the direction of the velocity but not its magnitude. An object undergoing circular motion experiences centripetal acceleration, as seen in [\[link\]](#). Thus, a_t and a_c are perpendicular and independent of one another.

Tangential acceleration a_t is directly related to the angular acceleration α and is linked to an increase or decrease in the velocity, but not its direction.



Centripetal acceleration a_c occurs as the direction of velocity changes; it is perpendicular to the circular motion.

Centripetal and tangential acceleration are thus perpendicular to each other.

Now we can find the exact relationship between linear acceleration a_t and angular acceleration α . Because linear acceleration is proportional to a change in the magnitude of the velocity, it is defined (as it was in [One-Dimensional Kinematics](#)) to be

Equation:

$$a_t = \frac{\Delta v}{\Delta t}.$$

For circular motion, note that $v = r\omega$, so that

Equation:

$$a_t = \frac{\Delta(r\omega)}{\Delta t}.$$

The radius r is constant for circular motion, and so $\Delta(r\omega) = r(\Delta\omega)$. Thus,

Equation:

$$a_t = r \frac{\Delta\omega}{\Delta t}.$$

By definition, $\alpha = \frac{\Delta\omega}{\Delta t}$. Thus,

Equation:

$$a_t = r\alpha,$$

or

Equation:

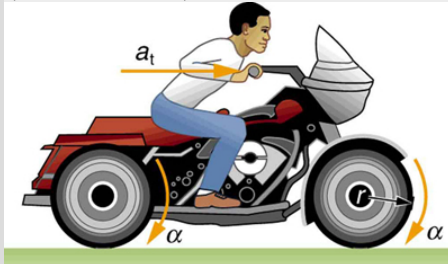
$$\alpha = \frac{a_t}{r}.$$

These equations mean that linear acceleration and angular acceleration are directly proportional. The greater the angular acceleration is, the larger the linear (tangential) acceleration is, and vice versa. For example, the greater the angular acceleration of a car's drive wheels, the greater the acceleration of the car. The radius also matters. For example, the smaller a wheel, the smaller its linear acceleration for a given angular acceleration α .

Example:**Calculating the Angular Acceleration of a Motorcycle Wheel**

A powerful motorcycle can accelerate from 0 to 30.0 m/s (about 108 km/h) in 4.20 s. What is the angular acceleration of its 0.320-m-radius wheels?

(See [\[link\]](#).)



The linear acceleration of a motorcycle is accompanied by an angular acceleration of its wheels.

Strategy

We are given information about the linear velocities of the motorcycle. Thus, we can find its linear acceleration a_t . Then, the expression $\alpha = \frac{a_t}{r}$ can be used to find the angular acceleration.

Solution

The linear acceleration is

Equation:

$$\begin{aligned} a_t &= \frac{\Delta v}{\Delta t} \\ &= \frac{30.0 \text{ m/s}}{4.20 \text{ s}} \\ &= 7.14 \text{ m/s}^2. \end{aligned}$$

We also know the radius of the wheels. Entering the values for a_t and r into $\alpha = \frac{a_t}{r}$, we get

Equation:

$$\begin{aligned}
 \alpha &= \frac{a_t}{r} \\
 &= \frac{7.14 \text{ m/s}^2}{0.320 \text{ m}} \\
 &= 22.3 \text{ rad/s}^2.
 \end{aligned}$$

Discussion

Units of radians are dimensionless and appear in any relationship between angular and linear quantities.

So far, we have defined three rotational quantities— θ , ω , and α . These quantities are analogous to the translational quantities x , v , and a . [\[link\]](#) displays rotational quantities, the analogous translational quantities, and the relationships between them.

Rotational	Translational	Relationship
θ	x	$\theta = \frac{x}{r}$
ω	v	$\omega = \frac{v}{r}$
α	a	$\alpha = \frac{a_t}{r}$

Rotational and Translational Quantities

Note:**Making Connections: Take-Home Experiment**

Sit down with your feet on the ground on a chair that rotates. Lift one of your legs such that it is unbent (straightened out). Using the other leg, begin to rotate yourself by pushing on the ground. Stop using your leg to push the ground but allow the chair to rotate. From the origin where you began, sketch the angle, angular velocity, and angular acceleration of your leg as a function of time in the form of three separate graphs. Estimate the magnitudes of these quantities.

Exercise:**Check Your Understanding****Problem:**

Angular acceleration is a vector, having both magnitude and direction. How do we denote its magnitude and direction? Illustrate with an example.

Solution:

The magnitude of angular acceleration is α and its most common units are rad/s^2 . The direction of angular acceleration along a fixed axis is denoted by a + or a – sign, just as the direction of linear acceleration in one dimension is denoted by a + or a – sign. For example, consider a gymnast doing a forward flip. Her angular momentum would be parallel to the mat and to her left. The magnitude of her angular acceleration would be proportional to her angular velocity (spin rate) and her moment of inertia about her spin axis.

Note:**PhET Explorations: Ladybug Revolution**

Join the ladybug in an exploration of rotational motion. Rotate the merry-go-round to change its angle, or choose a constant angular velocity or

angular acceleration. Explore how circular motion relates to the bug's x,y position, velocity, and acceleration using vectors or graphs.

[Ladybug](#)
[Revolutio](#)
[n](#)

Section Summary

- Uniform circular motion is the motion with a constant angular velocity $\omega = \frac{\Delta\theta}{\Delta t}$.
- In non-uniform circular motion, the velocity changes with time and the rate of change of angular velocity (i.e. angular acceleration) is $\alpha = \frac{\Delta\omega}{\Delta t}$.
- Linear or tangential acceleration refers to changes in the magnitude of velocity but not its direction, given as $a_t = \frac{\Delta v}{\Delta t}$.
- For circular motion, note that $v = r\omega$, so that

Equation:

$$a_t = \frac{\Delta(r\omega)}{\Delta t}.$$

- The radius r is constant for circular motion, and so $\Delta(r\omega) = r\Delta\omega$. Thus,

Equation:

$$a_t = r \frac{\Delta\omega}{\Delta t}.$$

- By definition, $\Delta\omega/\Delta t = \alpha$. Thus,

Equation:

$$a_t = r\alpha$$

or

Equation:

$$\alpha = \frac{a_t}{r}.$$

Conceptual Questions

Exercise:

Problem:

Analogies exist between rotational and translational physical quantities. Identify the rotational term analogous to each of the following: acceleration, force, mass, work, translational kinetic energy, linear momentum, impulse.

Exercise:

Problem:

Explain why centripetal acceleration changes the direction of velocity in circular motion but not its magnitude.

Exercise:

Problem:

In circular motion, a tangential acceleration can change the magnitude of the velocity but not its direction. Explain your answer.

Exercise:

Problem:

Suppose a piece of food is on the edge of a rotating microwave oven plate. Does it experience nonzero tangential acceleration, centripetal acceleration, or both when: (a) The plate starts to spin? (b) The plate rotates at constant angular velocity? (c) The plate slows to a halt?

Problems & Exercises**Exercise:****Problem:**

At its peak, a tornado is 60.0 m in diameter and carries 500 km/h winds. What is its angular velocity and the equivalent revolutions per second??

Solution:

$$\omega = 0.737 \text{ rev/s}$$

Exercise:**Problem: Integrated Concepts**

An ultracentrifuge accelerates from rest to 100,000 rpm in 2.00 min. (a) What is its angular acceleration in rad/s^2 ? (b) What is the tangential acceleration of a point 9.50 cm from the axis of rotation? (c) What is the radial acceleration in m/s^2 and multiples of g of this point at full rpm?

Exercise:**Problem: Integrated Concepts**

You have a grindstone (a disk) that is 90.0 kg, has a 0.340-m radius, and is turning at 90.0 rpm, and you press a steel axe against it with a

radial force of 20.0 N. (a) Assuming the kinetic coefficient of friction between steel and stone is 0.20, calculate the angular acceleration of the grindstone. (b) How many turns will the stone make before coming to rest?

Solution:

(a) -0.26 rad/s^2

(b) 27 rev

Exercise:

Problem: Unreasonable Results

You are told that a basketball player spins the ball with an angular acceleration of 100 rad/s^2 . (a) What is the ball's final angular velocity if the ball starts from rest and the acceleration lasts 2.00 s? (b) What is unreasonable about the result? (c) Which premises are unreasonable or inconsistent?

Glossary

angular acceleration

the rate of change of angular velocity with time

change in angular velocity

the difference between final and initial values of angular velocity

tangential acceleration

the acceleration in a direction tangent to the circle at the point of interest in circular motion

Kinematics of Rotational Motion

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Observe the kinematics of rotational motion.
- Derive rotational kinematic equations.
- Evaluate problem solving strategies for rotational kinematics.

Just by using our intuition, we can begin to see how rotational quantities like θ , ω , and α are related to one another. For example, if a motorcycle wheel has a large angular acceleration for a fairly long time, it ends up spinning rapidly and rotates through many revolutions. In more technical terms, if the wheel's angular acceleration α is large for a long period of time t , then the final angular velocity ω and angle of rotation θ are large. The wheel's rotational motion is exactly analogous to the fact that the motorcycle's large translational acceleration produces a large final velocity, and the distance traveled will also be large.

Kinematics is the description of motion. The **kinematics of rotational motion** describes the relationships among rotation angle, angular velocity, angular acceleration, and time. Let us start by finding an equation relating ω , α , and t . To determine this equation, we recall a familiar kinematic equation for translational, or straight-line, motion:

Equation:

$$v = v_0 + at \quad (\text{constant } a)$$

Note that in rotational motion $a = a_t$, and we shall use the symbol a for tangential or linear acceleration from now on. As in linear kinematics, we assume a is constant, which means that angular acceleration α is also a constant, because $a = r\alpha$. Now, let us substitute $v = r\omega$ and $a = r\alpha$ into the linear equation above:

Equation:

$$r\omega = r\omega_0 + r\alpha t.$$

The radius r cancels in the equation, yielding

Equation:

$$\omega = \omega_0 + \alpha t \quad (\text{constant } \alpha),$$

where ω_0 is the initial angular velocity. This last equation is a *kinematic relationship* among ω , α , and t —that is, it describes their relationship without reference to forces or masses that may affect rotation. It is also precisely analogous in form to its translational counterpart.

Note:

Making Connections

Kinematics for rotational motion is completely analogous to translational kinematics, first presented in [One-Dimensional Kinematics](#). Kinematics is concerned with the description of motion without regard to force or mass. We will find that translational kinematic quantities, such as displacement, velocity, and acceleration have direct analogs in rotational motion.

Starting with the four kinematic equations we developed in [One-Dimensional Kinematics](#), we can derive the following four rotational kinematic equations (presented together with their translational counterparts):

Rotational	Translational	
$\theta = \omega t$	$x = \bar{v}t$	
$\omega = \omega_0 + \alpha t$	$v = v_0 + at$	(constant α, a)
$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$	$x = v_0 t + \frac{1}{2}at^2$	(constant α, a)
$\omega^2 = \omega_0^2 + 2\alpha\theta$	$v^2 = v_0^2 + 2ax$	(constant α, a)

Rotational Kinematic Equations

In these equations, the subscript 0 denotes initial values (θ_0 , x_0 , and t_0 are initial values), and the average angular velocity ω and average velocity \bar{v} are defined as follows:

Equation:

$$\omega = \frac{\omega_0 + \omega}{2} \text{ and } v = \frac{v_0 + v}{2}.$$

The equations given above in [\[link\]](#) can be used to solve any rotational or translational kinematics problem in which a and α are constant.

Note:

Problem-Solving Strategy for Rotational Kinematics

1. *Examine the situation to determine that rotational kinematics (rotational motion) is involved.* Rotation must be involved, but without the need to consider forces or masses that affect the motion.
2. *Identify exactly what needs to be determined in the problem (identify the unknowns).* A sketch of the situation is useful.
3. *Make a list of what is given or can be inferred from the problem as stated (identify the knowns).*
4. *Solve the appropriate equation or equations for the quantity to be determined (the unknown).* It can be useful to think in terms of a translational analog because by now you are familiar with such motion.
5. *Substitute the known values along with their units into the appropriate equation, and obtain numerical solutions complete with units.* Be sure to use units of radians for angles.
6. *Check your answer to see if it is reasonable: Does your answer make sense?*

Example:

Calculating the Acceleration of a Fishing Reel

A deep-sea fisherman hooks a big fish that swims away from the boat pulling the fishing line from his fishing reel. The whole system is initially at rest and the fishing line unwinds from the reel at a radius of 4.50 cm from its axis of rotation. The reel is given an angular acceleration of 110 rad/s^2 for 2.00 s as seen in [\[link\]](#).

- (a) What is the final angular velocity of the reel?
- (b) At what speed is fishing line leaving the reel after 2.00 s elapses?
- (c) How many revolutions does the reel make?
- (d) How many meters of fishing line come off the reel in this time?

Strategy

In each part of this example, the strategy is the same as it was for solving problems in linear kinematics. In particular, known values are identified and a relationship is then sought that can be used to solve for the unknown.

Solution for (a)

Here α and t are given and ω needs to be determined. The most straightforward equation to use is $\omega = \omega_0 + \alpha t$ because the unknown is already on one side and all other terms are known. That equation states that

Equation:

$$\omega = \omega_0 + \alpha t.$$

We are also given that $\omega_0 = 0$ (it starts from rest), so that

Equation:

$$\omega = 0 + (110 \text{ rad/s}^2)(2.00\text{s}) = 220 \text{ rad/s}.$$

Solution for (b)

Now that ω is known, the speed v can most easily be found using the relationship

Equation:

$$v = r\omega,$$

where the radius r of the reel is given to be 4.50 cm; thus,

Equation:

$$v = (0.0450 \text{ m})(220 \text{ rad/s}) = 9.90 \text{ m/s}.$$

Note again that radians must always be used in any calculation relating linear and angular quantities. Also, because radians are dimensionless, we have $\text{m} \times \text{rad} = \text{m}$.

Solution for (c)

Here, we are asked to find the number of revolutions. Because $1 \text{ rev} = 2\pi \text{ rad}$, we can find the number of revolutions by finding θ in radians. We are given α and t , and we know ω_0 is zero, so that θ can be obtained using $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$.

Equation:

$$\begin{aligned}\theta &= \omega_0 t + \frac{1}{2}\alpha t^2 \\ &= 0 + (0.500)\left(110 \text{ rad/s}^2\right)(2.00 \text{ s})^2 = 220 \text{ rad}.\end{aligned}$$

Converting radians to revolutions gives

Equation:

$$\theta = (220 \text{ rad}) \frac{1 \text{ rev}}{2\pi \text{ rad}} = 35.0 \text{ rev.}$$

Solution for (d)

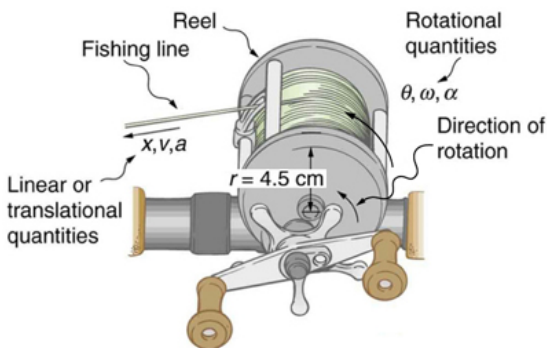
The number of meters of fishing line is x , which can be obtained through its relationship with θ :

Equation:

$$x = r\theta = (0.0450 \text{ m})(220 \text{ rad}) = 9.90 \text{ m.}$$

Discussion

This example illustrates that relationships among rotational quantities are highly analogous to those among linear quantities. We also see in this example how linear and rotational quantities are connected. The answers to the questions are realistic. After unwinding for two seconds, the reel is found to spin at 220 rad/s, which is 2100 rpm. (No wonder reels sometimes make high-pitched sounds.) The amount of fishing line played out is 9.90 m, about right for when the big fish bites.



Fishing line coming off a rotating reel moves linearly.

[\[link\]](#) and [\[link\]](#) consider relationships between rotational and linear

quantities associated with a fishing reel.

Example:

Calculating the Duration When the Fishing Reel Slows Down and Stops

Now let us consider what happens if the fisherman applies a brake to the spinning reel, achieving an angular acceleration of -300 rad/s^2 . How long does it take the reel to come to a stop?

Strategy

We are asked to find the time t for the reel to come to a stop. The initial and final conditions are different from those in the previous problem, which involved the same fishing reel. Now we see that the initial angular velocity is $\omega_0 = 220 \text{ rad/s}$ and the final angular velocity ω is zero. The angular acceleration is given to be $\alpha = -300 \text{ rad/s}^2$. Examining the available equations, we see all quantities but t are known in $\omega = \omega_0 + \alpha t$, making it easiest to use this equation.

Solution

The equation states

Equation:

$$\omega = \omega_0 + \alpha t.$$

We solve the equation algebraically for t , and then substitute the known values as usual, yielding

Equation:

$$t = \frac{\omega - \omega_0}{\alpha} = \frac{0 - 220 \text{ rad/s}}{-300 \text{ rad/s}^2} = 0.733 \text{ s}.$$

Discussion

Note that care must be taken with the signs that indicate the directions of various quantities. Also, note that the time to stop the reel is fairly small

because the acceleration is rather large. Fishing lines sometimes snap because of the accelerations involved, and fishermen often let the fish swim for a while before applying brakes on the reel. A tired fish will be slower, requiring a smaller acceleration.

Example:

Calculating the Slow Acceleration of Trains and Their Wheels

Large freight trains accelerate very slowly. Suppose one such train accelerates from rest, giving its 0.350-m-radius wheels an angular acceleration of 0.250 rad/s^2 . After the wheels have made 200 revolutions (assume no slippage): (a) How far has the train moved down the track? (b) What are the final angular velocity of the wheels and the linear velocity of the train?

Strategy

In part (a), we are asked to find x , and in (b) we are asked to find ω and v . We are given the number of revolutions, the radius of the wheels r , and the angular acceleration α .

Solution for (a)

The distance x is very easily found from the relationship between distance and rotation angle:

Equation:

$$\theta = \frac{x}{r}.$$

Solving this equation for x yields

Equation:

$$x = r\theta.$$

Before using this equation, we must convert the number of revolutions into radians, because we are dealing with a relationship between linear and rotational quantities:

Equation:

$$\theta = (200 \text{ rev}) \frac{2\pi \text{ rad}}{1 \text{ rev}} = 1257 \text{ rad}.$$

Now we can substitute the known values into $x = r\theta$ to find the distance the train moved down the track:

Equation:

$$x = r\theta = (0.350 \text{ m})(1257 \text{ rad}) = 440 \text{ m}.$$

Solution for (b)

We cannot use any equation that incorporates t to find ω , because the equation would have at least two unknown values. The equation $\omega^2 = \omega_0^2 + 2\alpha\theta$ will work, because we know the values for all variables except ω :

Equation:

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

Taking the square root of this equation and entering the known values gives

Equation:

$$\begin{aligned}\omega &= \left[0 + 2(0.250 \text{ rad/s}^2)(1257 \text{ rad}) \right]^{1/2} \\ &= 25.1 \text{ rad/s}.\end{aligned}$$

We can find the linear velocity of the train, v , through its relationship to ω :

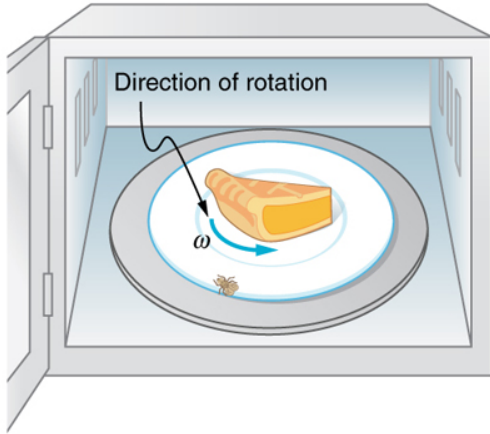
Equation:

$$v = r\omega = (0.350 \text{ m})(25.1 \text{ rad/s}) = 8.77 \text{ m/s}.$$

Discussion

The distance traveled is fairly large and the final velocity is fairly slow (just under 32 km/h).

There is translational motion even for something spinning in place, as the following example illustrates. [\[link\]](#) shows a fly on the edge of a rotating microwave oven plate. The example below calculates the total distance it travels.



The image shows a microwave plate. The fly makes revolutions while the food is heated (along with the fly).

Example:

Calculating the Distance Traveled by a Fly on the Edge of a Microwave Oven Plate

A person decides to use a microwave oven to reheat some lunch. In the process, a fly accidentally flies into the microwave and lands on the outer edge of the rotating plate and remains there. If the plate has a radius of 0.15 m and rotates at 6.0 rpm, calculate the total distance traveled by the fly during a 2.0-min cooking period. (Ignore the start-up and slow-down times.)

Strategy

First, find the total number of revolutions, and then the linear distance x traveled.

Solution

The number of revolutions is the product of rpm and time:

Equation:

$$(6.0 \text{ rpm})(2.0 \text{ min}) = 12 \text{ rev.}$$

Convert to radians to find

Equation:

$$\theta = (12 \text{ rev}) \left(\frac{2\pi \text{ rad}}{1 \text{ rev}} \right) = 75.4 \text{ rad.}$$

Now, using the relationship between x and θ , we can determine the distance traveled:

Equation:

$$x = r\theta = (0.15 \text{ m})(75.4 \text{ rad}) = 11 \text{ m.}$$

Discussion

Quite a trip (if it survives)! Note that this distance is the total distance traveled by the fly. Displacement is actually zero for complete revolutions because they bring the fly back to its original position. The distinction between total distance traveled and displacement was first noted in [One-Dimensional Kinematics](#).

Exercise:

Check Your Understanding

Problem:

Rotational kinematics has many useful relationships, often expressed in equation form. Are these relationships laws of physics or are they simply descriptive? (Hint: the same question applies to linear kinematics.)

Solution:

Rotational kinematics (just like linear kinematics) is descriptive and does not represent laws of nature. With kinematics, we can describe many things to great precision but kinematics does not consider causes. For example, a large angular acceleration describes a very rapid change in angular velocity without any consideration of its cause.

Section Summary

- Kinematics is the description of motion.
- The kinematics of rotational motion describes the relationships among rotation angle, angular velocity, angular acceleration, and time.
- Starting with the four kinematic equations we developed in the [One-Dimensional Kinematics](#), we can derive the four rotational kinematic equations (presented together with their translational counterparts) seen in [\[link\]](#).
- In these equations, the subscript 0 denotes initial values (x_0 and t_0 are initial values), and the average angular velocity ω and average velocity \bar{v} are defined as follows:

Equation:

$$\omega = \frac{\omega_0 + \omega}{2} \text{ and } v = \frac{v_0 + v}{2}.$$

Problems & Exercises

Exercise:

Problem:

With the aid of a string, a gyroscope is accelerated from rest to 32 rad/s in 0.40 s.

- (a) What is its angular acceleration in rad/s²?
- (b) How many revolutions does it go through in the process?

Solution:

(a) 80 rad/s^2

(b) 1.0 rev

Exercise:**Problem:**

Suppose a piece of dust finds itself on a CD. If the spin rate of the CD is 500 rpm, and the piece of dust is 4.3 cm from the center, what is the total distance traveled by the dust in 3 minutes? (Ignore accelerations due to getting the CD rotating.)

Exercise:**Problem:**

A gyroscope slows from an initial rate of 32.0 rad/s at a rate of 0.700 rad/s^2 .

(a) How long does it take to come to rest?

(b) How many revolutions does it make before stopping?

Solution:

(a) 45.7 s

(b) 116 rev

Exercise:

Problem: During a very quick stop, a car decelerates at 7.00 m/s^2 .

(a) What is the angular acceleration of its 0.280-m-radius tires, assuming they do not slip on the pavement?

- (b) How many revolutions do the tires make before coming to rest, given their initial angular velocity is 95.0 rad/s ?
- (c) How long does the car take to stop completely?
- (d) What distance does the car travel in this time?
- (e) What was the car's initial velocity?
- (f) Do the values obtained seem reasonable, considering that this stop happens very quickly?



Yo-yos are amusing toys that display significant physics and are engineered to enhance performance based on physical laws. (credit: Beyond Neon, Flickr)

Exercise:

Problem:

Everyday application: Suppose a yo-yo has a center shaft that has a 0.250 cm radius and that its string is being pulled.

- (a) If the string is stationary and the yo-yo accelerates away from it at a rate of 1.50 m/s^2 , what is the angular acceleration of the yo-yo?

(b) What is the angular velocity after 0.750 s if it starts from rest?

(c) The outside radius of the yo-yo is 3.50 cm. What is the tangential acceleration of a point on its edge?

Solution:

a) 600 rad/s^2

b) 450 rad/s

c) 21.0 m/s

Glossary

kinematics of rotational motion

describes the relationships among rotation angle, angular velocity, angular acceleration, and time

Dynamics of Rotational Motion: Rotational Inertia

LEARNING OBJECTIVES

By the end of this section, you will be able to:

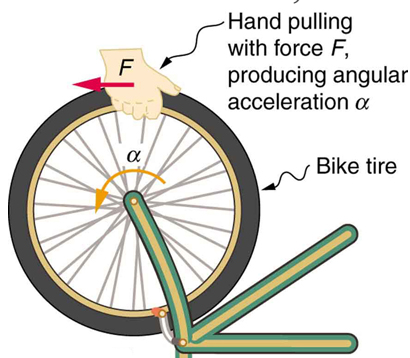
- Understand the relationship between force, mass, and acceleration.
- Study the turning effect of force.
- Study the analogy between force and torque, mass and moment of inertia, and linear acceleration and angular acceleration.

The information presented in this section supports the following AP® learning objectives and science practices:

- **4.D.1.1** The student is able to describe a representation and use it to analyze a situation in which several forces exerted on a rotating system of rigidly connected objects change the angular velocity and angular momentum of the system. (S.P. 1.2, 1.4)
- **4.D.1.2** The student is able to plan data collection strategies designed to establish that torque, angular velocity, angular acceleration, and angular momentum can be predicted accurately when the variables are treated as being clockwise or counterclockwise with respect to a well-defined axis of rotation, and refine the research question based on the examination of data. (S.P. 3.2, 4.1, 5.1, 5.3)
- **5.E.2.1** The student is able to describe or calculate the angular momentum and rotational inertia of a system in terms of the locations and velocities of objects that make up the system. Students are expected to do qualitative reasoning with compound objects. Students are expected to do calculations

with a fixed set of extended objects and point masses. (S.P. 2.2)

If you have ever spun a bike wheel or pushed a merry-go-round, you know that force is needed to change angular velocity as seen in [\[link\]](#). In fact, your intuition is reliable in predicting many of the factors that are involved. For example, we know that a door opens slowly if we push too close to its hinges. Furthermore, we know that the more massive the door, the more slowly it opens. The first example implies that the farther the force is applied from the pivot, the greater the angular acceleration; another implication is that angular acceleration is inversely proportional to mass. These relationships should seem very similar to the familiar relationships among force, mass, and acceleration embodied in Newton's second law of motion. There are, in fact, precise rotational analogs to both force and mass.



Force is required to spin the bike wheel.

The greater the force, the greater the angular acceleration produced. The more massive the wheel, the smaller the angular

acceleration. If you
push on a spoke
closer to the axle,
the angular
acceleration will be
smaller.

To develop the precise relationship among force, mass, radius, and angular acceleration, consider what happens if we exert a force F on a point mass m that is at a distance r from a pivot point, as shown in [\[link\]](#). Because the force is perpendicular to r , an acceleration $a = \frac{F}{m}$ is obtained in the direction of F . We can rearrange this equation such that $F = ma$ and then look for ways to relate this expression to expressions for rotational quantities. We note that $a = r\alpha$, and we substitute this expression into $F = ma$, yielding

Equation:

$$F = mr\alpha.$$

Recall that **torque** is the turning effectiveness of a force. In this case, because \mathbf{F} is perpendicular to r , torque is simply $\tau = Fr$. So, if we multiply both sides of the equation above by r , we get torque on the left-hand side. That is,

Equation:

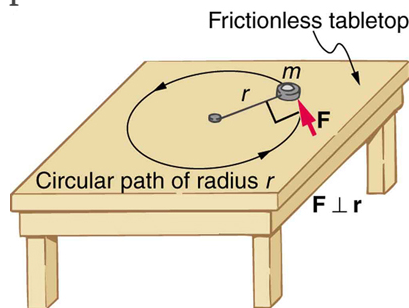
$$rF = mr^2\alpha$$

or

Equation:

$$\tau = mr^2\alpha.$$

This last equation is the rotational analog of Newton's second law ($F = ma$), where torque is analogous to force, angular acceleration is analogous to translational acceleration, and mr^2 is analogous to mass (or inertia). The quantity mr^2 is called the **rotational inertia** or **moment of inertia** of a point mass m a distance r from the center of rotation.



An object is supported by a horizontal frictionless table and is attached to a pivot point by a cord that supplies centripetal force. A force F is applied to the object perpendicular to the radius r , causing it to accelerate about the pivot point. The force is kept perpendicular to r .

Note:

Making Connections: Rotational Motion Dynamics

Dynamics for rotational motion is completely analogous to linear or translational dynamics. Dynamics is concerned with force and mass and their effects on motion. For rotational motion, we will find direct analogs to force and mass that behave just as we would expect from our earlier experiences.

Rotational Inertia and Moment of Inertia

Before we can consider the rotation of anything other than a point mass like the one in [\[link\]](#), we must extend the idea of rotational inertia to all types of objects. To expand our concept of rotational inertia, we define the **moment of inertia** I of an object to be the sum of mr^2 for all the point masses of which it is composed. That is, $I = \sum mr^2$. Here I is analogous to m in translational motion. Because of the distance r , the moment of inertia for any object depends on the chosen axis. Actually, calculating I is beyond the scope of this text except for one simple case—that of a hoop, which has all its mass at the same distance from its axis. A hoop's moment of inertia around its axis is therefore MR^2 , where M is its total mass and R its radius. (We use M and R for an entire object to distinguish them from m and r for point masses.) In all other cases, we must consult [\[link\]](#) (note that the table is piece of artwork that has shapes as well as formulae) for formulas for I that have been derived from integration over the continuous body. Note that I has units of mass multiplied by distance squared ($\text{kg} \cdot \text{m}^2$), as we might expect from its definition.

The general relationship among torque, moment of inertia, and angular acceleration is

Equation:

$$\text{net } \tau = I\alpha$$

or

Equation:

$$\alpha = \frac{\text{net } \tau}{I},$$

where net τ is the total torque from all forces relative to a chosen axis. For simplicity, we will only consider torques exerted by forces in the plane of the rotation. Such torques are either positive or negative and add like ordinary numbers. The relationship in $\tau = I\alpha$, $\alpha = \frac{\text{net } \tau}{I}$ is the rotational analog to Newton's second law and is very generally applicable. This equation is actually valid for *any* torque, applied to *any* object, relative to *any* axis.

As we might expect, the larger the torque is, the larger the angular acceleration is. For example, the harder a child pushes on a merry-go-round, the faster it accelerates. Furthermore, the more massive a merry-go-round, the slower it accelerates for the same torque. The basic relationship between moment of inertia and angular acceleration is that the larger the moment of inertia, the smaller is the angular acceleration. But there is an additional twist. The moment of inertia depends not only on the mass of an object, but also on its *distribution* of mass relative to the axis around which it rotates. For example, it will be much easier to accelerate a merry-go-round full of children if they stand close to its axis than if they all stand at the outer edge. The mass is the same in both cases; but the moment of inertia is much larger when the children are at the edge.

Note:

Take-Home Experiment

Cut out a circle that has about a 10 cm radius from stiff cardboard. Near the edge of the circle, write numbers 1 to 12 like hours on a clock face. Position the circle so that it can rotate freely about a horizontal axis through its center, like a wheel. (You could loosely nail the circle to a wall.) Hold the circle stationary and with the number 12 positioned at the top, attach a lump of blue putty (sticky material used for fixing posters to walls) at the number 3. How large does the lump need to be to just rotate the circle? Describe how you can change the moment of inertia of the

circle. How does this change affect the amount of blue putty needed at the number 3 to just rotate the circle? Change the circle's moment of inertia and then try rotating the circle by using different amounts of blue putty. Repeat this process several times.

In what direction did the circle rotate when you added putty at the number 3 (clockwise or counterclockwise)? In which of these directions was the resulting angular velocity? Was the angular velocity constant? What can we say about the direction (clockwise or counterclockwise) of the angular acceleration? How could you change the placement of the putty to create angular velocity in the opposite direction?

Note:

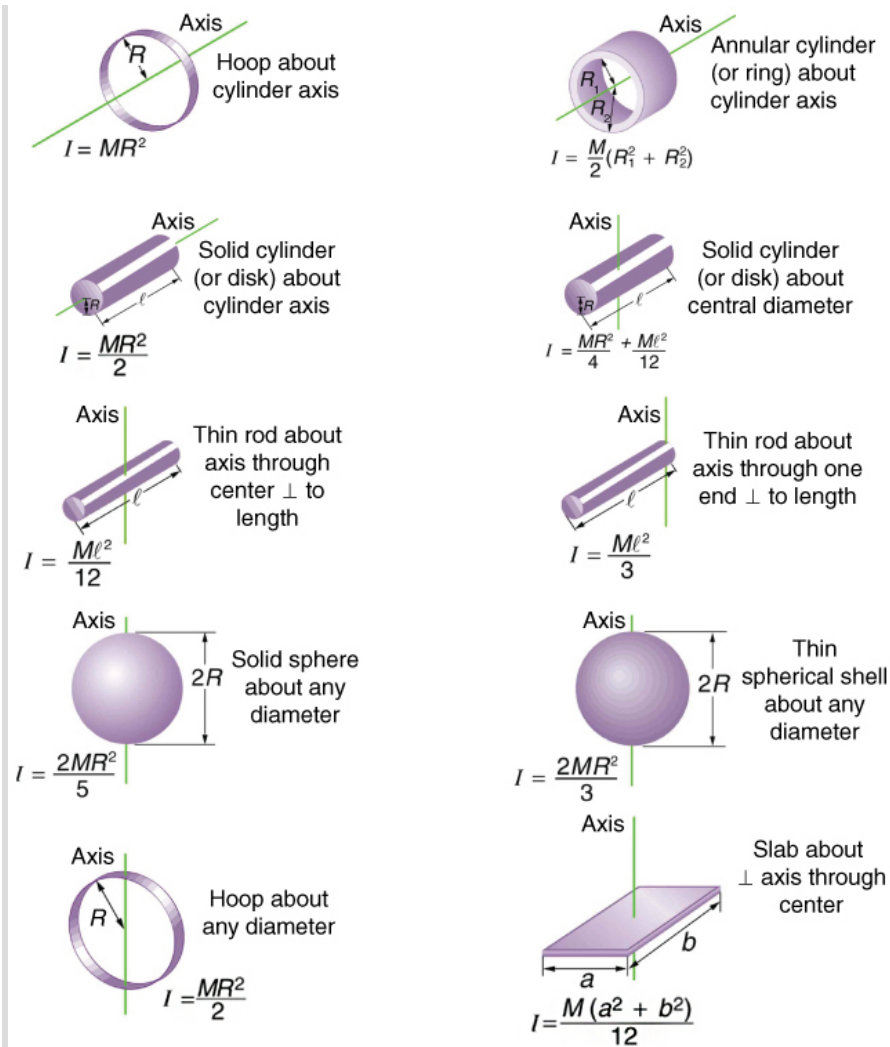
Problem-Solving Strategy for Rotational Dynamics

1. *Examine the situation to determine that torque and mass are involved in the rotation. Draw a careful sketch of the situation.*
2. *Determine the system of interest.*
3. *Draw a free body diagram. That is, draw and label all external forces acting on the system of interest.*
4. *Apply $\text{net } \tau = I\alpha$, $\alpha = \frac{\text{net } \tau}{I}$, the rotational equivalent of Newton's second law, to solve the problem. Care must be taken to use the correct moment of inertia and to consider the torque about the point of rotation.*
5. *As always, check the solution to see if it is reasonable.*

Note:

Making Connections

In statics, the net torque is zero, and there is no angular acceleration. In rotational motion, net torque is the cause of angular acceleration, exactly as in Newton's second law of motion for rotation.



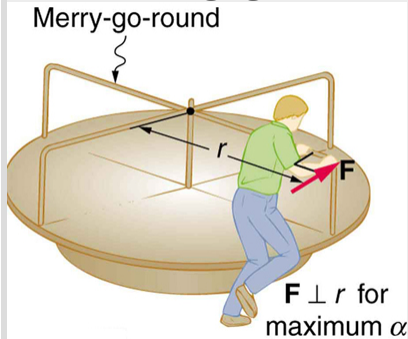
Some rotational inertias.

Example:

Calculating the Effect of Mass Distribution on a Merry-Go-Round

Consider the father pushing a playground merry-go-round in [\[link\]](#). He exerts a force of 250 N at the edge of the 50.0-kg merry-go-round, which has a 1.50 m radius. Calculate the angular acceleration produced (a) when no one is on the merry-go-round and (b) when an 18.0-kg child sits 1.25 m

away from the center. Consider the merry-go-round itself to be a uniform disk with negligible retarding friction.



A father pushes a playground merry-go-round at its edge and perpendicular to its radius to achieve maximum torque.

Strategy

Angular acceleration is given directly by the expression $\alpha = \frac{\text{net } \tau}{I}$:

Equation:

$$\alpha = \frac{\tau}{I}.$$

To solve for α , we must first calculate the torque τ (which is the same in both cases) and moment of inertia I (which is greater in the second case). To find the torque, we note that the applied force is perpendicular to the radius and friction is negligible, so that

Equation:

$$\tau = rF \sin \theta = (1.50 \text{ m})(250 \text{ N}) = 375 \text{ N} \cdot \text{m}.$$

Solution for (a)

The moment of inertia of a solid disk about this axis is given in [\[link\]](#) to be

Equation:

$$\frac{1}{2}MR^2,$$

where $M = 50.0 \text{ kg}$ and $R = 1.50 \text{ m}$, so that

Equation:

$$I = (0.500)(50.0 \text{ kg})(1.50 \text{ m})^2 = 56.25 \text{ kg} \cdot \text{m}^2.$$

Now, after we substitute the known values, we find the angular acceleration to be

Equation:

$$\alpha = \frac{\tau}{I} = \frac{375 \text{ N} \cdot \text{m}}{56.25 \text{ kg} \cdot \text{m}^2} = 6.67 \frac{\text{rad}}{\text{s}^2}.$$

Solution for (b)

We expect the angular acceleration for the system to be less in this part, because the moment of inertia is greater when the child is on the merry-go-round. To find the total moment of inertia I , we first find the child's moment of inertia I_c by considering the child to be equivalent to a point mass at a distance of 1.25 m from the axis. Then,

Equation:

$$I_c = MR^2 = (18.0 \text{ kg})(1.25 \text{ m})^2 = 28.13 \text{ kg} \cdot \text{m}^2.$$

The total moment of inertia is the sum of moments of inertia of the merry-go-round and the child (about the same axis). To justify this sum to yourself, examine the definition of I :

Equation:

$$I = 28.13 \text{ kg} \cdot \text{m}^2 + 56.25 \text{ kg} \cdot \text{m}^2 = 84.38 \text{ kg} \cdot \text{m}^2.$$

Substituting known values into the equation for α gives

Equation:

$$\alpha = \frac{\tau}{I} = \frac{375 \text{ N} \cdot \text{m}}{84.38 \text{ kg} \cdot \text{m}^2} = 4.44 \frac{\text{rad}}{\text{s}^2}.$$

Discussion

The angular acceleration is less when the child is on the merry-go-round than when the merry-go-round is empty, as expected. The angular accelerations found are quite large, partly due to the fact that friction was considered to be negligible. If, for example, the father kept pushing perpendicularly for 2.00 s, he would give the merry-go-round an angular velocity of 13.3 rad/s when it is empty but only 8.89 rad/s when the child is on it. In terms of revolutions per second, these angular velocities are 2.12 rev/s and 1.41 rev/s, respectively. The father would end up running at about 50 km/h in the first case. Summer Olympics, here he comes! Confirmation of these numbers is left as an exercise for the reader.

Note:

Making Connections: Multiple Forces on One System

A large potter's wheel has a diameter of 60.0 cm and a mass of 8.0 kg. It is powered by a 20.0 N motor acting on the outer edge. There is also a brake capable of exerting a 15.0 N force at a radius of 12.0 cm from the axis of rotation, on the underside.

What is the angular acceleration when the motor is in use?

The torque is found by $\tau = rF \sin \theta = (0.300 \text{ m})(20.0 \text{ N}) = 6.00 \text{ N}\cdot\text{m}$.

The moment of inertia is calculated as

$$I = \frac{1}{2} MR^2 = \frac{1}{2} (8.0 \text{ kg})(0.300 \text{ m})^2 = 0.36 \text{ kg}\cdot\text{m}^2.$$

Thus, the angular acceleration would be

$$\alpha = \frac{\tau}{I} = \frac{6.00 \text{ N}\cdot\text{m}}{0.36 \text{ kg}\cdot\text{m}^2} = 17 \text{ rad/s}^2.$$

Note that the friction is always acting in a direction opposite to the rotation that is currently happening in this system. If the potter makes a mistake and has both the brake and motor on simultaneously, the friction force of the brake will exert a torque opposite that of the motor.

The torque from the brake is

$$\tau = rF \sin \theta = (0.120 \text{ m})(15.0 \text{ N}) = 1.80 \text{ N}\cdot\text{m}.$$

Thus, the net torque is $6.00 \text{ N}\cdot\text{m} - 1.80 \text{ N}\cdot\text{m} = 4.20 \text{ N}\cdot\text{m}$.

And the angular acceleration is $\alpha = \frac{\tau}{I} = \frac{4.20 \text{ N}\cdot\text{m}}{0.36 \text{ kg}\cdot\text{m}^2} = 12 \text{ rad/s}^2$.

Exercise:
Check Your Understanding

Problem:

Torque is the analog of force and moment of inertia is the analog of mass. Force and mass are physical quantities that depend on only one factor. For example, mass is related solely to the numbers of atoms of various types in an object. Are torque and moment of inertia similarly simple?

Solution:

No. Torque depends on three factors: force magnitude, force direction, and point of application. Moment of inertia depends on both mass and its distribution relative to the axis of rotation. So, while the analogies are precise, these rotational quantities depend on more factors.

Test Prep for AP Courses

Exercise:

Problem:

A piece of wood can be carved by spinning it on a motorized lathe and holding a sharp chisel to the edge of the wood as it spins. How does the angular velocity of a piece of wood with a radius of 0.2 m spinning on a lathe change when a chisel is held to the wood's edge with a force of 50 N?

- a. It increases by $0.1 \text{ N}\cdot\text{m}$ multiplied by the moment of inertia of the wood.
 - b. It decreases by $0.1 \text{ N}\cdot\text{m}$ divided by the moment of inertia of the wood-and-lathe system.
 - c. It decreases by $0.1 \text{ N}\cdot\text{m}$ multiplied by the moment of inertia of the wood.
 - d. It decreases by 0.1 m/s^2 .
-

Solution:

(b)

Exercise:**Problem:**

A Ferris wheel is loaded with people in the chairs at the following positions: 4 o'clock, 1 o'clock, 9 o'clock, and 6 o'clock. As the wheel begins to turn, what forces are acting on the system? How will each force affect the angular velocity and angular momentum?

Exercise:**Problem:**

A lever is placed on a fulcrum. A rock is placed on the left end of the lever and a downward (clockwise) force is applied to the right end of the lever. What measurements would be most effective to help you determine the angular momentum of the system? (Assume the lever itself has negligible mass.)

- a. the angular velocity and mass of the rock
- b. the angular velocity and mass of the rock, and the radius of the lever
- c. the velocity of the force, the radius of the lever, and the mass of the rock
- d. the mass of the rock, the length of the lever on both sides of the fulcrum, and the force applied on the right side of the lever

Solution:

(d)

Exercise:

Problem:

You can use the following setup to determine angular acceleration and angular momentum: A lever is placed on a fulcrum. A rock is placed on the left end of the lever and a known downward (clockwise) force is applied to the right end of the lever. What calculations would you perform? How would you account for gravity in your calculations?

Exercise:**Problem:**

Consider two sizes of disk, both of mass M . One size of disk has radius R ; the other has radius $2R$. System A consists of two of the larger disks rigidly connected to each other with a common axis of rotation. System B consists of one of the larger disks and a number of the smaller disks rigidly connected with a common axis of rotation. If the moment of inertia for system A equals the moment of inertia for system B, how many of the smaller disks are in system B?

- a. 1
- b. 2
- c. 3
- d. 4

Solution:

(d)

You are given a thin rod of length 1.0 m and mass 2.0 kg, a small lead weight of 0.50 kg, and a not-so-small lead weight of 1.0 kg. The rod has three holes, one in each end and one through the middle, which may either hold a pivot point or one of the small lead weights.

Exercise:

Problem:

You are given a thin rod of length 1.0 m and mass 2.0 kg, a small lead weight of 0.50 kg, and a not-so-small lead weight of 1.0 kg. The rod has three holes, one in each end and one through the middle, which may either hold a pivot point or one of the small lead weights. How do you arrange these objects so that the resulting system has the maximum possible moment of inertia? What is that moment of inertia? do you arrange these objects so that the resulting system has the maximum possible moment of inertia? What is that moment of inertia?

Section Summary

- The farther the force is applied from the pivot, the greater is the angular acceleration; angular acceleration is inversely proportional to mass.
- If we exert a force F on a point mass m that is at a distance r from a pivot point and because the force is perpendicular to r , an acceleration $a = F/m$ is obtained in the direction of F . We can rearrange this equation such that

Equation:

$$F = ma,$$

and then look for ways to relate this expression to expressions for rotational quantities. We note that $a = r\alpha$, and we substitute this expression into $F=ma$, yielding

Equation:

$$F = mr\alpha$$

- Torque is the turning effectiveness of a force. In this case, because F is perpendicular to r , torque is simply $\tau = rF$. If we multiply both sides of the equation above by r , we get torque on the left-hand side. That is,

Equation:

$$rF = mr^2\alpha$$

or

Equation:

$$\tau = mr^2\alpha.$$

- The moment of inertia I of an object is the sum of MR^2 for all the point masses of which it is composed. That is,

Equation:

$$I = \sum mr^2.$$

- The general relationship among torque, moment of inertia, and angular acceleration is

Equation:

$$\tau = I\alpha$$

or

Equation:

$$\alpha = \frac{\text{net } \tau}{I} .$$

Conceptual Questions

Exercise:

Problem:

The moment of inertia of a long rod spun around an axis through one end perpendicular to its length is $ML^2/3$. Why is this moment of inertia greater than it would be if you spun a point mass M at the location of the center of mass of the rod (at $L/2$)? (That would be $ML^2/4$.)

Exercise:**Problem:**

Why is the moment of inertia of a hoop that has a mass M and a radius R greater than the moment of inertia of a disk that has the same mass and radius? Why is the moment of inertia of a spherical shell that has a mass M and a radius R greater than that of a solid sphere that has the same mass and radius?

Exercise:**Problem:**

Give an example in which a small force exerts a large torque. Give another example in which a large force exerts a small torque.

Exercise:**Problem:**

While reducing the mass of a racing bike, the greatest benefit is realized from reducing the mass of the tires and wheel rims. Why does this allow a racer to achieve greater accelerations than would an identical reduction in the mass of the bicycle's frame?



The image shows a side view of a racing bicycle. Can you see evidence in the design of the wheels on this racing bicycle that their moment of inertia has been purposely reduced? (credit: Jesús Rodríguez)

Exercise:

Problem:

A ball slides up a frictionless ramp. It is then rolled without slipping and with the same initial velocity up another frictionless ramp (with the same slope angle). In which case does it reach a greater height, and why?

Problems & Exercises

Exercise:

Problem:

This problem considers additional aspects of example [Calculating the Effect of Mass Distribution on a Merry-Go-Round](#). (a) How long does it take the father to give the merry-go-round an angular velocity of 1.50 rad/s ? (b) How many revolutions must he go through to generate this velocity? (c) If he exerts a slowing force of 300 N at a radius of 1.35 m , how long would it take him to stop them?

Solution:

(a) 0.338 s

(b) 0.0403 rev

(c) 0.313 s

Exercise:

Problem:

Calculate the moment of inertia of a skater given the following information. (a) The 60.0-kg skater is approximated as a cylinder that has a 0.110-m radius. (b) The skater with arms extended is approximately a cylinder that is 52.5 kg, has a 0.110-m radius, and has two 0.900-m-long arms which are 3.75 kg each and extend straight out from the cylinder like rods rotated about their ends.

Exercise:

Problem:

The triceps muscle in the back of the upper arm extends the forearm. This muscle in a professional boxer exerts a force of 2.00×10^3 N with an effective perpendicular lever arm of 3.00 cm, producing an angular acceleration of the forearm of 120 rad/s^2 . What is the moment of inertia of the boxer's forearm?

Solution:

$0.50 \text{ kg} \cdot \text{m}^2$

Exercise:

Problem:

A soccer player extends her lower leg in a kicking motion by exerting a force with the muscle above the knee in the front of her leg. She produces an angular acceleration of 30.00 rad/s^2 and her lower leg has a moment of inertia of $0.750 \text{ kg} \cdot \text{m}^2$. What is the force exerted by the muscle if its effective perpendicular lever arm is 1.90 cm?

Exercise:

Problem:

Suppose you exert a force of 180 N tangential to a 0.280-m-radius 75.0-kg grindstone (a solid disk).

(a) What torque is exerted? (b) What is the angular acceleration assuming negligible opposing friction? (c) What is the angular acceleration if there is an opposing frictional force of 20.0 N exerted 1.50 cm from the axis?

Solution:

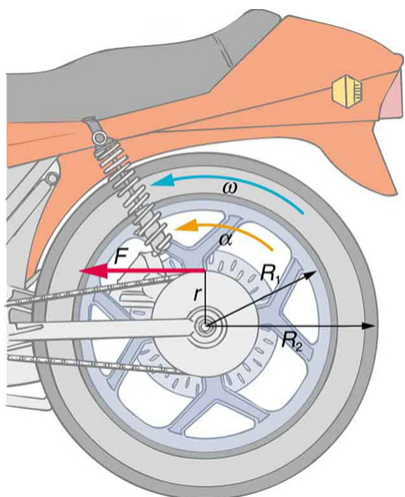
(a) $50.4 \text{ N} \cdot \text{m}$

(b) 17.1 rad/s^2

(c) 17.0 rad/s^2

Exercise:**Problem:**

Consider the 12.0 kg motorcycle wheel shown in [\[link\]](#). Assume it to be approximately an annular ring with an inner radius of 0.280 m and an outer radius of 0.330 m. The motorcycle is on its center stand, so that the wheel can spin freely. (a) If the drive chain exerts a force of 2200 N at a radius of 5.00 cm, what is the angular acceleration of the wheel? (b) What is the tangential acceleration of a point on the outer edge of the tire? (c) How long, starting from rest, does it take to reach an angular velocity of 80.0 rad/s?



A motorcycle wheel has a moment of inertia approximately that of an annular ring.

Exercise:

Problem:

Zorch, an archenemy of Superman, decides to slow Earth's rotation to once per 28.0 h by exerting an opposing force at and parallel to the equator. Superman is not immediately concerned, because he knows Zorch can only exert a force of 4.00×10^7 N (a little greater than a Saturn V rocket's thrust). How long must Zorch push with this force to accomplish his goal? (This period gives Superman time to devote to other villains.) Explicitly show how you follow the steps found in [Problem-Solving Strategy for Rotational Dynamics](#).

Solution:

$$3.96 \times 10^{18} \text{ s}$$

$$\text{or } 1.26 \times 10^{11} \text{ y}$$

Exercise:**Problem:**

An automobile engine can produce $200 \text{ N} \cdot \text{m}$ of torque. Calculate the angular acceleration produced if 95.0% of this torque is applied to the drive shaft, axle, and rear wheels of a car, given the following information. The car is suspended so that the wheels can turn freely. Each wheel acts like a 15.0 kg disk that has a 0.180 m radius. The walls of each tire act like a 2.00-kg annular ring that has inside radius of 0.180 m and outside radius of 0.320 m . The tread of each tire acts like a 10.0-kg hoop of radius 0.330 m . The 14.0-kg axle acts like a rod that has a 2.00-cm radius. The 30.0-kg drive shaft acts like a rod that has a 3.20-cm radius.

Exercise:**Problem:**

Starting with the formula for the moment of inertia of a rod rotated around an axis through one end perpendicular to its length ($I = M\ell^2/3$), prove that the moment of inertia of a rod rotated about an axis through its center perpendicular to its length is $I = M\ell^2/12$. You will find the graphics in [\[link\]](#) useful in visualizing these rotations.

Solution:

$$I_{\text{end}} = I_{\text{center}} + m\left(\frac{l}{2}\right)^2$$

$$\text{Thus, } I_{\text{center}} = I_{\text{end}} - \frac{1}{4}ml^2 = \frac{1}{3}ml^2 - \frac{1}{4}ml^2 = \frac{1}{12}ml^2$$

Exercise:**Problem: Unreasonable Results**

A gymnast doing a forward flip lands on the mat and exerts a $500\text{-N} \cdot \text{m}$ torque to slow and then reverse her angular velocity. Her initial angular velocity is 10.0 rad/s , and her moment of inertia is

$0.050 \text{ kg} \cdot \text{m}^2$. (a) What time is required for her to exactly reverse her spin? (b) What is unreasonable about the result? (c) Which premises are unreasonable or inconsistent?

Solution:

(a) 2.0 ms

(b) The time interval is too short.

(c) The moment of inertia is much too small, by one to two orders of magnitude. A torque of $500 \text{ N} \cdot \text{m}$ is reasonable.

Exercise:

Problem: Unreasonable Results

An advertisement claims that an 800-kg car is aided by its 20.0-kg flywheel, which can accelerate the car from rest to a speed of 30.0 m/s. The flywheel is a disk with a 0.150-m radius. (a) Calculate the angular velocity the flywheel must have if 95.0% of its rotational energy is used to get the car up to speed. (b) What is unreasonable about the result? (c) Which premise is unreasonable or which premises are inconsistent?

Solution:

(a) 17,500 rpm

(b) This angular velocity is very high for a disk of this size and mass. The radial acceleration at the edge of the disk is $> 50,000 \text{ gs}$.

(c) Flywheel mass and radius should both be much greater, allowing for a lower spin rate (angular velocity).

Glossary

torque

the turning effectiveness of a force

rotational inertia

resistance to change of rotation. The more rotational inertia an object has, the harder it is to rotate

moment of inertia

mass times the square of perpendicular distance from the rotation axis; for a point mass, it is $I = mr^2$ and, because any object can be built up from a collection of point masses, this relationship is the basis for all other moments of inertia

Rotational Kinetic Energy: Work and Energy Revisited

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Derive the equation for rotational work.
- Calculate rotational kinetic energy.
- Demonstrate the law of conservation of energy.

The information presented in this section supports the following AP® learning objectives and science practices:

- **3.F.2.1** The student is able to make predictions about the change in the angular velocity about an axis for an object when forces exerted on the object cause a torque about that axis. **(S.P. 6.4)**
- **3.F.2.2** The student is able to plan data collection and analysis strategies designed to test the relationship between a torque exerted on an object and the change in angular velocity of that object about an axis. **(S.P. 4.1, 4.2, 5.1)**

In this module, we will learn about work and energy associated with rotational motion. [\[link\]](#) shows a worker using an electric grindstone propelled by a motor. Sparks are flying, and noise and vibration are created as layers of steel are pared from the pole. The stone continues to turn even after the motor is turned off, but it is eventually brought to a stop by friction. Clearly, the motor had to work to get the stone spinning. This work

went into heat, light, sound, vibration, and considerable **rotational kinetic energy**.



The motor works in spinning the grindstone, giving it rotational kinetic energy. That energy is then converted to heat, light, sound, and vibration. (credit: U.S. Navy photo by Mass Communication Specialist Seaman Zachary David Bell)

Work must be done to rotate objects such as grindstones or merry-go-rounds. Work was defined in [Uniform Circular Motion and Gravitation](#) for translational motion, and we can build on that knowledge when considering work done in rotational motion. The simplest rotational situation is one in which the net force is exerted perpendicular to the radius of a disk (as shown in [\[link\]](#)) and remains perpendicular as the disk starts to rotate. The force is parallel to the displacement, and so the net work done is the product of the force times the arc length traveled:

Equation:

$$\text{net } W = (\text{net } F)\Delta s.$$

To get torque and other rotational quantities into the equation, we multiply and divide the right-hand side of the equation by r , and gather terms:

Equation:

$$\text{net } W = (r \text{ net } F) \frac{\Delta s}{r}.$$

We recognize that $r \text{ net } F = \text{net } \tau$ and $\Delta s/r = \theta$, so that

Equation:

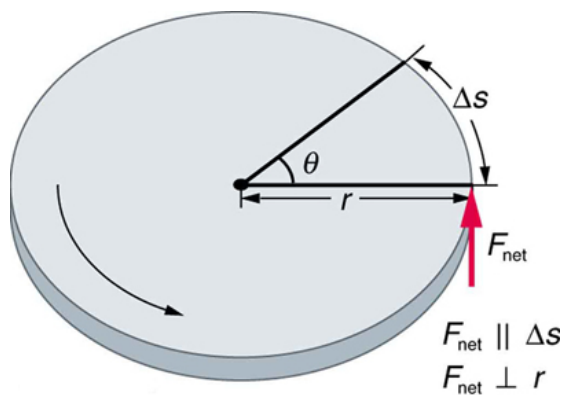
$$\text{net } W = (\text{net } \tau)\theta.$$

This equation is the expression for rotational work. It is very similar to the familiar definition of translational work as force multiplied by distance. Here, torque is analogous to force, and angle is analogous to distance. The equation $\text{net } W = (\text{net } \tau)\theta$ is valid in general, even though it was derived for a special case.

To get an expression for rotational kinetic energy, we must again perform some algebraic manipulations. The first step is to note that $\text{net } \tau = I\alpha$, so that

Equation:

$$\text{net } W = I\alpha\theta.$$



The net force on this disk is
kept perpendicular to its

radius as the force causes the disk to rotate. The net work done is thus $(\text{net } F)\Delta s$. The net work goes into rotational kinetic energy.

Note:

Making Connections

q Work and energy in rotational motion are completely analogous to work and energy in translational motion, first presented in [Uniform Circular Motion and Gravitation](#).

Now, we solve one of the rotational kinematics equations for $\alpha\theta$. We start with the equation

Equation:

$$\omega^2 = \omega_0^2 + 2\alpha\theta.$$

Next, we solve for $\alpha\theta$:

Equation:

$$\alpha\theta = \frac{\omega^2 - \omega_0^2}{2}.$$

Substituting this into the equation for net W and gathering terms yields

Equation:

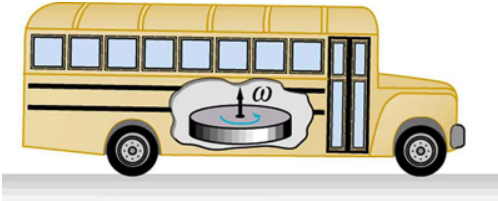
$$\text{net } W = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2.$$

This equation is the **work-energy theorem** for rotational motion only. As you may recall, net work changes the kinetic energy of a system. Through an analogy with translational motion, we define the term $\left(\frac{1}{2}\right)I\omega^2$ to be **rotational kinetic energy** KE_{rot} for an object with a moment of inertia I and an angular velocity ω :

Equation:

$$KE_{\text{rot}} = \frac{1}{2}I\omega^2.$$

The expression for rotational kinetic energy is exactly analogous to translational kinetic energy, with I being analogous to m and ω to v . Rotational kinetic energy has important effects. Flywheels, for example, can be used to store large amounts of rotational kinetic energy in a vehicle, as seen in [\[link\]](#).



Experimental vehicles, such as this bus, have been constructed in which rotational kinetic energy is stored in a large flywheel. When the bus goes down a hill, its transmission converts its gravitational potential energy into KE_{rot} . It can also convert translational kinetic energy, when the bus stops, into KE_{rot} . The flywheel's energy can then be used to accelerate, to go up

another hill, or to keep the bus from going against friction.

Example:

Calculating the Work and Energy for Spinning a Grindstone

Consider a person who spins a large grindstone by placing her hand on its edge and exerting a force through part of a revolution as shown in [\[link\]](#). In this example, we verify that the work done by the torque she exerts equals the change in rotational energy. (a) How much work is done if she exerts a force of 200 N through a rotation of 1.00 rad(57.3°)? The force is kept perpendicular to the grindstone's 0.320-m radius at the point of application, and the effects of friction are negligible. (b) What is the final angular velocity if the grindstone has a mass of 85.0 kg? (c) What is the final rotational kinetic energy? (It should equal the work.)

Strategy

To find the work, we can use the equation $\text{net } W = (\text{net } \tau)\theta$. We have enough information to calculate the torque and are given the rotation angle. In the second part, we can find the final angular velocity using one of the kinematic relationships. In the last part, we can calculate the rotational kinetic energy from its expression in $\text{KE}_{\text{rot}} = \frac{1}{2} I\omega^2$.

Solution for (a)

The net work is expressed in the equation

Equation:

$$\text{net } W = (\text{net } \tau)\theta,$$

where $\text{net } \tau$ is the applied force multiplied by the radius (rF) because there is no retarding friction, and the force is perpendicular to r . The angle θ is given. Substituting the given values in the equation above yields

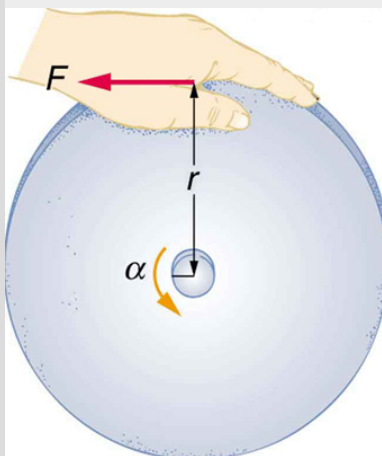
Equation:

$$\begin{aligned}\text{net } W &= rF\theta = (0.320 \text{ m})(200 \text{ N})(1.00 \text{ rad}) \\ &= 64.0 \text{ N} \cdot \text{m}.\end{aligned}$$

Noting that $1 \text{ N} \cdot \text{m} = 1 \text{ J}$,

Equation:

$$\text{net } W = 64.0 \text{ J}.$$



A large grindstone is given a spin by a person grasping its outer edge.

Solution for (b)

To find ω from the given information requires more than one step. We start with the kinematic relationship in the equation

Equation:

$$\omega^2 = \omega_0^2 + 2\alpha\theta.$$

Note that $\omega_0 = 0$ because we start from rest. Taking the square root of the resulting equation gives

Equation:

$$\omega = (2\alpha\theta)^{1/2}.$$

Now we need to find α . One possibility is

Equation:

$$\alpha = \frac{\text{net } \tau}{I},$$

where the torque is

Equation:

$$\text{net } \tau = rF = (0.320 \text{ m})(200 \text{ N}) = 64.0 \text{ N} \cdot \text{m}.$$

The formula for the moment of inertia for a disk is found in [\[link\]](#):

Equation:

$$I = \frac{1}{2}MR^2 = 0.5(85.0 \text{ kg})(0.320 \text{ m})^2 = 4.352 \text{ kg} \cdot \text{m}^2.$$

Substituting the values of torque and moment of inertia into the expression for α , we obtain

Equation:

$$\alpha = \frac{64.0 \text{ N} \cdot \text{m}}{4.352 \text{ kg} \cdot \text{m}^2} = 14.7 \frac{\text{rad}}{\text{s}^2}.$$

Now, substitute this value and the given value for θ into the above expression for ω :

Equation:

$$\omega = (2\alpha\theta)^{1/2} = \left[2 \left(14.7 \frac{\text{rad}}{\text{s}^2} \right) (1.00 \text{ rad}) \right]^{1/2} = 5.42 \frac{\text{rad}}{\text{s}}.$$

Solution for (c)

The final rotational kinetic energy is

Equation:

$$\text{KE}_{\text{rot}} = \frac{1}{2}I\omega^2.$$

Both I and ω were found above. Thus,

Equation:

$$KE_{\text{rot}} = (0.5)(4.352 \text{ kg} \cdot \text{m}^2)(5.42 \text{ rad/s})^2 = 64.0 \text{ J}.$$

Discussion

The final rotational kinetic energy equals the work done by the torque, which confirms that the work done went into rotational kinetic energy. We could, in fact, have used an expression for energy instead of a kinematic relation to solve part (b). We will do this in later examples.

Helicopter pilots are quite familiar with rotational kinetic energy. They know, for example, that a point of no return will be reached if they allow their blades to slow below a critical angular velocity during flight. The blades lose lift, and it is impossible to immediately get the blades spinning fast enough to regain it. Rotational kinetic energy must be supplied to the blades to get them to rotate faster, and enough energy cannot be supplied in time to avoid a crash. Because of weight limitations, helicopter engines are too small to supply both the energy needed for lift and to replenish the rotational kinetic energy of the blades once they have slowed down. The rotational kinetic energy is put into them before takeoff and must not be allowed to drop below this crucial level. One possible way to avoid a crash is to use the gravitational potential energy of the helicopter to replenish the rotational kinetic energy of the blades by losing altitude and aligning the blades so that the helicopter is spun up in the descent. Of course, if the helicopter's altitude is too low, then there is insufficient time for the blade to regain lift before reaching the ground.

Note:

Take-Home Experiment

Rotational motion can be observed in wrenches, clocks, wheels or spools on axels, and seesaws. Choose an object or system that exhibits rotational motion and plan an experiment to test how torque affects angular velocity. How will you create and measure different amounts of torque? How will

you measure angular velocity? Remember that
net $\tau = I\alpha$, $I \propto mr^2$, and $\omega = \frac{v}{r}$.

Note:

Problem-Solving Strategy for Rotational Energy

1. *Determine that energy or work is involved in the rotation.*
2. *Determine the system of interest. A sketch usually helps.*
3. *Analyze the situation to determine the types of work and energy involved.*
4. *For closed systems, mechanical energy is conserved. That is, $KE_i + PE_i = KE_f + PE_f$. Note that KE_i and KE_f may each include translational and rotational contributions.*
5. *For open systems, mechanical energy may not be conserved, and other forms of energy (referred to previously as OE), such as heat transfer, may enter or leave the system. Determine what they are, and calculate them as necessary.*
6. *Eliminate terms wherever possible to simplify the algebra.*
7. *Check the answer to see if it is reasonable.*

Example:

Calculating Helicopter Energies

A typical small rescue helicopter, similar to the one in [\[link\]](#), has four blades. Each is 4.00 m long and has a mass of 50.0 kg. The blades can be approximated as thin rods that rotate about one end of an axis perpendicular to their length. The helicopter has a total loaded mass of 1000 kg. (a) Calculate the rotational kinetic energy in the blades when they rotate at 300 rpm. (b) Calculate the translational kinetic energy of the helicopter when it flies at 20.0 m/s, and compare it with the rotational energy in the blades. (c) To what height could the helicopter be raised if all of the rotational kinetic energy could be used to lift it?

Strategy

Rotational and translational kinetic energies can be calculated from their definitions. The last part of the problem relates to the idea that energy can change form, in this case from rotational kinetic energy to gravitational potential energy.

Solution for (a)

The rotational kinetic energy is

Equation:

$$\text{KE}_{\text{rot}} = \frac{1}{2} I \omega^2.$$

We must calculate the angular velocity from rpm and calculate the moment of inertia before we can find KE_{rot} . The angular velocity ω is

Equation:

$$\omega = \frac{300 \text{ rev}}{1.00 \text{ min}} \cdot \frac{2\pi \text{ rad}}{1 \text{ rev}} \cdot \frac{1.00 \text{ min}}{60.0 \text{ s}} = 31.4 \frac{\text{rad}}{\text{s}}.$$

The moment of inertia of one blade will be that of a thin rod rotated about its end, found in [\[link\]](#). The total I is four times this moment of inertia, because there are four blades. Thus,

Equation:

$$I = 4 \frac{M\ell^2}{3} = 4 \times \frac{(50.0 \text{ kg})(4.00 \text{ m})^2}{3} = 1067 \text{ kg} \cdot \text{m}^2.$$

Entering ω and I into the expression for rotational kinetic energy gives

Equation:

$$\begin{aligned} \text{KE}_{\text{rot}} &= 0.5(1067 \text{ kg} \cdot \text{m}^2)(31.4 \text{ rad/s})^2 \\ &= 5.26 \times 10^5 \text{ J} \end{aligned}$$

Solution for (b)

Translational kinetic energy was defined in [Uniform Circular Motion and Gravitation](#). Entering the given values of mass and velocity, we obtain

Equation:

$$\text{KE}_{\text{trans}} = \frac{1}{2}mv^2 = (0.5)(1000 \text{ kg})(20.0 \text{ m/s})^2 = 2.00 \times 10^5 \text{ J}.$$

To compare kinetic energies, we take the ratio of translational kinetic energy to rotational kinetic energy. This ratio is

Equation:

$$\frac{2.00 \times 10^5 \text{ J}}{5.26 \times 10^5 \text{ J}} = 0.380.$$

Solution for (c)

At the maximum height, all rotational kinetic energy will have been converted to gravitational energy. To find this height, we equate those two energies:

Equation:

$$\text{KE}_{\text{rot}} = \text{PE}_{\text{grav}}$$

or

Equation:

$$\frac{1}{2}I\omega^2 = mgh.$$

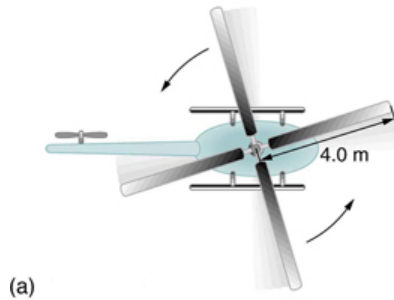
We now solve for h and substitute known values into the resulting equation

Equation:

$$h = \frac{\frac{1}{2}I\omega^2}{mg} = \frac{5.26 \times 10^5 \text{ J}}{(1000 \text{ kg})(9.80 \text{ m/s}^2)} = 53.7 \text{ m}.$$

Discussion

The ratio of translational energy to rotational kinetic energy is only 0.380. This ratio tells us that most of the kinetic energy of the helicopter is in its spinning blades—something you probably would not suspect. The 53.7 m height to which the helicopter could be raised with the rotational kinetic energy is also impressive, again emphasizing the amount of rotational kinetic energy in the blades.



The first image shows how helicopters store large amounts of rotational kinetic energy in their blades. This energy must be put into the blades before takeoff and maintained until the end of the flight. The engines do not have enough power to simultaneously provide lift and put significant rotational energy into the blades. The second image shows a helicopter from the Auckland

Westpac Rescue
Helicopter Service.
Over 50,000 lives
have been saved
since its operations
beginning in 1973.
Here, a water
rescue operation is
shown. (credit: 111
Emergency, Flickr)

Note:

Making Connections

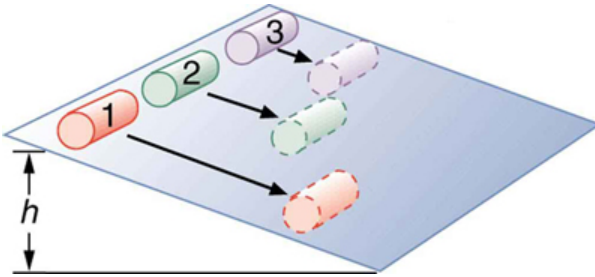
Conservation of energy includes rotational motion, because rotational kinetic energy is another form of KE . [Uniform Circular Motion and Gravitation](#) has a detailed treatment of conservation of energy.

How Thick Is the Soup? Or Why Don't All Objects Roll Downhill at the Same Rate?

One of the quality controls in a tomato soup factory consists of rolling filled cans down a ramp. If they roll too fast, the soup is too thin. Why should cans of identical size and mass roll down an incline at different rates? And why should the thickest soup roll the slowest?

The easiest way to answer these questions is to consider energy. Suppose each can starts down the ramp from rest. Each can starting from rest means each starts with the same gravitational potential energy PE_{grav} , which is converted entirely to KE, provided each rolls without slipping. KE, however, can take the form of KE_{trans} or KE_{rot} , and total KE is the sum of the two. If a can rolls down a ramp, it puts part of its energy into rotation,

leaving less for translation. Thus, the can goes slower than it would if it slid down. Furthermore, the thin soup does not rotate, whereas the thick soup does, because it sticks to the can. The thick soup thus puts more of the can's original gravitational potential energy into rotation than the thin soup, and the can rolls more slowly, as seen in [\[link\]](#).



Three cans of soup with identical masses race down an incline. The first can has a low friction coating and does not roll but just slides down the incline. It wins because it converts its entire PE into translational KE. The second and third cans both roll down the incline without slipping. The second can contains thin soup and comes in second because part of its initial PE goes into rotating the can (but not the thin soup). The third can contains thick soup. It comes in third because the soup rotates along with the can, taking even more of the initial PE for rotational KE, leaving less for translational KE.

Assuming no losses due to friction, there is only one force doing work—gravity. Therefore the total work done is the change in kinetic energy. As the cans start moving, the potential energy is changing into kinetic energy. Conservation of energy gives

Equation:

$$PE_i = KE_f.$$

More specifically,

Equation:

$$PE_{\text{grav}} = KE_{\text{trans}} + KE_{\text{rot}}$$

or

Equation:

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2.$$

So, the initial mgh is divided between translational kinetic energy and rotational kinetic energy; and the greater I is, the less energy goes into translation. If the can slides down without friction, then $\omega = 0$ and all the energy goes into translation; thus, the can goes faster.

Note:

Take-Home Experiment

Locate several cans each containing different types of food. First, predict which can will win the race down an inclined plane and explain why. See if your prediction is correct. You could also do this experiment by collecting several empty cylindrical containers of the same size and filling them with different materials such as wet or dry sand.

Example:**Calculating the Speed of a Cylinder Rolling Down an Incline**

Calculate the final speed of a solid cylinder that rolls down a 2.00-m-high incline. The cylinder starts from rest, has a mass of 0.750 kg, and has a radius of 4.00 cm.

Strategy

We can solve for the final velocity using conservation of energy, but we must first express rotational quantities in terms of translational quantities to end up with v as the only unknown.

Solution

Conservation of energy for this situation is written as described above:

Equation:

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2.$$

Before we can solve for v , we must get an expression for I from [\[link\]](#). Because v and ω are related (note here that the cylinder is rolling without slipping), we must also substitute the relationship $\omega = v/R$ into the expression. These substitutions yield

Equation:

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\left(\frac{v^2}{R^2}\right).$$

Interestingly, the cylinder's radius R and mass m cancel, yielding

Equation:

$$gh = \frac{1}{2}v^2 + \frac{1}{4}v^2 = \frac{3}{4}v^2.$$

Solving algebraically, the equation for the final velocity v gives

Equation:

$$v = \left(\frac{4gh}{3}\right)^{1/2}.$$

Substituting known values into the resulting expression yields

Equation:

$$v = \frac{4 \left(9.80 \text{ m/s}^2 \right) (2.00 \text{ m})}{3}^{1/2} = 5.11 \text{ m/s}.$$

Discussion

Because m and R cancel, the result $v = \left(\frac{4}{3} gh \right)^{1/2}$ is valid for any solid cylinder, implying that all solid cylinders will roll down an incline at the same rate independent of their masses and sizes. (Rolling cylinders down inclines is what Galileo actually did to show that objects fall at the same rate independent of mass.) Note that if the cylinder slid without friction down the incline without rolling, then the entire gravitational potential energy would go into translational kinetic energy. Thus, $\frac{1}{2}mv^2 = mgh$ and $v = (2gh)^{1/2}$, which is 22% greater than $(4gh/3)^{1/2}$. That is, the cylinder would go faster at the bottom.

Exercise:

Check Your Understanding

Problem:

Analogy of Rotational and Translational Kinetic Energy

Is rotational kinetic energy completely analogous to translational kinetic energy? What, if any, are their differences? Give an example of each type of kinetic energy.

Solution:

Yes, rotational and translational kinetic energy are exact analogs. They both are the energy of motion involved with the coordinated (non-random) movement of mass relative to some reference frame. The only difference between rotational and translational kinetic energy is that translational is straight line motion while rotational is not. An example

of both kinetic and translational kinetic energy is found in a bike tire while being ridden down a bike path. The rotational motion of the tire means it has rotational kinetic energy while the movement of the bike along the path means the tire also has translational kinetic energy. If you were to lift the front wheel of the bike and spin it while the bike is stationary, then the wheel would have only rotational kinetic energy relative to the Earth.

Note:

PhET Explorations: My Solar System

Build your own system of heavenly bodies and watch the gravitational ballet. With this orbit simulator, you can set initial positions, velocities, and masses of 2, 3, or 4 bodies, and then see them orbit each other.

[My
Solar
System](#)

Test Prep for AP Courses

Exercise:

Problem:

Gear A, which turns clockwise, meshes with gear B, which turns counterclockwise. When more force is applied through gear A, torque is created. How does the angular velocity of gear B change as a result?

- a. It increases in magnitude.
- b. It decreases in magnitude.

- c. It changes direction.
- d. It stays the same.

Solution:

(a)

Exercise:

Problem:

Which will cause a greater increase in the angular velocity of a disk: doubling the torque applied or halving the radius at which the torque is applied? Explain.

Exercise:

Problem:

Which measure would not be useful to help you determine the change in angular velocity when the torque on a fishing reel is increased?

- a. the radius of the reel
- b. the amount of line that unspools
- c. the angular momentum of the fishing line
- d. the time it takes the line to unspool

Solution:

(c)

Exercise:

Problem:

What data could you collect to study the change in angular velocity when two people push a merry-go-round instead of one, providing twice as much torque? How would you use the data you collect?

Section Summary

- The rotational kinetic energy KE_{rot} for an object with a moment of inertia I and an angular velocity ω is given by

Equation:

$$\text{KE}_{\text{rot}} = \frac{1}{2} I \omega^2.$$

- Helicopters store large amounts of rotational kinetic energy in their blades. This energy must be put into the blades before takeoff and maintained until the end of the flight. The engines do not have enough power to simultaneously provide lift and put significant rotational energy into the blades.
 - Work and energy in rotational motion are completely analogous to work and energy in translational motion.
 - The equation for the **work-energy theorem** for rotational motion is,
- Equation:**

$$\text{net } W = \frac{1}{2} I \omega^2 - \frac{1}{2} I \omega_0^2.$$

Conceptual Questions

Exercise:

Problem:

Describe the energy transformations involved when a yo-yo is thrown downward and then climbs back up its string to be caught in the user's hand.

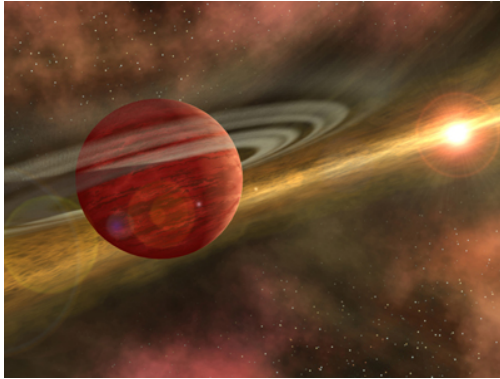
Exercise:

Problem:

What energy transformations are involved when a dragster engine is revved, its clutch let out rapidly, its tires spun, and it starts to accelerate forward? Describe the source and transformation of energy at each step.

Exercise:**Problem:**

The Earth has more rotational kinetic energy now than did the cloud of gas and dust from which it formed. Where did this energy come from?



An immense cloud of rotating gas and dust contracted under the influence of gravity to form the Earth and in the process rotational kinetic energy increased. (credit: NASA)

Problems & Exercises**Exercise:**

Problem:

This problem considers energy and work aspects of [\[link\]](#)—use data from that example as needed. (a) Calculate the rotational kinetic energy in the merry-go-round plus child when they spin at a frequency of 20.0 rpm. (b) Using energy considerations, find the number of revolutions the father will have to push to achieve this angular velocity starting from rest. (c) Again, using energy considerations, calculate the force the father must exert to stop the merry-go-round in two revolutions

Solution:

(a) 185 J

(b) 0.0785 rev

(c) $W = 9.81 \text{ N}$

Exercise:**Problem:**

What is the final velocity of a hoop that rolls without slipping down a 5.00-m-high hill, starting from rest?

Exercise:**Problem:**

(a) Calculate the rotational kinetic energy of Earth on its axis. (b) What is the rotational kinetic energy of Earth in its orbit around the Sun?

Solution:

(a) $2.57 \times 10^{29} \text{ J}$

(b) $\text{KE}_{\text{rot}} = 2.65 \times 10^{33} \text{ J}$

Exercise:

Problem:

Calculate the rotational kinetic energy in the motorcycle wheel ([link](#)) if its angular velocity is 120 rad/s. Assume $M = 12.0 \text{ kg}$, $R_1 = 0.280 \text{ m}$, and $R_2 = 0.330 \text{ m}$.

Exercise:**Problem:**

A baseball pitcher throws the ball in a motion where there is rotation of the forearm about the elbow joint as well as other movements. If the linear velocity of the ball relative to the elbow joint is 20.0 m/s at a distance of 0.480 m from the joint and the moment of inertia of the forearm is $0.500 \text{ kg} \cdot \text{m}^2$, what is the rotational kinetic energy of the forearm?

Solution:

$$KE_{\text{rot}} = 434 \text{ J}$$

Exercise:**Problem:**

While punting a football, a kicker rotates his leg about the hip joint. The moment of inertia of the leg is $3.75 \text{ kg} \cdot \text{m}^2$ and its rotational kinetic energy is 175 J. (a) What is the angular velocity of the leg? (b) What is the velocity of tip of the punter's shoe if it is 1.05 m from the hip joint? (c) Explain how the football can be given a velocity greater than the tip of the shoe (necessary for a decent kick distance).

Exercise:

Problem:

A bus contains a 1500 kg flywheel (a disk that has a 0.600 m radius) and has a total mass of 10,000 kg. (a) Calculate the angular velocity the flywheel must have to contain enough energy to take the bus from rest to a speed of 20.0 m/s, assuming 90.0% of the rotational kinetic energy can be transformed into translational energy. (b) How high a hill can the bus climb with this stored energy and still have a speed of 3.00 m/s at the top of the hill? Explicitly show how you follow the steps in the [Problem-Solving Strategy for Rotational Energy](#).

Solution:

(a) 128 rad/s

(b) 19.9 m

Exercise:**Problem:**

A ball with an initial velocity of 8.00 m/s rolls up a hill without slipping. Treating the ball as a spherical shell, calculate the vertical height it reaches. (b) Repeat the calculation for the same ball if it slides up the hill without rolling.

Exercise:**Problem:**

While exercising in a fitness center, a man lies face down on a bench and lifts a weight with one lower leg by contacting the muscles in the back of the upper leg. (a) Find the angular acceleration produced given the mass lifted is 10.0 kg at a distance of 28.0 cm from the knee joint, the moment of inertia of the lower leg is $0.900 \text{ kg} \cdot \text{m}^2$, the muscle force is 1500 N, and its effective perpendicular lever arm is 3.00 cm. (b) How much work is done if the leg rotates through an angle of 20.0° with a constant force exerted by the muscle?

Solution:

(a) 10.4 rad/s^2

(b) $\text{net } W = 6.11 \text{ J}$

Exercise:**Problem:**

To develop muscle tone, a woman lifts a 2.00-kg weight held in her hand. She uses her biceps muscle to flex the lower arm through an angle of 60.0° . (a) What is the angular acceleration if the weight is 24.0 cm from the elbow joint, her forearm has a moment of inertia of $0.250 \text{ kg} \cdot \text{m}^2$, and the net force she exerts is 750 N at an effective perpendicular lever arm of 2.00 cm? (b) How much work does she do?

Exercise:**Problem:**

Consider two cylinders that start down identical inclines from rest except that one is frictionless. Thus one cylinder rolls without slipping, while the other slides frictionlessly without rolling. They both travel a short distance at the bottom and then start up another incline. (a) Show that they both reach the same height on the other incline, and that this height is equal to their original height. (b) Find the ratio of the time the rolling cylinder takes to reach the height on the second incline to the time the sliding cylinder takes to reach the height on the second incline. (c) Explain why the time for the rolling motion is greater than that for the sliding motion.

Exercise:**Problem:**

What is the moment of inertia of an object that rolls without slipping down a 2.00-m-high incline starting from rest, and has a final velocity of 6.00 m/s? Express the moment of inertia as a multiple of MR^2 , where M is the mass of the object and R is its radius.

Exercise:**Problem:**

Suppose a 200-kg motorcycle has two wheels like, the one described in [\[link\]](#) and is heading toward a hill at a speed of 30.0 m/s. (a) How high can it coast up the hill, if you neglect friction? (b) How much energy is lost to friction if the motorcycle only gains an altitude of 35.0 m before coming to rest?

Exercise:**Problem:**

In softball, the pitcher throws with the arm fully extended (straight at the elbow). In a fast pitch the ball leaves the hand with a speed of 139 km/h. (a) Find the rotational kinetic energy of the pitcher's arm given its moment of inertia is $0.720 \text{ kg} \cdot \text{m}^2$ and the ball leaves the hand at a distance of 0.600 m from the pivot at the shoulder. (b) What force did the muscles exert to cause the arm to rotate if their effective perpendicular lever arm is 4.00 cm and the ball is 0.156 kg?

Solution:

(a) 1.49 kJ

(b) $2.52 \times 10^4 \text{ N}$

Exercise:**Problem: Construct Your Own Problem**

Consider the work done by a spinning skater pulling her arms in to increase her rate of spin. Construct a problem in which you calculate the work done with a “force multiplied by distance” calculation and compare it to the skater's increase in kinetic energy.

Glossary

work-energy theorem

if one or more external forces act upon a rigid object, causing its kinetic energy to change from KE_1 to KE_2 , then the work W done by the net force is equal to the change in kinetic energy

rotational kinetic energy

the kinetic energy due to the rotation of an object. This is part of its total kinetic energy

Angular Momentum and Its Conservation

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Understand the analogy between angular momentum and linear momentum.
- Observe the relationship between torque and angular momentum.
- Apply the law of conservation of angular momentum.

The information presented in this section supports the following AP® learning objectives and science practices:

- **4.D.2.1** The student is able to describe a model of a rotational system and use that model to analyze a situation in which angular momentum changes due to interaction with other objects or systems. **(S.P. 1.2, 1.4)**
- **4.D.2.2** The student is able to plan a data collection and analysis strategy to determine the change in angular momentum of a system and relate it to interactions with other objects and systems. **(S.P. 2.2)**
- **4.D.3.1** The student is able to use appropriate mathematical routines to calculate values for initial or final angular momentum, or change in angular momentum of a system, or average torque or time during which the torque is exerted in analyzing a situation involving torque and angular momentum. **(S.P. 2.2)**
- **4.D.3.2** The student is able to plan a data collection strategy designed to test the relationship between the change in angular momentum of a system and the product of the average torque

applied to the system and the time interval during which the torque is exerted. (S.P. 4.1, 4.2)

- **5.E.1.1** The student is able to make qualitative predictions about the angular momentum of a system for a situation in which there is no net external torque. (S.P. 6.4, 7.2)
- **5.E.1.2** The student is able to make calculations of quantities related to the angular momentum of a system when the net external torque on the system is zero. (S.P. 2.1, 2.2)
- **5.E.2.1** The student is able to describe or calculate the angular momentum and rotational inertia of a system in terms of the locations and velocities of objects that make up the system. Students are expected to do qualitative reasoning with compound objects. Students are expected to do calculations with a fixed set of extended objects and point masses. (S.P. 2.2)

Why does Earth keep on spinning? What started it spinning to begin with? And how does an ice skater manage to spin faster and faster simply by pulling her arms in? Why does she not have to exert a torque to spin faster? Questions like these have answers based in angular momentum, the rotational analog to linear momentum.

By now the pattern is clear—every rotational phenomenon has a direct translational analog. It seems quite reasonable, then, to define **angular momentum** L as

Equation:

$$L = I\omega.$$

This equation is an analog to the definition of linear momentum as $p = mv$. Units for linear momentum are $\text{kg} \cdot \text{m/s}$ while units for angular momentum are $\text{kg} \cdot \text{m}^2/\text{s}$. As we would expect, an object that has a large moment of inertia I , such as Earth, has a very large angular momentum. An object that

has a large angular velocity ω , such as a centrifuge, also has a rather large angular momentum.

Note:

Making Connections

Angular momentum is completely analogous to linear momentum, first presented in [Uniform Circular Motion and Gravitation](#). It has the same implications in terms of carrying rotation forward, and it is conserved when the net external torque is zero. Angular momentum, like linear momentum, is also a property of the atoms and subatomic particles.

Example:

Calculating Angular Momentum of the Earth

Strategy

No information is given in the statement of the problem; so we must look up pertinent data before we can calculate $L = I\omega$. First, according to [\[link\]](#), the formula for the moment of inertia of a sphere is

Equation:

$$I = \frac{2MR^2}{5}$$

so that

Equation:

$$L = I\omega = \frac{2MR^2\omega}{5}.$$

Earth's mass M is 5.979×10^{24} kg and its radius R is 6.376×10^6 m. Earth experiences, of course, exactly one revolution per day, but we must express ω in radians per second to do the calculation in SI units.

Solution

Substituting known information into the expression for L and converting ω to radians per second gives

Equation:

$$\begin{aligned}
 L &= 0.4(5.979 \times 10^{24} \text{ kg})(6.376 \times 10^6 \text{ m})^2 \left(\frac{1 \text{ rev}}{\text{d}}\right) \\
 &= 9.72 \times 10^{37} \text{ kg} \cdot \text{m}^2 \cdot \text{rev/d}.
 \end{aligned}$$

Substituting $2\pi \text{ rad}$ for 1 rev and $8.64 \times 10^4 \text{ s}$ for 1 day gives

Equation:

$$\begin{aligned}
 L &= (9.72 \times 10^{37} \text{ kg} \cdot \text{m}^2) \left(\frac{2\pi \text{ rad/rev}}{8.64 \times 10^4 \text{ s/d}}\right) (1 \text{ rev/d}) \\
 &= 7.07 \times 10^{33} \text{ kg} \cdot \text{m}^2/\text{s}.
 \end{aligned}$$

Discussion

This number is large, demonstrating that Earth, as expected, has a tremendous angular momentum. The answer is approximate, because we have assumed a constant density for Earth in order to estimate its moment of inertia.

When you push a merry-go-round, spin a bike wheel, or open a door, you exert a torque. If the torque you exert is greater than opposing torques, then the rotation accelerates, and angular momentum increases. The greater the net torque, the more rapid the increase in L . The relationship between torque and angular momentum is

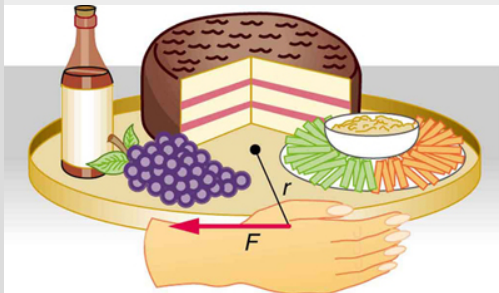
Equation:

$$\text{net } \tau = \frac{\Delta L}{\Delta t}.$$

This expression is exactly analogous to the relationship between force and linear momentum, $F = \Delta p / \Delta t$. The equation $\text{net } \tau = \frac{\Delta L}{\Delta t}$ is very fundamental and broadly applicable. It is, in fact, the rotational form of Newton's second law.

Example:**Calculating the Torque Putting Angular Momentum Into a Lazy Susan**

[\[link\]](#) shows a Lazy Susan food tray being rotated by a person in quest of sustenance. Suppose the person exerts a 2.50 N force perpendicular to the lazy Susan's 0.260-m radius for 0.150 s. (a) What is the final angular momentum of the lazy Susan if it starts from rest, assuming friction is negligible? (b) What is the final angular velocity of the lazy Susan, given that its mass is 4.00 kg and assuming its moment of inertia is that of a disk?



A partygoer exerts a torque on a lazy Susan to make it rotate. The equation $\tau = \frac{\Delta L}{\Delta t}$ gives the relationship between torque and the angular momentum produced.

Strategy

We can find the angular momentum by solving $\tau = \frac{\Delta L}{\Delta t}$ for ΔL , and using the given information to calculate the torque. The final angular momentum equals the change in angular momentum, because the lazy Susan starts from rest. That is, $\Delta L = L$. To find the final velocity, we must calculate ω from the definition of L in $L = I\omega$.

Solution for (a)

Solving $\tau = \frac{\Delta L}{\Delta t}$ for ΔL gives

Equation:

$$\Delta L = (\text{net } \tau) \Delta t.$$

Because the force is perpendicular to r , we see that $\text{net } \tau = rF$, so that

Equation:

$$\begin{aligned} L &= rF \Delta t = (0.260 \text{ m})(2.50 \text{ N})(0.150 \text{ s}) \\ &= 9.75 \times 10^{-2} \text{ kg} \cdot \text{m}^2/\text{s}. \end{aligned}$$

Solution for (b)

The final angular velocity can be calculated from the definition of angular momentum,

Equation:

$$L = I\omega.$$

Solving for ω and substituting the formula for the moment of inertia of a disk into the resulting equation gives

Equation:

$$\omega = \frac{L}{I} = \frac{L}{\frac{1}{2}MR^2}.$$

And substituting known values into the preceding equation yields

Equation:

$$\omega = \frac{9.75 \times 10^{-2} \text{ kg} \cdot \text{m}^2/\text{s}}{(0.500)(4.00 \text{ kg})(0.260 \text{ m})} = 0.721 \text{ rad/s}.$$

Discussion

Note that the imparted angular momentum does not depend on any property of the object but only on torque and time. The final angular velocity is equivalent to one revolution in 8.71 s (determination of the time period is left as an exercise for the reader), which is about right for a lazy Susan.

Note:

Take-Home Experiment

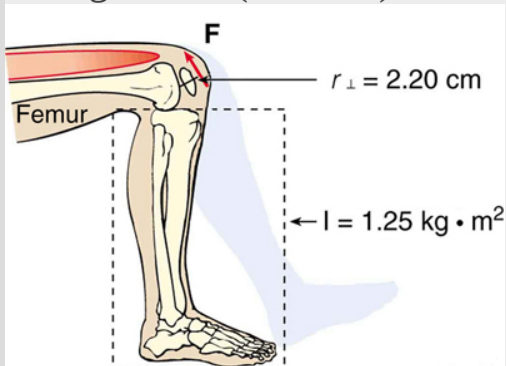
Plan an experiment to analyze changes to a system's angular momentum. Choose a system capable of rotational motion such as a lazy Susan or a merry-go-round. Predict how the angular momentum of this system will change when you add an object to the lazy Susan or jump onto the merry-go-round. What variables can you control? What are you measuring? In other words, what are your independent and dependent variables? Are there any independent variables that it would be useful to keep constant (angular velocity, perhaps)? Collect data in order to calculate or estimate the angular momentum of your system when in motion. What do you observe? Collect data in order to calculate the change in angular momentum as a result of the interaction you performed.

Using your data, how does the angular momentum vary with the size and location of an object added to the rotating system?

Example:

Calculating the Torque in a Kick

The person whose leg is shown in [\[link\]](#) kicks his leg by exerting a 2000-N force with his upper leg muscle. The effective perpendicular lever arm is 2.20 cm. Given the moment of inertia of the lower leg is $1.25 \text{ kg} \cdot \text{m}^2$, (a) find the angular acceleration of the leg. (b) Neglecting the gravitational force, what is the rotational kinetic energy of the leg after it has rotated through 57.3° (1.00 rad)?



The muscle in the upper leg gives the lower leg an angular acceleration and

imparts rotational kinetic energy to it by exerting a torque about the knee. \mathbf{F} is a vector that is perpendicular to r . This example examines the situation.

Strategy

The angular acceleration can be found using the rotational analog to Newton's second law, or $\alpha = \text{net } \tau / I$. The moment of inertia I is given and the torque can be found easily from the given force and perpendicular lever arm. Once the angular acceleration α is known, the final angular velocity and rotational kinetic energy can be calculated.

Solution to (a)

From the rotational analog to Newton's second law, the angular acceleration α is

Equation:

$$\alpha = \frac{\text{net } \tau}{I}.$$

Because the force and the perpendicular lever arm are given and the leg is vertical so that its weight does not create a torque, the net torque is thus

Equation:

$$\begin{aligned}\text{net } \tau &= r_{\perp} F \\ &= (0.0220 \text{ m})(2000 \text{ N}) \\ &= 44.0 \text{ N} \cdot \text{m}.\end{aligned}$$

Substituting this value for the torque and the given value for the moment of inertia into the expression for α gives

Equation:

$$\alpha = \frac{44.0 \text{ N} \cdot \text{m}}{1.25 \text{ kg} \cdot \text{m}^2} = 35.2 \text{ rad/s}^2.$$

Solution to (b)

The final angular velocity can be calculated from the kinematic expression

Equation:

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

or

Equation:

$$\omega^2 = 2\alpha\theta$$

because the initial angular velocity is zero. The kinetic energy of rotation is

Equation:

$$\text{KE}_{\text{rot}} = \frac{1}{2} I \omega^2$$

so it is most convenient to use the value of ω^2 just found and the given value for the moment of inertia. The kinetic energy is then

Equation:

$$\begin{aligned}\text{KE}_{\text{rot}} &= 0.5(1.25 \text{ kg} \cdot \text{m}^2)(70.4 \text{ rad}^2/\text{s}^2) \\ &= 44.0 \text{ J}\end{aligned}$$

Discussion

These values are reasonable for a person kicking his leg starting from the position shown. The weight of the leg can be neglected in part (a) because it exerts no torque when the center of gravity of the lower leg is directly beneath the pivot in the knee. In part (b), the force exerted by the upper leg is so large that its torque is much greater than that created by the weight of the lower leg as it rotates. The rotational kinetic energy given to the lower leg is enough that it could give a ball a significant velocity by transferring some of this energy in a kick.

Note:

Making Connections: Conservation Laws

Angular momentum, like energy and linear momentum, is conserved. This universally applicable law is another sign of underlying unity in physical laws. Angular momentum is conserved when net external torque is zero, just as linear momentum is conserved when the net external force is zero.

Conservation of Angular Momentum

We can now understand why Earth keeps on spinning. As we saw in the previous example, $\Delta L = (\text{net } \tau)\Delta t$. This equation means that, to change angular momentum, a torque must act over some period of time. Because Earth has a large angular momentum, a large torque acting over a long time is needed to change its rate of spin. So what external torques are there? Tidal friction exerts torque that is slowing Earth's rotation, but tens of millions of years must pass before the change is very significant. Recent research indicates the length of the day was 18 h some 900 million years ago. Only the tides exert significant retarding torques on Earth, and so it will continue to spin, although ever more slowly, for many billions of years.

What we have here is, in fact, another conservation law. If the net torque is zero, then angular momentum is constant or *conserved*. We can see this rigorously by considering $\text{net } \tau = \frac{\Delta L}{\Delta t}$ for the situation in which the net torque is zero. In that case,

Equation:

$$\text{net } \tau = 0$$

implying that

Equation:

$$\frac{\Delta L}{\Delta t} = 0.$$

If the change in angular momentum ΔL is zero, then the angular momentum is constant; thus,

Equation:

$$L = \text{constant} \text{ (net } \tau = 0 \text{)}$$

or

Equation:

$$L = L' \text{ (net } \tau = 0 \text{)}.$$

These expressions are the **law of conservation of angular momentum**. Conservation laws are as scarce as they are important.

An example of conservation of angular momentum is seen in [\[link\]](#), in which an ice skater is executing a spin. The net torque on her is very close to zero, because there is relatively little friction between her skates and the ice and because the friction is exerted very close to the pivot point. (Both r and F are small, and so τ is negligibly small.) Consequently, she can spin for quite some time. She can do something else, too. She can increase her rate of spin by pulling her arms and legs in. Why does pulling her arms and legs in increase her rate of spin? The answer is that her angular momentum is constant, so that

Equation:

$$L = L'.$$

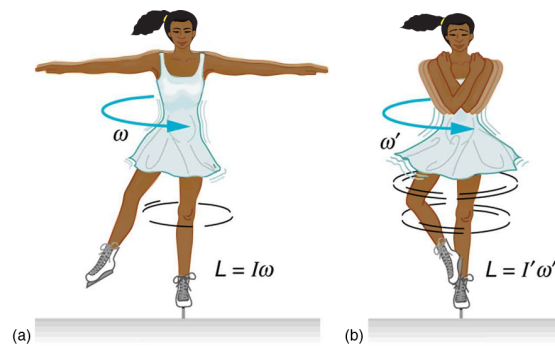
Expressing this equation in terms of the moment of inertia,

Equation:

$$I\omega = I'\omega',$$

where the primed quantities refer to conditions after she has pulled in her arms and reduced her moment of inertia. Because I' is smaller, the angular

velocity ωI must increase to keep the angular momentum constant. The change can be dramatic, as the following example shows.



(a) An ice skater is spinning on the tip of her skate with her arms extended. Her angular momentum is conserved because the net torque on her is negligibly small. In the next image, her rate of spin increases greatly when she pulls in her arms, decreasing her moment of inertia. The work she does to pull in her arms results in an increase in rotational kinetic energy.

Example:

Calculating the Angular Momentum of a Spinning Skater

Suppose an ice skater, such as the one in [\[link\]](#), is spinning at 0.800 rev/ s with her arms extended. She has a moment of inertia of $2.34 \text{ kg} \cdot \text{m}^2$ with her arms extended and of $0.363 \text{ kg} \cdot \text{m}^2$ with her arms close to her body.

(These moments of inertia are based on reasonable assumptions about a 60.0-kg skater.) (a) What is her angular velocity in revolutions per second after she pulls in her arms? (b) What is her rotational kinetic energy before and after she does this?

Strategy

In the first part of the problem, we are looking for the skater's angular velocity ω' after she has pulled in her arms. To find this quantity, we use the conservation of angular momentum and note that the moments of inertia and initial revolutions per second are given. To find the initial and final kinetic energies, we use the definition of rotational kinetic energy given by

Equation:

$$\text{KE}_{\text{rot}} = \frac{1}{2} I \omega^2.$$

Solution for (a)

Because torque is negligible (as discussed above), the conservation of angular momentum given in $I\omega = I'\omega'$ is applicable. Thus,

Equation:

$$L = L'$$

or

Equation:

$$I\omega = I'\omega'$$

Solving for ω' and substituting known values into the resulting equation gives

Equation:

$$\begin{aligned}\omega' &= \frac{I}{I'}\omega = \left(\frac{2.34 \text{ kg}\cdot\text{m}^2}{0.363 \text{ kg}\cdot\text{m}^2}\right)(0.800 \text{ rev/s})(2\pi \text{ rad/rev}) \\ &= 32.4 \text{ rad/s}.\end{aligned}$$

Solution for (b)

Rotational kinetic energy is given by

Equation:

$$KE_{\text{rot}} = \frac{1}{2} I \omega^2.$$

The initial value is found by substituting known values into the equation and converting the angular velocity to rad/s:

Equation:

$$\begin{aligned} KE_{\text{rot}} &= (0.5)(2.34 \text{ kg} \cdot \text{m}^2)((0.800 \text{ rev/s})(2\pi \text{ rad/rev}))^2 \\ &= 29.6 \text{ J.} \end{aligned}$$

The final rotational kinetic energy is

Equation:

$$KE_{\text{rot}'} = \frac{1}{2} I' \omega'^2.$$

Substituting known values into this equation gives

Equation:

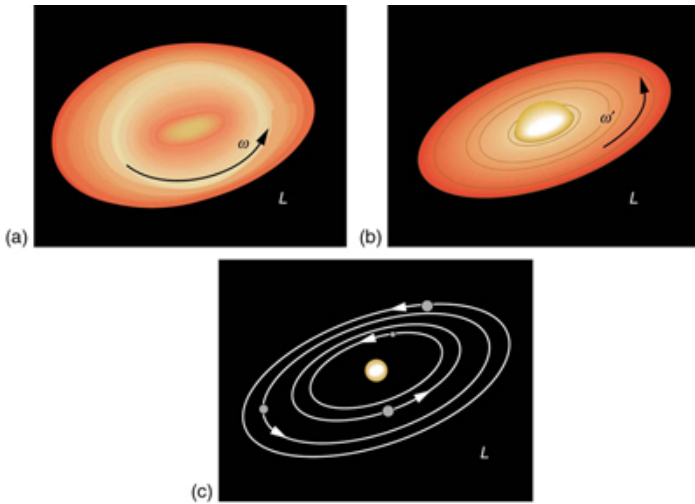
$$\begin{aligned} KE_{\text{rot}'} &= (0.5)(0.363 \text{ kg} \cdot \text{m}^2)(32.4 \text{ rad/s})^2 \\ &= 191 \text{ J.} \end{aligned}$$

Discussion

In both parts, there is an impressive increase. First, the final angular velocity is large, although most world-class skaters can achieve spin rates about this great. Second, the final kinetic energy is much greater than the initial kinetic energy. The increase in rotational kinetic energy comes from work done by the skater in pulling in her arms. This work is internal work that depletes some of the skater's food energy.

There are several other examples of objects that increase their rate of spin because something reduced their moment of inertia. Tornadoes are one example. Storm systems that create tornadoes are slowly rotating. When the radius of rotation narrows, even in a local region, angular velocity increases, sometimes to the furious level of a tornado. Earth is another

example. Our planet was born from a huge cloud of gas and dust, the rotation of which came from turbulence in an even larger cloud. Gravitational forces caused the cloud to contract, and the rotation rate increased as a result. (See [\[link\]](#).)



The Solar System coalesced from a cloud of gas and dust that was originally rotating. The orbital motions and spins of the planets are in the same direction as the original spin and conserve the angular momentum of the parent cloud.

In case of human motion, one would not expect angular momentum to be conserved when a body interacts with the environment as its foot pushes off the ground. Astronauts floating in space aboard the International Space Station have no angular momentum relative to the inside of the ship if they are motionless. Their bodies will continue to have this zero value no matter how they twist about as long as they do not give themselves a push off the side of the vessel.

Exercise:

Check Your Understanding

Problem:

Is angular momentum completely analogous to linear momentum?
What, if any, are their differences?

Solution:

Yes, angular and linear momentums are completely analogous. While they are exact analogs they have different units and are not directly inter-convertible like forms of energy are.

Test Prep for AP Courses**Exercise:****Problem:**

Which rotational system would be best to use as a model to measure how angular momentum changes when forces on the system are changed?

- a. a fishing reel
- b. a planet and its moon
- c. a figure skater spinning
- d. a person's lower leg

Solution:

(a)

Exercise:**Problem:**

You are collecting data to study changes in the angular momentum of a bicycle wheel when a force is applied to it. Which of the following measurements would be least helpful to you?

- a. the time for which the force is applied
- b. the radius at which the force is applied
- c. the angular velocity of the wheel when the force is applied
- d. the direction of the force

Exercise:

Problem:

Which torque applied to a disk with radius 7.0 cm for 3.5 s will produce an angular momentum of 25 N•m•s?

- a. 7.1 N•m
- b. 357.1 N•m
- c. 3.6 N•m
- d. 612.5 N•m

Solution:

(a)

Exercise:

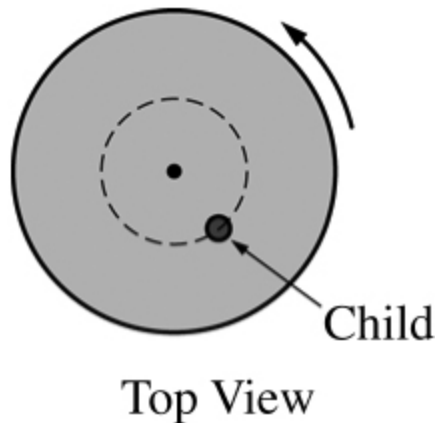
Problem:

Which of the following would be the best way to produce measurable amounts of torque on a system to test the relationship between the angular momentum of the system, the average torque applied to the system, and the time for which the torque is applied?

- a. having different numbers of people push on a merry-go-round
- b. placing known masses on one end of a seesaw
- c. touching the outer edge of a bicycle wheel to a treadmill that is moving at different speeds
- d. hanging known masses from a string that is wound around a spool suspended horizontally on an axle

Exercise:

Problem:



A curved arrow lies at the side of a gray disk. There is a point at the center of the disk, and around the point there is a dashed circle. There is a point labeled “Child” on the dashed circle. Below the disc is a label saying “Top View”.

The diagram above shows a top view of a child of mass M on a circular platform of mass $2M$ that is rotating counterclockwise. Assume the platform rotates without friction. Which of the following describes an action by the child that will increase the angular speed of the platform-child system and why?

- a. The child moves toward the center of the platform, increasing the total angular momentum of the system.
- b. The child moves toward the center of the platform, decreasing the rotational inertia of the system.

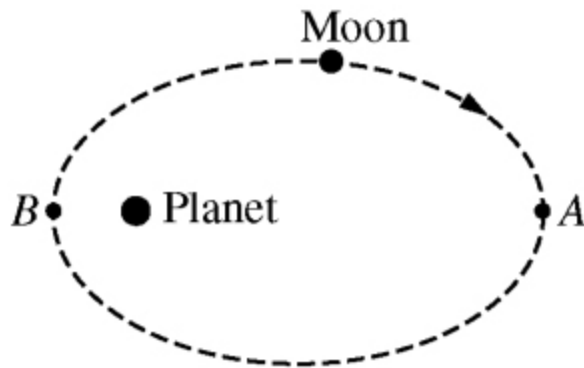
- c. The child moves away from the center of the platform, increasing the total angular momentum of the system.
 - d. The child moves away from the center of the platform, decreasing the rotational inertia of the system.
-

Solution:

(b)

Exercise:

Problem:



A point labeled "Moon" lies on a dashed ellipse. Two other points, labeled "A" and "B", lie at opposite ends of the ellipse. A point labeled "Planet" lies inside the ellipse.

A moon is in an elliptical orbit about a planet as shown above. At point A the moon has speed u_A and is at distance RA from the planet. At point B the moon has speed u_B . Has the moon's angular momentum changed? Explain your answer.

Exercise:

Problem:

A hamster sits 0.10 m from the center of a lazy Susan of negligible mass. The wheel spins with a frequency of 1.0 rev/s. How will the frequency of the lazy Susan change if the hamster walks to 0.30 m from the center of rotation? Assume zero friction and no external torque.

- a. It will speed up to 2.0 rev/s.
- b. It will speed up to 9.0 rev/s.
- c. It will slow to 0.11 rev/s.
- d. It will slow to 0.22 rev/s.

Solution:

(c)

Exercise:**Problem:**

Earth has a mass of 6.0×10^{24} kg, a radius of 6.4×10^6 m, and a rotational frequency of 1.2×10^{-5} rev/s. How would the planet's rotational frequency change if a layer of Earth with mass 1.0×10^{23} kg broke off of the Earth, decreasing Earth's radius by 0.2×10^6 m? Assume no friction.

Exercise:**Problem:**

Consider system A, consisting of two disks of radius R , with both rotating clockwise. Now consider system B, consisting of one disk of radius R rotating counterclockwise and another disk of radius $2R$ rotating clockwise. All of the disks have the same mass, and all have the same magnitude of angular velocity.

Which system has the greatest angular momentum?

- a. A
 - b. B
 - c. They're equal.
 - d. Not enough information
-

Solution:

(b)

Exercise:

Problem:

Assume that a baseball bat being swung at 3π rad/s by a batting machine is equivalent to a 1.1 m thin rod with a mass of 1.0 kg. How fast would a 0.15 kg baseball that squarely hits the very tip of the bat have to be going for the net angular momentum of the bat-ball system to be zero?

Section Summary

- Every rotational phenomenon has a direct translational analog , likewise angular momentum L can be defined as $L = I\omega$.
- This equation is an analog to the definition of linear momentum as $p = mv$. The relationship between torque and angular momentum is $\tau_{\text{net}} = \frac{\Delta L}{\Delta t}$.
- Angular momentum, like energy and linear momentum, is conserved. This universally applicable law is another sign of underlying unity in physical laws. Angular momentum is conserved when net external torque is zero, just as linear momentum is conserved when the net external force is zero.

Conceptual Questions

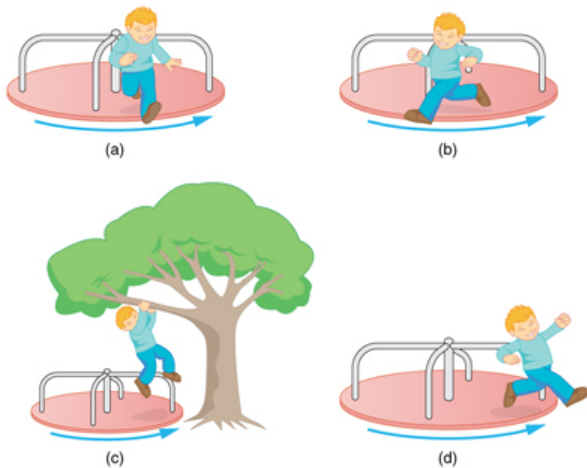
Exercise:

Problem:

When you start the engine of your car with the transmission in neutral, you notice that the car rocks in the opposite sense of the engine's rotation. Explain in terms of conservation of angular momentum. Is the angular momentum of the car conserved for long (for more than a few seconds)?

Exercise:**Problem:**

Suppose a child walks from the outer edge of a rotating merry-go-round to the inside. Does the angular velocity of the merry-go-round increase, decrease, or remain the same? Explain your answer.



A child may jump off a merry-go-round in a variety of directions.

Exercise:

Problem:

Suppose a child gets off a rotating merry-go-round. Does the angular velocity of the merry-go-round increase, decrease, or remain the same if: (a) He jumps off radially? (b) He jumps backward to land motionless? (c) He jumps straight up and hangs onto an overhead tree branch? (d) He jumps off forward, tangential to the edge? Explain your answers. (Refer to [\[link\]](#)).

Exercise:**Problem:**

Helicopters have a small propeller on their tail to keep them from rotating in the opposite direction of their main lifting blades. Explain in terms of Newton's third law why the helicopter body rotates in the opposite direction to the blades.

Exercise:**Problem:**

Whenever a helicopter has two sets of lifting blades, they rotate in opposite directions (and there will be no tail propeller). Explain why it is best to have the blades rotate in opposite directions.

Exercise:**Problem:**

Describe how work is done by a skater pulling in her arms during a spin. In particular, identify the force she exerts on each arm to pull it in and the distance each moves, noting that a component of the force is in the direction moved. Why is angular momentum not increased by this action?

Exercise:

Problem:

When there is a global heating trend on Earth, the atmosphere expands and the length of the day increases very slightly. Explain why the length of a day increases.

Exercise:**Problem:**

Nearly all conventional piston engines have flywheels on them to smooth out engine vibrations caused by the thrust of individual piston firings. Why does the flywheel have this effect?

Exercise:**Problem:**

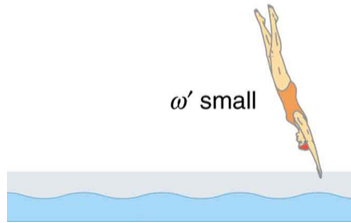
Jet turbines spin rapidly. They are designed to fly apart if something makes them seize suddenly, rather than transfer angular momentum to the plane's wing, possibly tearing it off. Explain how flying apart conserves angular momentum without transferring it to the wing.

Exercise:**Problem:**

An astronaut tightens a bolt on a satellite in orbit. He rotates in a direction opposite to that of the bolt, and the satellite rotates in the same direction as the bolt. Explain why. If a handhold is available on the satellite, can this counter-rotation be prevented? Explain your answer.

Exercise:**Problem:**

Competitive divers pull their limbs in and curl up their bodies when they do flips. Just before entering the water, they fully extend their limbs to enter straight down. Explain the effect of both actions on their angular velocities. Also explain the effect on their angular momenta.



The diver spins rapidly when curled up and slows when she extends her limbs before entering the water.

Exercise:

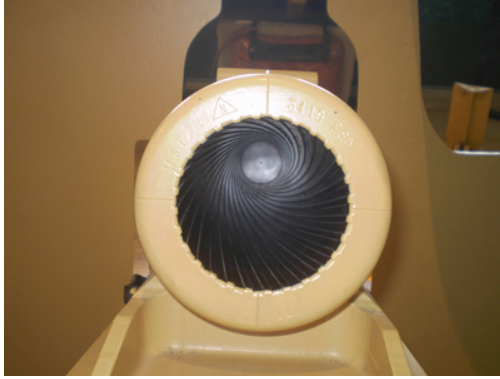
Problem:

Draw a free body diagram to show how a diver gains angular momentum when leaving the diving board.

Exercise:

Problem:

In terms of angular momentum, what is the advantage of giving a football or a rifle bullet a spin when throwing or releasing it?



The image shows a view down the barrel of a cannon, emphasizing its rifling. Rifling in the barrel of a canon causes the projectile to spin just as is the case for rifles (hence the name for the grooves in the barrel).
(credit: Elsie esq., Flickr)

Problems & Exercises

Exercise:

Problem:

- (a) Calculate the angular momentum of the Earth in its orbit around the Sun.
- (b) Compare this angular momentum with the angular momentum of Earth on its axis.

Solution:

- (a) $2.66 \times 10^{40} \text{ kg} \cdot \text{m}^2/\text{s}$

(b) $7.07 \times 10^{33} \text{ kg} \cdot \text{m}^2/\text{s}$

The angular momentum of the Earth in its orbit around the Sun is 3.77×10^6 times larger than the angular momentum of the Earth around its axis.

Exercise:

Problem:

(a) What is the angular momentum of the Moon in its orbit around Earth?

(b) How does this angular momentum compare with the angular momentum of the Moon on its axis? Remember that the Moon keeps one side toward Earth at all times.

(c) Discuss whether the values found in parts (a) and (b) seem consistent with the fact that tidal effects with Earth have caused the Moon to rotate with one side always facing Earth.

Exercise:

Problem:

Suppose you start an antique car by exerting a force of 300 N on its crank for 0.250 s. What angular momentum is given to the engine if the handle of the crank is 0.300 m from the pivot and the force is exerted to create maximum torque the entire time?

Solution:

$22.5 \text{ kg} \cdot \text{m}^2/\text{s}$

Exercise:

Problem:

A playground merry-go-round has a mass of 120 kg and a radius of 1.80 m and it is rotating with a frequency of 0.500 rev/s. What is its angular velocity after a 22.0-kg child gets onto it by grabbing its outer edge? The child is initially at rest.

Exercise:**Problem:**

Three children are riding on the edge of a merry-go-round that is 100 kg, has a 1.60-m radius, and is spinning at 20.0 rpm. The children have masses of 22.0, 28.0, and 33.0 kg. If the child who has a mass of 28.0 kg moves to the center of the merry-go-round, what is the new frequency in rpm?

Solution:

25.3 rpm

Exercise:**Problem:**

(a) Calculate the angular momentum of an ice skater spinning at 6.00 rev/s given his moment of inertia is $0.400 \text{ kg} \cdot \text{m}^2$. (b) He reduces his rate of spin (his angular velocity) by extending his arms and increasing his moment of inertia. Find the value of his moment of inertia if his angular velocity decreases to 1.25 rev/s. (c) Suppose instead he keeps his arms in and allows friction of the ice to slow him to 3.00 rev/s. What average torque was exerted if this takes 15.0 s?

Exercise:**Construct Your Own Problem**

Problem:

Consider the Earth-Moon system. Construct a problem in which you calculate the total angular momentum of the system including the spins of the Earth and the Moon on their axes and the orbital angular momentum of the Earth-Moon system in its nearly monthly rotation. Calculate what happens to the Moon's orbital radius if the Earth's rotation decreases due to tidal drag. Among the things to be considered are the amount by which the Earth's rotation slows and the fact that the Moon will continue to have one side always facing the Earth.

Glossary

angular momentum

the product of moment of inertia and angular velocity

law of conservation of angular momentum

angular momentum is conserved, i.e., the initial angular momentum is equal to the final angular momentum when no external torque is applied to the system

Collisions of Extended Bodies in Two Dimensions

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Observe collisions of extended bodies in two dimensions.
- Examine collisions at the point of percussion.

The information presented in this section supports the following AP[®] learning objectives and science practices:

- **3.F.3.1** The student is able to predict the behavior of rotational collision situations by the same processes that are used to analyze linear collision situations using an analogy between impulse and change of linear momentum and angular impulse and change of angular momentum. **(S.P. 6.4, 7.2)**
- **3.F.3.2** In an unfamiliar context or using representations beyond equations, the student is able to justify the selection of a mathematical routine to solve for the change in angular momentum of an object caused by torques exerted on the object. **(S.P. 2.1)**
- **3.F.3.3** The student is able to plan data collection and analysis strategies designed to test the relationship between torques exerted on an object and the change in angular momentum of that object. **(S.P. 4.1, 4.2, 5.1, 5.3)**
- **4.D.2.1** The student is able to describe a model of a rotational system and use that model to analyze a situation in which angular momentum changes due to interaction with other objects or systems. **(S.P. 1.2, 1.4)**
- **4.D.2.2** The student is able to plan a data collection and analysis strategy to determine the change in angular

momentum of a system and relate it to interactions with other objects and systems. (S.P. 2.2)

Bowling pins are sent flying and spinning when hit by a bowling ball—angular momentum as well as linear momentum and energy have been imparted to the pins. (See [\[link\]](#)). Many collisions involve angular momentum. Cars, for example, may spin and collide on ice or a wet surface. Baseball pitchers throw curves by putting spin on the baseball. A tennis player can put a lot of top spin on the tennis ball which causes it to dive down onto the court once it crosses the net. We now take a brief look at what happens when objects that can rotate collide.

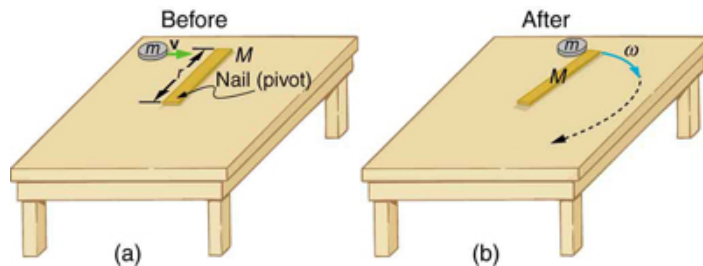
Consider the relatively simple collision shown in [\[link\]](#), in which a disk strikes and adheres to an initially motionless stick nailed at one end to a frictionless surface. After the collision, the two rotate about the nail. There is an unbalanced external force on the system at the nail. This force exerts no torque because its lever arm r is zero. Angular momentum is therefore conserved in the collision. Kinetic energy is not conserved, because the collision is inelastic. It is possible that momentum is not conserved either because the force at the nail may have a component in the direction of the disk's initial velocity. Let us examine a case of rotation in a collision in [\[link\]](#).



The bowling ball causes
the pins to fly, some of

them spinning violently.

(credit: Tinou Bao,
Flickr)



(a) A disk slides toward a motionless stick on a frictionless surface. (b) The disk hits the stick at one end and adheres to it, and they rotate together, pivoting around the nail. Angular momentum is conserved for this inelastic collision because the surface is frictionless and the unbalanced external force at the nail exerts no torque.

Example:

Rotation in a Collision

Suppose the disk in [\[link\]](#) has a mass of 50.0 g and an initial velocity of 30.0 m/s when it strikes the stick that is 1.20 m long and 2.00 kg.

- (a) What is the angular velocity of the two after the collision?
- (b) What is the kinetic energy before and after the collision?
- (c) What is the total linear momentum before and after the collision?

Strategy for (a)

We can answer the first question using conservation of angular momentum as noted. Because angular momentum is $I\omega$, we can solve for angular

velocity.

Solution for (a)

Conservation of angular momentum states

Equation:

$$L = L',$$

where primed quantities stand for conditions after the collision and both momenta are calculated relative to the pivot point. The initial angular momentum of the system of stick-disk is that of the disk just before it strikes the stick. That is,

Equation:

$$L = I\omega,$$

where I is the moment of inertia of the disk and ω is its angular velocity around the pivot point. Now, $I = mr^2$ (taking the disk to be approximately a point mass) and $\omega = v/r$, so that

Equation:

$$L = mr^2 \frac{v}{r} = mvr.$$

After the collision,

Equation:

$$L' = I'\omega'.$$

It is ω' that we wish to find. Conservation of angular momentum gives

Equation:

$$I'\omega' = mvr.$$

Rearranging the equation yields

Equation:

$$\omega' = \frac{mvr}{I'}$$

where I' is the moment of inertia of the stick and disk stuck together, which is the sum of their individual moments of inertia about the nail.

[\[link\]](#) gives the formula for a rod rotating around one end to be

$I = Mr^2/3$. Thus,

Equation:

$$I' = mr^2 + \frac{Mr^2}{3} = \left(m + \frac{M}{3}\right)r^2.$$

Entering known values in this equation yields,

Equation:

$$I' = (0.0500 \text{ kg} + 0.667 \text{ kg})(1.20 \text{ m})^2 = 1.032 \text{ kg} \cdot \text{m}^2.$$

The value of I' is now entered into the expression for ω' , which yields

Equation:

$$\begin{aligned}\omega' &= \frac{mvr}{I'} = \frac{(0.0500 \text{ kg})(30.0 \text{ m/s})(1.20 \text{ m})}{1.032 \text{ kg} \cdot \text{m}^2} \\ &= 1.744 \text{ rad/s} \approx 1.74 \text{ rad/s}.\end{aligned}$$

Strategy for (b)

The kinetic energy before the collision is the incoming disk's translational kinetic energy, and after the collision, it is the rotational kinetic energy of the two stuck together.

Solution for (b)

First, we calculate the translational kinetic energy by entering given values for the mass and speed of the incoming disk.

Equation:

$$\text{KE} = \frac{1}{2}mv^2 = (0.500)(0.0500 \text{ kg})(30.0 \text{ m/s})^2 = 22.5 \text{ J}$$

After the collision, the rotational kinetic energy can be found because we now know the final angular velocity and the final moment of inertia. Thus, entering the values into the rotational kinetic energy equation gives

Equation:

$$\begin{aligned}\text{KE}' &= \frac{1}{2}I'\omega'^2 = (0.5)(1.032 \text{ kg} \cdot \text{m}^2)\left(1.744 \frac{\text{rad}}{\text{s}}\right)^2 \\ &= 1.57 \text{ J}.\end{aligned}$$

Strategy for (c)

The linear momentum before the collision is that of the disk. After the collision, it is the sum of the disk's momentum and that of the center of mass of the stick.

Solution of (c)

Before the collision, then, linear momentum is

Equation:

$$p = mv = (0.0500 \text{ kg})(30.0 \text{ m/s}) = 1.50 \text{ kg} \cdot \text{m/s}.$$

After the collision, the disk and the stick's center of mass move in the same direction. The total linear momentum is that of the disk moving at a new velocity $v' = r\omega'$ plus that of the stick's center of mass, which moves at half this speed because $v_{\text{CM}} = \left(\frac{r}{2}\right)\omega' = \frac{v'}{2}$. Thus,

Equation:

$$p' = mv' + Mv_{\text{CM}} = mv' + \frac{Mv'}{2}.$$

Gathering similar terms in the equation yields,

Equation:

$$p' = \left(m + \frac{M}{2}\right)v'$$

so that

Equation:

$$p' = \left(m + \frac{M}{2}\right)r\omega'.$$

Substituting known values into the equation,

Equation:

$$p' = (1.050 \text{ kg})(1.20 \text{ m})(1.744 \text{ rad/s}) = 2.20 \text{ kg} \cdot \text{m/s}.$$

Discussion

First note that the kinetic energy is less after the collision, as predicted, because the collision is inelastic. More surprising is that the momentum after the collision is actually greater than before the collision. This result can be understood if you consider how the nail affects the stick and vice versa. Apparently, the stick pushes backward on the nail when first struck by the disk. The nail's reaction (consistent with Newton's third law) is to push forward on the stick, imparting momentum to it in the same direction in which the disk was initially moving, thereby increasing the momentum of the system.

Note:

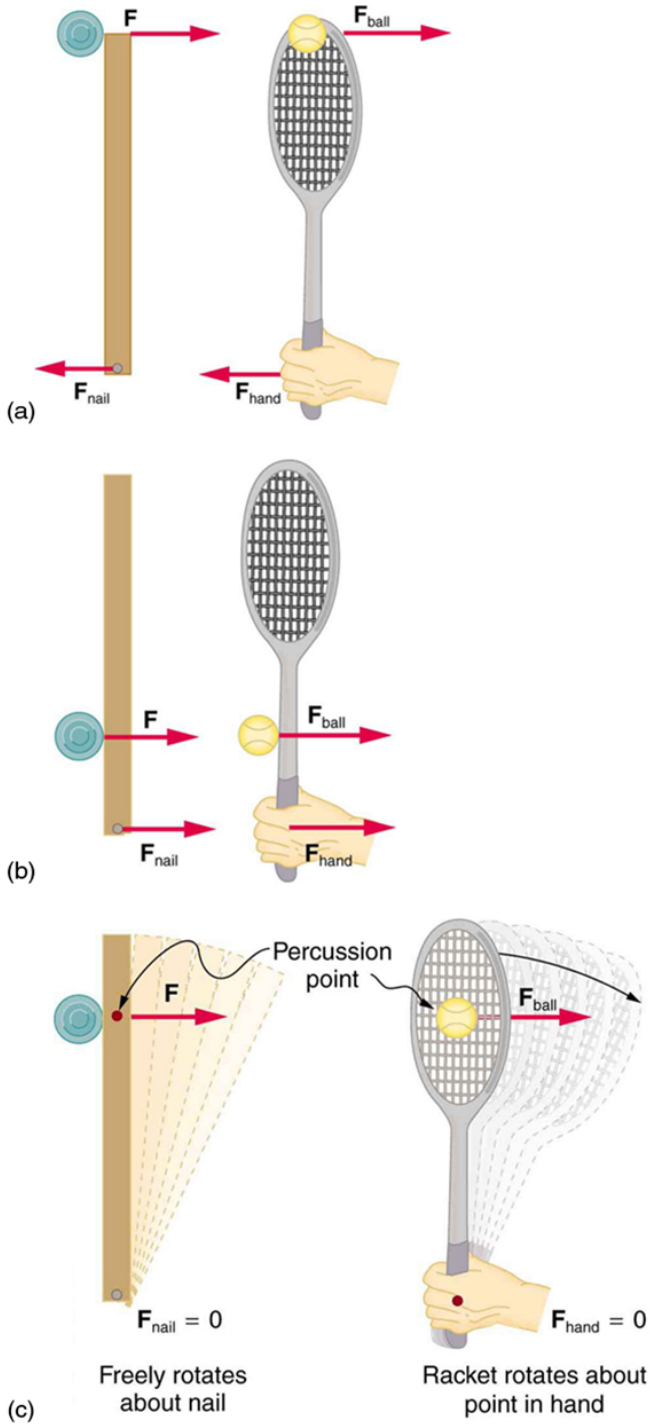
Applying the Science Practices: Rotational Collisions

When the disk in [\[link\]](#) strikes the stick, it exerts a torque on the stick. It is this torque that changes the angular momentum of the stick. A greater torque would produce a greater increase in angular momentum. As you saw in [\[link\]](#), the relationship between net torque and angular momentum can be expressed as $\tau = \frac{\Delta L}{\Delta t}$, where ΔL represents the change in angular momentum and Δt represents the time interval that it took for the angular momentum to change. How can you test this? Design an experiment in which you apply different measurable torques to a simple system. The torque applied to the stick can be varied by changing the mass of the disk, the initial velocity of the disk, or the radius of the impact of the disk on the stick. How will you measure changes to angular momentum? Recall that angular momentum is defined as $L = I\omega$.

The above example has other implications. For example, what would happen if the disk hit very close to the nail? Obviously, a force would be exerted on the nail in the forward direction. So, when the stick is struck at the end farthest from the nail, a backward force is exerted on the nail, and when it is hit at the end nearest the nail, a forward force is exerted on the

nail. Thus, striking it at a certain point in between produces no force on the nail. This intermediate point is known as the *percussion point*.

An analogous situation occurs in tennis as seen in [\[link\]](#). If you hit a ball with the end of your racquet, the handle is pulled away from your hand. If you hit a ball much farther down, for example, on the shaft of the racquet, the handle is pushed into your palm. And if you hit the ball at the racquet's percussion point (what some people call the “sweet spot”), then little or *no* force is exerted on your hand, and there is less vibration, reducing chances of a tennis elbow. The same effect occurs for a baseball bat.



A disk hitting a stick is compared to a tennis ball being hit by a racquet. (a) When the ball strikes the racquet near the end, a backward force is exerted on the hand. (b) When the

racquet is struck much farther down, a forward force is exerted on the hand. (c) When the racquet is struck at the percussion point, no force is delivered to the hand.

Exercise:

Check Your Understanding

Problem: Is rotational kinetic energy a vector? Justify your answer.

Solution:

No, energy is always scalar whether motion is involved or not. No form of energy has a direction in space and you can see that rotational kinetic energy does not depend on the direction of motion just as linear kinetic energy is independent of the direction of motion.

Test Prep for AP Courses

Exercise:

Problem:

A box with a mass of 2.0 kg rests on one end of a seesaw. The seesaw is 6.0 m long, and we can assume it has negligible mass.

Approximately what angular momentum will the box have if someone with a mass of 65 kg sits on the other end of the seesaw quickly, with a velocity of 1.2 m/s?

- a. $702 \text{ kg}\cdot\text{m}^2/\text{s}$
- b. $39 \text{ kg}\cdot\text{m}^2/\text{s}$
- c. $18 \text{ kg}\cdot\text{m}^2/\text{s}$
- d. $1.2 \text{ kg}\cdot\text{m}^2/\text{s}$

Solution:

(b)

Exercise:**Problem:**

A spinner in a board game can be thought of as a thin rod that spins about an axis at its center. The spinner in a certain game is 12 cm long and has a mass of 10 g. How will its angular velocity change when it is flicked at one end with a force equivalent to 15 g travelling at 5.0 m/s if all the energy of the collision is transferred to the spinner? (You can use the table in [Figure 10.12](#) to estimate the rotational inertia of the spinner.)

Exercise:**Problem:**

A cyclist pedals to exert a torque on the rear wheel of the bicycle. When the cyclist changes to a higher gear, the torque increases. Which of the following would be the most effective strategy to help you determine the change in angular momentum of the bicycle wheel?

- a. multiplying the ratio between the two torques by the mass of the bicycle and rider
- b. adding the two torques together, and multiplying by the time for which both torques are applied
- c. multiplying the difference in the two torques by the time for which the new torque is applied
- d. multiplying both torques by the mass of the bicycle and rider

Solution:

(c)

Exercise:

Problem:

An electric screwdriver has two speeds, each of which exerts a different torque on a screw. Describe what calculations you could use to help you compare the angular momentum of a screw at each speed. What measurements would you need to make in order to calculate this?

Exercise:**Problem:**

Why is it important to consider the shape of an object when determining the object's angular momentum?

- a. The shape determines the location of the center of mass. The location of the center of mass in turn determines the angular velocity of the object.
- b. The shape helps you determine the location of the object's outer edge, where rotational velocity will be greatest.
- c. The shape helps you determine the location of the center of rotation.
- d. The shape determines the location of the center of mass. The location of the center of mass contributes to the object's rotational inertia, which contributes to its angular momentum.

Solution:

(d)

Exercise:**Problem:**

How could you collect and analyze data to test the difference between the torques provided by two speeds on a tabletop fan?

Exercise:

Problem:

Describe a rotational system you could use to demonstrate the effect on the system's angular momentum of applying different amounts of external torque.

Solution:

A door on hinges is a rotational system. When you push or pull on the door handle, the angular momentum of the system changes. If a weight is hung on the door handle, then pushing on the door with the same force will cause a different increase in angular momentum. If you push or pull near the hinges with the same force, the resulting angular momentum of the system will also be different.

Exercise:**Problem:**

How could you use simple equipment such as balls and string to study the changes in angular momentum of a system when it interacts with another system?

Section Summary

- Angular momentum L is analogous to linear momentum and is given by $L = I\omega$.
- Angular momentum is changed by torque, following the relationship $\tau_{\text{net}} = \frac{\Delta L}{\Delta t}$.
- Angular momentum is conserved if the net torque is zero $L = \text{constant}$ (net $\tau = 0$) or $L = L'$ (net $\tau = 0$). This equation is known as the law of conservation of angular momentum, which may be conserved in collisions.

Conceptual Questions**Exercise:**

Problem:

Describe two different collisions—one in which angular momentum is conserved, and the other in which it is not. Which condition determines whether or not angular momentum is conserved in a collision?

Exercise:**Problem:**

Suppose an ice hockey puck strikes a hockey stick that lies flat on the ice and is free to move in any direction. Which quantities are likely to be conserved: angular momentum, linear momentum, or kinetic energy (assuming the puck and stick are very resilient)?

Exercise:**Problem:**

While driving his motorcycle at highway speed, a physics student notices that pulling back lightly on the right handlebar tips the cycle to the left and produces a left turn. Explain why this happens.

Problems & Exercises**Exercise:****Problem:**

Repeat [\[link\]](#) in which the disk strikes and adheres to the stick 0.100 m from the nail.

Solution:

(a) 0.156 rad/s

(b) 1.17×10^{-2} J

(c) $0.188 \text{ kg} \cdot \text{m/s}$

Exercise:

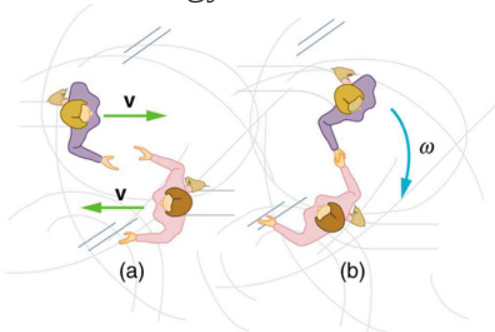
Problem:

Repeat [\[link\]](#) in which the disk originally spins clockwise at 1000 rpm and has a radius of 1.50 cm.

Exercise:

Problem:

Twin skaters approach one another as shown in [\[link\]](#) and lock hands. (a) Calculate their final angular velocity, given each had an initial speed of 2.50 m/s relative to the ice. Each has a mass of 70.0 kg, and each has a center of mass located 0.800 m from their locked hands. You may approximate their moments of inertia to be that of point masses at this radius. (b) Compare the initial kinetic energy and final kinetic energy.



Twin skaters approach each other with identical speeds. Then, the skaters lock hands and spin.

Solution:

(a) 3.13 rad/s

(b) Initial KE = 438 J, final KE = 438 J

Exercise:

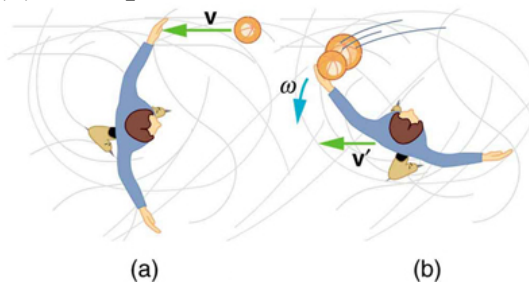
Problem:

Suppose a 0.250-kg ball is thrown at 15.0 m/s to a motionless person standing on ice who catches it with an outstretched arm as shown in [\[link\]](#).

(a) Calculate the final linear velocity of the person, given his mass is 70.0 kg.

(b) What is his angular velocity if each arm is 5.00 kg? You may treat the ball as a point mass and treat the person's arms as uniform rods (each has a length of 0.900 m) and the rest of his body as a uniform cylinder of radius 0.180 m. Neglect the effect of the ball on his center of mass so that his center of mass remains in his geometrical center.

(c) Compare the initial and final total kinetic energies.



The figure shows the overhead view of a person standing motionless on ice about to catch a ball. Both arms are outstretched. After catching the ball, the skater recoils and rotates.

Exercise:

Problem:

Repeat [\[link\]](#) in which the stick is free to have translational motion as well as rotational motion.

Solution:

- (a) 1.70 rad/s
- (b) Initial KE = 22.5 J, final KE = 2.04 J
- (c) 1.50 kg · m/s

Gyroscopic Effects: Vector Aspects of Angular Momentum

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Describe the **right-hand rule** to find the direction of angular velocity, momentum, and torque.
- Explain the gyroscopic effect.
- Study how Earth acts like a gigantic gyroscope.

The information presented in this section supports the following AP® learning objectives and science practices:

- **4.D.3.1** The student is able to use appropriate mathematical routines to calculate values for initial or final angular momentum, or change in angular momentum of a system, or average torque or time during which the torque is exerted in analyzing a situation involving torque and angular momentum. **(S.P. 2.2)**
- **4.D.3.2** The student is able to plan a data collection strategy designed to test the relationship between the change in angular momentum of a system and the product of the average torque applied to the system and the time interval during which the torque is exerted. **(S.P. 4.1, 4.2)**

Angular momentum is a vector and, therefore, *has direction as well as magnitude*. Torque affects both the direction and the magnitude of angular momentum. What is the direction of the angular momentum of a rotating

object like the disk in [\[link\]](#)? The figure shows the **right-hand rule** used to find the direction of both angular momentum and angular velocity. Both \mathbf{L} and $\boldsymbol{\omega}$ are vectors—each has direction and magnitude. Both can be represented by arrows. The right-hand rule defines both to be perpendicular to the plane of rotation in the direction shown. Because angular momentum is related to angular velocity by $\mathbf{L} = I\boldsymbol{\omega}$, the direction of \mathbf{L} is the same as the direction of $\boldsymbol{\omega}$. Notice in the figure that both point along the axis of rotation.

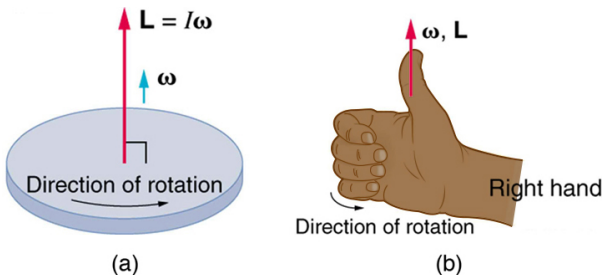


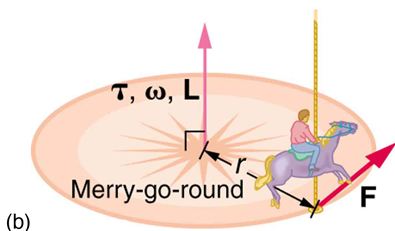
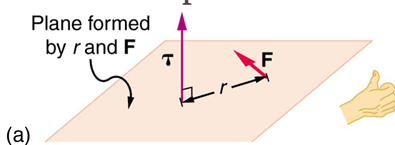
Figure (a) shows a disk is rotating counterclockwise when viewed from above. Figure (b) shows the right-hand rule. The direction of angular velocity $\boldsymbol{\omega}$ size and angular momentum \mathbf{L} are defined to be the direction in which the thumb of your right hand points when you curl your fingers in the direction of the disk's rotation as shown.

Now, recall that torque changes angular momentum as expressed by **Equation:**

$$\text{net } \boldsymbol{\tau} = \frac{\Delta \mathbf{L}}{\Delta t}.$$

This equation means that the direction of $\Delta \mathbf{L}$ is the same as the direction of the torque $\boldsymbol{\tau}$ that creates it. This result is illustrated in [\[link\]](#), which shows the direction of torque and the angular momentum it creates.

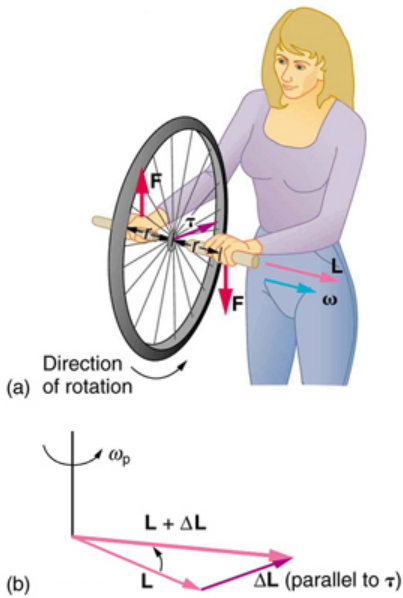
Let us now consider a bicycle wheel with a couple of handles attached to it, as shown in [\[link\]](#). (This device is popular in demonstrations among physicists, because it does unexpected things.) With the wheel rotating as shown, its angular momentum is to the woman's left. Suppose the person holding the wheel tries to rotate it as in the figure. Her natural expectation is that the wheel will rotate in the direction she pushes it—but what happens is quite different. The forces exerted create a torque that is horizontal toward the person, as shown in [\[link\]](#)(a). This torque creates a change in angular momentum \mathbf{L} in the same direction, perpendicular to the original angular momentum \mathbf{L} , thus changing the direction of \mathbf{L} but not the magnitude of \mathbf{L} . [\[link\]](#) shows how $\Delta \mathbf{L}$ and \mathbf{L} add, giving a new angular momentum with direction that is inclined more toward the person than before. The axis of the wheel has thus moved *perpendicular to the forces exerted on it*, instead of in the expected direction.



In figure (a), the torque is perpendicular to the plane formed by r and \mathbf{F} and is the direction your right thumb would point to if you curled your fingers in the

direction of \mathbf{F} .

Figure (b) shows that the direction of the torque is the same as that of the angular momentum it produces.



In figure (a), a person holding the spinning bike wheel lifts it with her right hand and pushes down with her left hand in an attempt to rotate the wheel. This action creates a torque directly toward her. This torque causes a change in angular momentum $\Delta\mathbf{L}$ in

exactly the same direction. Figure (b) shows a vector diagram depicting how $\Delta \mathbf{L}$ and \mathbf{L} add, producing a new angular momentum pointing more toward the person. The wheel moves toward the person, perpendicular to the forces she exerts on it.

Note:

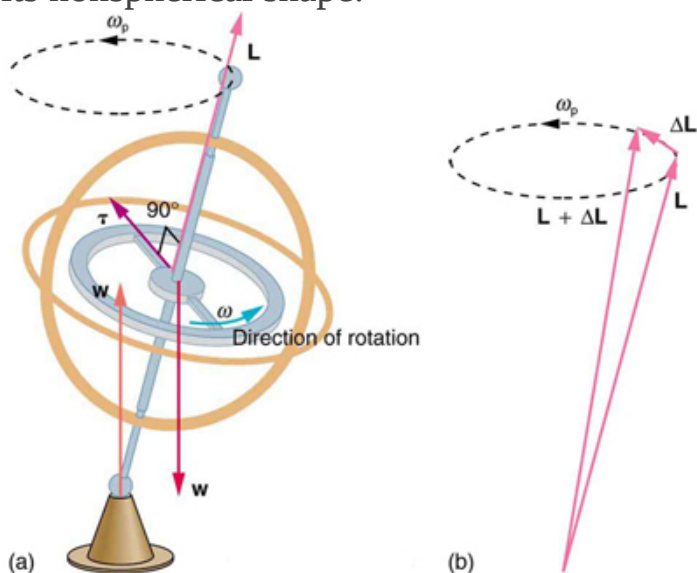
Applying Science Practices: Angular Momentum and Torque

You have seen that change in angular momentum depends on the average torque applied and the time interval during which the torque is applied. Plan an experiment similar to the one shown in [Figure 10.30](#) to test the relationship between the change in angular momentum of a system and the product of the average torque applied to the system and the time interval during which the torque is exerted. What would you use as your test system? How could you measure applied torque? What observations could you make to help you analyze changes in angular momentum? Remember that, since angular momentum is a vector, changes can relate to its magnitude or its direction.

This same logic explains the behavior of gyroscopes. [\[link\]](#) shows the two forces acting on a spinning gyroscope. The torque produced is perpendicular to the angular momentum, thus the direction of the torque is

changed, but not its magnitude. The gyroscope *precesses* around a vertical axis, since the torque is always horizontal and perpendicular to \mathbf{L} . If the gyroscope is *not* spinning, it acquires angular momentum in the direction of the torque ($\mathbf{L} = \Delta\mathbf{L}$), and it rotates around a horizontal axis, falling over just as we would expect.

Earth itself acts like a gigantic gyroscope. Its angular momentum is along its axis and points at Polaris, the North Star. But Earth is slowly precessing (once in about 26,000 years) due to the torque of the Sun and the Moon on its nonspherical shape.



As seen in figure (a), the forces on a spinning gyroscope are its weight and the supporting force from the stand.

These forces create a horizontal torque on the gyroscope, which create a change in angular momentum $\Delta\mathbf{L}$ that is also horizontal. In figure (b),

$\Delta\mathbf{L}$ and \mathbf{L} add to produce a new angular momentum with the same magnitude, but different direction, so that the gyroscope precesses in the direction shown instead of falling over.

Exercise:
Check Your Understanding

Problem:

Rotational kinetic energy is associated with angular momentum? Does that mean that rotational kinetic energy is a vector?

Solution:

No, energy is always a scalar whether motion is involved or not. No form of energy has a direction in space and you can see that rotational kinetic energy does not depend on the direction of motion just as linear kinetic energy is independent of the direction of motion.

Test Prep for AP Courses

Exercise:

Problem:

A globe (model of the Earth) is a hollow sphere with a radius of 16 cm. By wrapping a cord around the equator of a globe and pulling on it, a person exerts a torque on the globe of $120 \text{ N} \cdot \text{m}$ for 1.2 s. What angular momentum does the globe have after 1.2 s?

Solution:

Since the globe is stationary to start with,

$$\tau = \frac{\Delta L}{\Delta t}$$

$$\tau \cdot \Delta t = \Delta L$$

By substituting,

$$120 \text{ N} \cdot \text{m} \cdot 1.2 \text{ s} = 144 \text{ N} \cdot \text{m} \cdot \text{s}.$$

The angular momentum of the globe after 1.2 s is $144 \text{ N}\cdot\text{m}\cdot\text{s}$.

Exercise:

Problem:

How could you use a fishing reel to test the relationship between the torque applied to a system, the time for which the torque was applied, and the resulting angular momentum of the system? How would you measure angular momentum?

Section Summary

- Torque is perpendicular to the plane formed by \mathbf{r} and \mathbf{F} and is the direction your right thumb would point if you curled the fingers of your right hand in the direction of \mathbf{F} . The direction of the torque is thus the same as that of the angular momentum it produces.
- The gyroscope precesses around a vertical axis, since the torque is always horizontal and perpendicular to \mathbf{L} . If the gyroscope is not spinning, it acquires angular momentum in the direction of the torque ($\mathbf{L} = \Delta\mathbf{L}$), and it rotates about a horizontal axis, falling over just as we would expect.
- Earth itself acts like a gigantic gyroscope. Its angular momentum is along its axis and points at Polaris, the North Star.

Conceptual Questions

Exercise:

Problem:

While driving his motorcycle at highway speed, a physics student notices that pulling back lightly on the right handlebar tips the cycle to the left and produces a left turn. Explain why this happens.

Exercise:

Problem:

Gyroscopes used in guidance systems to indicate directions in space must have an angular momentum that does not change in direction. Yet they are often subjected to large forces and accelerations. How can the direction of their angular momentum be constant when they are accelerated?

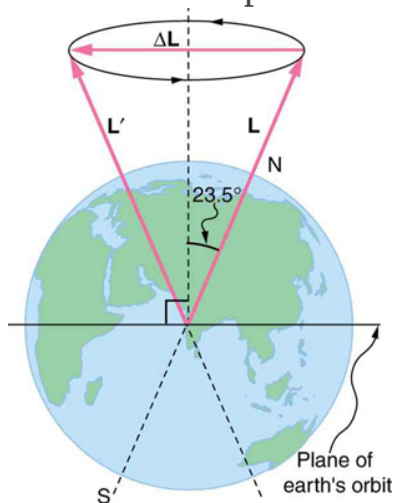
Problem Exercises

Exercise:

Problem: Integrated Concepts

The axis of Earth makes a 23.5° angle with a direction perpendicular to the plane of Earth's orbit. As shown in [\[link\]](#), this axis precesses, making one complete rotation in 25,780 y.

- (a) Calculate the change in angular momentum in half this time.
- (b) What is the average torque producing this change in angular momentum?
- (c) If this torque were created by a single force (it is not) acting at the most effective point on the equator, what would its magnitude be?



The Earth's axis slowly precesses, always making an angle of 23.5° with the direction perpendicular to the plane of Earth's orbit. The change in angular momentum for the two shown positions is quite large, although the magnitude \mathbf{L} is unchanged.

Solution:

(a) $5.64 \times 10^{33} \text{ kg} \cdot \text{m}^2/\text{s}$

(b) $1.39 \times 10^{22} \text{ N} \cdot \text{m}$

(c) $2.17 \times 10^{15} \text{ N}$

Glossary

right-hand rule

direction of angular velocity ω and angular momentum \mathbf{L} in which the thumb of your right hand points when you curl your fingers in the direction of the disk's rotation

Connection for AP® Courses
class="introduction"

There
are at
least
four
types
of
waves
in this
picture
—only
the
water
waves
are
evident
. There
are also
sound
waves,
light
waves,
and
waves
on the
guitar
strings.
(credit:
John
Norton
)



In this chapter, students are introduced to oscillation, the regular variation in the position of a system about a central point accompanied by transfer of energy and momentum, and to **waves**. A child's swing, a pendulum, a spring, and a vibrating string are all examples of oscillations. This chapter will address simple harmonic motion and periods of vibration, aspects of oscillation that produce waves, a common phenomenon in everyday life. Waves carry energy from one place to another." This chapter will show how harmonic oscillations produce waves that transport energy across space and through time. The information and examples presented support Big Ideas 1, 2, and 3 of the AP® Physics Curriculum Framework.

The chapter opens by discussing the forces that govern oscillations and waves. It goes on to discuss important concepts such as **simple harmonic motion**, uniform harmonic motion, and damped harmonic motion. You will also learn about energy in simple harmonic motion and how it changes from kinetic to potential, and how the total sum, which would be the mechanical energy of the oscillator, remains constant or conserved at all times. The chapter also discusses characteristics of waves, such as their frequency, period of oscillation, and the forms in which they can exist, i.e., transverse or longitudinal. The chapter ends by discussing what happens when two or more waves overlap and how the **amplitude** of the resultant wave changes, leading to the phenomena of **superposition** and **interference**.

The concepts in this chapter support:

Big Idea 3 The interactions of an object with other objects can be described by forces.

Enduring Understanding 3.B Classically, the acceleration of an object interacting with other objects can be predicted by using $\vec{a} = \sum \frac{\vec{F}}{m}$.

Essential Knowledge 3.B.3 Restoring forces can result in oscillatory motion. When a linear restoring force is exerted on an object displaced from an equilibrium position, the object will undergo a special type of motion called simple harmonic motion. Examples should include gravitational force exerted by the Earth on a simple pendulum and a mass-spring oscillator.

Big Idea 4 Interactions between systems can result in changes in those systems.

Enduring Understanding 4.C Interactions with other objects or systems can change the total energy of a system.

Essential Knowledge 4.C.1 The energy of a system includes its kinetic energy, potential energy, and microscopic internal energy. Examples should include gravitational potential energy, elastic potential energy, and kinetic energy.

Essential Knowledge 4.C.2 Mechanical energy (the sum of kinetic and potential energy) is transferred into or out of a system when an external force is exerted on a system such that a component of the force is parallel to its displacement. The process through which the energy is transferred is called work.

Big Idea 5 Changes that occur as a result of interactions are constrained by conservation laws.

Enduring Understanding 5.B The energy of a system is conserved.

Essential Knowledge 5.B.2 A system with internal structure can have internal energy, and changes in a system's internal structure can result in changes in internal energy. [Physics 1: includes mass-spring oscillators and simple pendulums. Physics 2: includes charged object in electric fields and examining changes in internal energy with changes in configuration.]

Big Idea 6 Waves can transfer energy and momentum from one location to another without the permanent transfer of mass and serve as a mathematical model for the description of other phenomena.

Enduring Understanding 6.A A wave is a traveling disturbance that transfers energy and momentum.

Essential Knowledge 6.A.1 Waves can propagate via different oscillation modes such as transverse and longitudinal.

Essential Knowledge 6.A.2 For propagation, mechanical waves require a medium, while electromagnetic waves do not require a physical medium. Examples should include light traveling through a vacuum and sound not traveling through a vacuum.

Essential Knowledge 6.A.3 The amplitude is the maximum displacement of a wave from its equilibrium value.

Essential Knowledge 6.A.4 Classically, the energy carried by a wave depends on and increases with amplitude. Examples should include sound waves.

Enduring Understanding 6.B A periodic wave is one that repeats as a function of both time and position and can be described by its amplitude, frequency, wavelength, speed, and energy.

Essential Knowledge 6.B.1 The period is the repeat time of the wave. The frequency is the number of repetitions over a period of time.

Essential Knowledge 6.B.2 The wavelength is the repeat distance of the wave.

Essential Knowledge 6.B.3 A simple wave can be described by an equation involving one sine or cosine function involving the wavelength, amplitude, and frequency of the wave.

Essential Knowledge 6.B.4 The wavelength is the ratio of speed over frequency.

Enduring Understanding 6.C Only waves exhibit interference and diffraction.

Essential Knowledge 6.C.1 When two waves cross, they travel through each other; they do not bounce off each other. Where the waves overlap, the resulting displacement can be determined by adding the displacements of the two waves. This is called superposition.

Enduring Understanding 6.D Interference and superposition lead to standing waves and beats.

Essential Knowledge 6.D.1 Two or more wave pulses can interact in such a way as to produce amplitude variations in the resultant wave. When two pulses cross, they travel through each other; they do not bounce off each other. Where the pulses overlap, the resulting displacement can be determined by adding the displacements of the two pulses. This is called superposition.

Essential Knowledge 6.D.2 Two or more traveling waves can interact in such a way as to produce amplitude variations in the resultant wave.

Essential Knowledge 6.D.3 Standing waves are the result of the addition of incident and reflected waves that are confined to a region and have nodes and antinodes. Examples should include waves on a fixed length of string, and sound waves in both closed and open tubes.

Essential Knowledge 6.D.4 The possible wavelengths of a standing wave are determined by the size of the region to which it is confined.

Essential Knowledge 6.D.5 Beats arise from the addition of waves of slightly different frequency.

Glossary

oscillate

moving back and forth regularly between two points

wave

a disturbance that moves from its source and carries energy

Hooke's Law: Stress and Strain Revisited

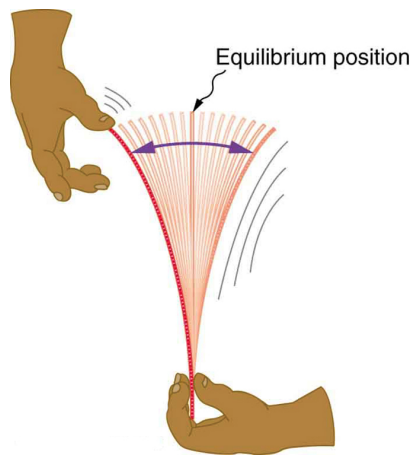
LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain Newton's third law of motion with respect to stress and deformation.
- Describe the restoring force and displacement.
- Use Hooke's law of deformation, and calculate stored energy in a spring.

The information presented in this section supports the following AP® learning objectives and science practices:

- **3.B.3.3** The student can analyze data to identify qualitative or quantitative relationships between given values and variables (i.e., force, displacement, acceleration, velocity, period of motion, frequency, spring constant, string length, mass) associated with objects in oscillatory motion to use that data to determine the value of an unknown. **(S.P. 2.2, 5.1)**
- **3.B.3.4** The student is able to construct a qualitative and/or a quantitative explanation of oscillatory behavior given evidence of a restoring force. **(S.P. 2.2, 6.2)**



When displaced from its vertical equilibrium position, this plastic ruler oscillates back and forth because of the restoring force opposing displacement.

When the ruler is on the left, there is a force to the right, and vice versa.

Newton's first law implies that an object oscillating back and forth is experiencing forces. Without force, the object would move in a straight line at a constant speed rather than oscillate. Consider, for example, plucking a plastic ruler to the left as shown in [\[link\]](#). The deformation of the ruler creates a force in the opposite direction, known as a **restoring force**. Once released, the restoring force causes the ruler to move back toward its stable equilibrium position, where the net force on it is zero. However, by the time the ruler gets there, it gains momentum and continues to move to the right, producing the opposite deformation. It is then forced to the left, back

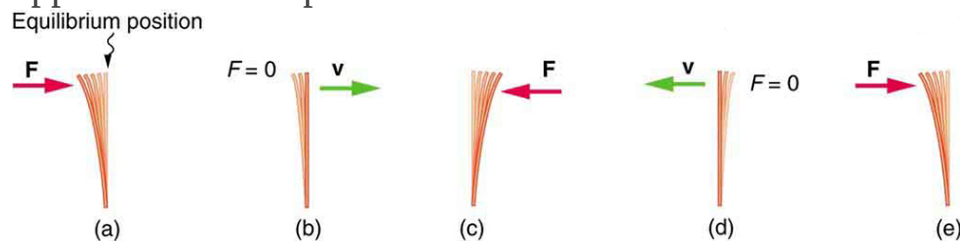
through equilibrium, and the process is repeated until dissipative forces dampen the motion. These forces remove mechanical energy from the system, gradually reducing the motion until the ruler comes to rest.

The simplest oscillations occur when the restoring force is directly proportional to displacement. When stress and strain were covered in [Newton's Third Law of Motion](#), the name was given to this relationship between force and displacement was Hooke's law:

Equation:

$$F = -kx.$$

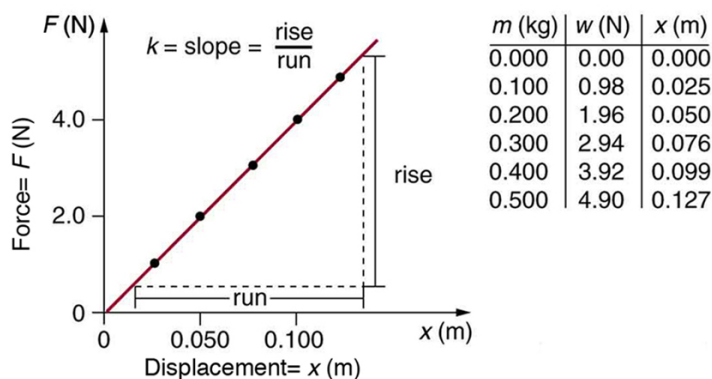
Here, F is the restoring force, x is the displacement from equilibrium or **deformation**, and k is a constant related to the difficulty in deforming the system. The minus sign indicates the restoring force is in the direction opposite to the displacement.



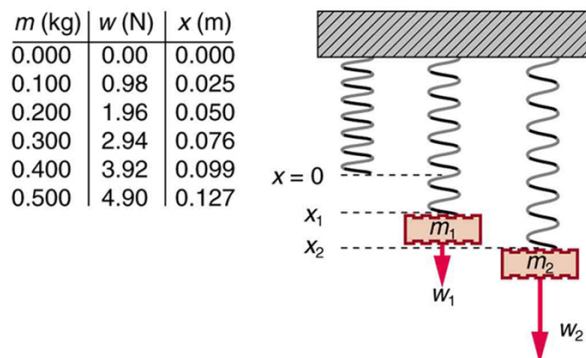
(a) The plastic ruler has been released, and the restoring force is returning the ruler to its equilibrium position. (b) The net force is zero at the equilibrium position, but the ruler has momentum and continues to move to the right. (c) The restoring force is in the opposite direction. It stops the ruler and moves it back toward equilibrium again. (d) Now the ruler has momentum to the left. (e) In the absence of damping (caused by frictional forces), the ruler reaches its original position. From there, the motion will repeat itself.

The **force constant** k is related to the rigidity (or stiffness) of a system—the larger the force constant, the greater the restoring force, and the stiffer the system. The units of k are newtons per meter (N/m). For example, k is directly related to Young's modulus when we stretch a string. [\[link\]](#) shows a graph of the absolute value of the restoring force versus the displacement for a system that can be described by Hooke's law—a simple spring in this case. The slope of the graph equals the force constant k in newtons per meter. A common physics laboratory exercise is to measure restoring forces created by springs, determine if they follow Hooke's law, and calculate their force constants if they do.

a)



b)



(a) A graph of absolute value of the restoring force versus displacement is displayed. The fact that the graph is a straight line means that the system obeys Hooke's law. The slope of the graph is the force constant k . (b) The data in the graph were generated by measuring the displacement of a spring from

equilibrium while supporting various weights. The restoring force equals the weight supported, if the mass is stationary.

Example:
How Stiff Are Car Springs?



The mass of a car increases due to the introduction of a passenger. This affects the displacement of the car on its suspension system. (credit: exfordy on Flickr)

What is the force constant for the suspension system of a car that settles 1.20 cm when an 80.0-kg person gets in?

Strategy

Consider the car to be in its equilibrium position $x = 0$ before the person gets in. The car then settles down 1.20 cm, which means it is displaced to a position $x = -1.20 \times 10^{-2}$ m. At that point, the springs supply a restoring force F equal to the person's weight

$w = mg = (80.0 \text{ kg})(9.80 \text{ m/s}^2) = 784 \text{ N}$. We take this force to be F in Hooke's law. Knowing F and x , we can then solve the force constant k .

Solution

1. Solve Hooke's law, $F = -kx$, for k :

Equation:

$$k = -\frac{F}{x}.$$

Substitute known values and solve k :

Equation:

$$\begin{aligned} k &= -\frac{784 \text{ N}}{-1.20 \times 10^{-2} \text{ m}} \\ &= 6.53 \times 10^4 \text{ N/m.} \end{aligned}$$

Discussion

Note that F and x have opposite signs because they are in opposite directions—the restoring force is up, and the displacement is down. Also, note that the car would oscillate up and down when the person got in if it were not for damping (due to frictional forces) provided by shock absorbers. Bouncing cars are a sure sign of bad shock absorbers.

Energy in Hooke's Law of Deformation

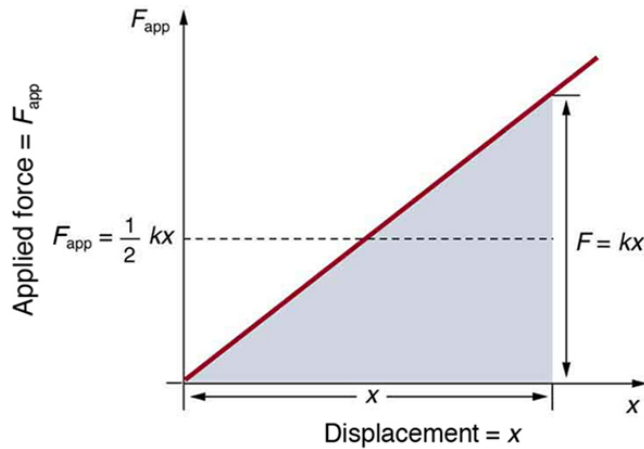
In order to produce a deformation, work must be done. That is, a force must be exerted through a distance, whether you pluck a guitar string or compress a car spring. If the only result is deformation, and no work goes into thermal, sound, or kinetic energy, then all the work is initially stored in the deformed object as some form of potential energy. The potential energy stored in a spring is $PE_{\text{el}} = \frac{1}{2}kx^2$. Here, we generalize the idea to elastic potential energy for a deformation of any system that can be described by Hooke's law. Hence,

Equation:

$$PE_{\text{el}} = \frac{1}{2}kx^2,$$

where PE_{el} is the **elastic potential energy** stored in any deformed system that obeys Hooke's law and has a displacement x from equilibrium and a force constant k .

It is possible to find the work done in deforming a system in order to find the energy stored. This work is performed by an applied force F_{app} . The applied force is exactly opposite to the restoring force (action-reaction), and so $F_{\text{app}} = kx$. [\[link\]](#) shows a graph of the applied force versus deformation x for a system that can be described by Hooke's law. Work done on the system is force multiplied by distance, which equals the area under the curve or $(1/2)kx^2$ (Method A in the figure). Another way to determine the work is to note that the force increases linearly from 0 to kx , so that the average force is $(1/2)kx$, the distance moved is x , and thus $W = F_{\text{app}}d = [(1/2)kx](x) = (1/2)kx^2$ (Method B in the figure).



Method A

$$W = \frac{1}{2} bh = \frac{1}{2} kxx$$

$$W = \frac{1}{2} kx^2$$

Method B

$$W = f \cdot x = \left(\frac{1}{2} kx \right) (x)$$

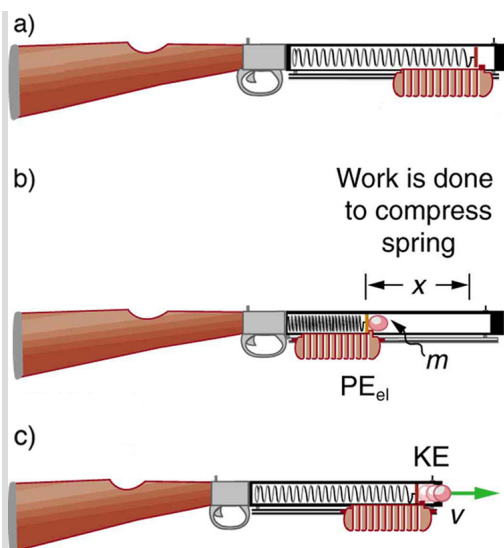
$$W = \frac{1}{2} kx^2$$

A graph of applied force versus distance for the deformation of a system that can be described by Hooke's law is displayed. The work done on the system equals the area under the graph or the area of the triangle, which is half its base multiplied by its height, or $W = (1/2)kx^2$.

Example:

Calculating Stored Energy: A Tranquilizer Gun Spring

We can use a toy gun's spring mechanism to ask and answer two simple questions: (a) How much energy is stored in the spring of a tranquilizer gun that has a force constant of 50.0 N/m and is compressed 0.150 m? (b) If you neglect friction and the mass of the spring, at what speed will a 2.00-g projectile be ejected from the gun?



(a) In this image of the gun, the spring is uncompressed before being cocked. (b) The spring has been compressed a distance x , and the projectile is in place. (c) When released, the spring converts elastic potential energy PE_{el} into kinetic energy.

Strategy for a

(a): The energy stored in the spring can be found directly from elastic potential energy equation, because k and x are given.

Solution for a

Entering the given values for k and x yields

Equation:

$$\begin{aligned} PE_{el} &= \frac{1}{2} kx^2 = \frac{1}{2} (50.0 \text{ N/m})(0.150 \text{ m})^2 = 0.563 \text{ N} \cdot \text{m} \\ &= 0.563 \text{ J} \end{aligned}$$

Strategy for b

Because there is no friction, the potential energy is converted entirely into kinetic energy. The expression for kinetic energy can be solved for the projectile's speed.

Solution for b

1. Identify known quantities:

Equation:

$$KE_f = PE_{el} \text{ or } \frac{1}{2}mv^2 = \frac{1}{2}kx^2 = PE_{el} = 0.563 \text{ J}$$

2. Solve for v :

Equation:

$$v = \left[\frac{2PE_{el}}{m} \right]^{1/2} = \left[\frac{2(0.563 \text{ J})}{0.002 \text{ kg}} \right]^{1/2} = 23.7(\text{J/kg})^{1/2}$$

3. Convert units: 23.7 m/s

Discussion

(a) and (b): This projectile speed is impressive for a tranquilizer gun (more than 80 km/h). The numbers in this problem seem reasonable. The force needed to compress the spring is small enough for an adult to manage, and the energy imparted to the dart is small enough to limit the damage it might do. Yet, the speed of the dart is great enough for it to travel an acceptable distance.

Exercise:

Check your Understanding

Problem:

Envision holding the end of a ruler with one hand and deforming it with the other. When you let go, you can see the oscillations of the ruler. In what way could you modify this simple experiment to increase the rigidity of the system?

Solution:

Answer

You could hold the ruler at its midpoint so that the part of the ruler that oscillates is half as long as in the original experiment.

Exercise:

Check your Understanding

Problem:

If you apply a deforming force on an object and let it come to equilibrium, what happened to the work you did on the system?

Solution:

Answer

It was stored in the object as potential energy.

Test Prep for AP Courses

Exercise:

Problem:

Which of the following represents the distance (how much ground the particle covers) moved by a particle in a simple harmonic motion in one time period? (Here, A represents the amplitude of the oscillation.)

- a. 0 cm
- b. A cm
- c. $2A$ cm
- d. $4A$ cm

Solution:

(d)

Exercise:**Problem:**

A spring has a spring constant of $80 \text{ N}\cdot\text{m}^{-1}$. What is the force required to (a) compress the spring by 5 cm and (b) expand the spring by 15 cm?

Exercise:**Problem:**

In the formula $F = -kx$, what does the minus sign indicate?

- a. It indicates that the restoring force is in the direction of the displacement.
- b. It indicates that the restoring force is in the direction opposite the displacement.
- c. It indicates that mechanical energy in the system decreases when a system undergoes oscillation.
- d. None of the above

Solution:

(b)

Exercise:**Problem:**

The splashing of a liquid resembles an oscillation. The restoring force in this scenario will be due to which of the following?

- a. Potential energy
- b. Kinetic energy
- c. Gravity
- d. Mechanical energy

Section Summary

- An oscillation is a back and forth motion of an object between two points of deformation.
- An oscillation may create a wave, which is a disturbance that propagates from where it was created.
- The simplest type of oscillations and waves are related to systems that can be described by Hooke's law:

Equation:

$$F = -kx,$$

where F is the restoring force, x is the displacement from equilibrium or deformation, and k is the force constant of the system.

- Elastic potential energy PE_{el} stored in the deformation of a system that can be described by Hooke's law is given by

Equation:

$$PE_{\text{el}} = (1/2)kx^2.$$

Conceptual Questions

Exercise:

Problem:

Describe a system in which elastic potential energy is stored.

Problems & Exercises

Exercise:

Problem:

Fish are hung on a spring scale to determine their mass (most fishermen feel no obligation to truthfully report the mass).

- (a) What is the force constant of the spring in such a scale if it the spring stretches 8.00 cm for a 10.0 kg load?
- (b) What is the mass of a fish that stretches the spring 5.50 cm?
- (c) How far apart are the half-kilogram marks on the scale?
-

Solution:

- (a) $1.23 \times 10^3 \text{ N/m}$
- (b) 6.88 kg
- (c) 4.00 mm

Exercise:

Problem:

It is weigh-in time for the local under-85-kg rugby team. The bathroom scale used to assess eligibility can be described by Hooke's law and is depressed 0.75 cm by its maximum load of 120 kg. (a) What is the spring's effective spring constant? (b) A player stands on the scales and depresses it by 0.48 cm. Is he eligible to play on this under-85 kg team?

Exercise:

Problem:

One type of BB gun uses a spring-driven plunger to blow the BB from its barrel. (a) Calculate the force constant of its plunger's spring if you must compress it 0.150 m to drive the 0.0500-kg plunger to a top speed of 20.0 m/s. (b) What force must be exerted to compress the spring?

Solution:

- (a) 889 N/m
- (b) 133 N

Exercise:**Problem:**

(a) The springs of a pickup truck act like a single spring with a force constant of $1.30 \times 10^5 \text{ N/m}$. By how much will the truck be depressed by its maximum load of 1000 kg?

(b) If the pickup truck has four identical springs, what is the force constant of each?

Exercise:**Problem:**

When an 80.0-kg man stands on a pogo stick, the spring is compressed 0.120 m.

(a) What is the force constant of the spring? (b) Will the spring be compressed more when he hops down the road?

Solution:

(a) $6.53 \times 10^3 \text{ N/m}$

(b) Yes

Exercise:**Problem:**

A spring has a length of 0.200 m when a 0.300-kg mass hangs from it, and a length of 0.750 m when a 1.95-kg mass hangs from it. (a) What is the force constant of the spring? (b) What is the unloaded length of the spring?

Glossary

deformation

displacement from equilibrium

elastic potential energy

potential energy stored as a result of deformation of an elastic object, such as the stretching of a spring

force constant

a constant related to the rigidity of a system: the larger the force constant, the more rigid the system; the force constant is represented by k

restoring force

force acting in opposition to the force caused by a deformation

Period and Frequency in Oscillations

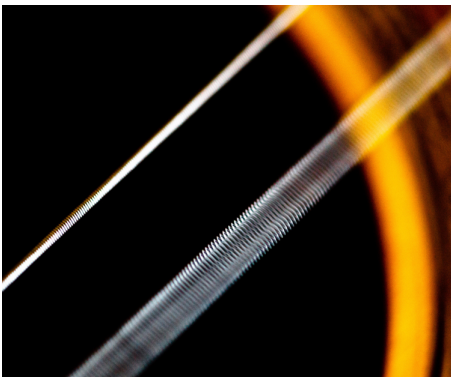
LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Relate recurring mechanical vibrations to the frequency and period of harmonic motion, such as the motion of a guitar string.
- Compute the frequency and period of an oscillation.

The information presented in this section supports the following AP[®] learning objectives and science practices:

- **3.B.3.3** The student can analyze data to identify qualitative or quantitative relationships between given values and variables (i.e., force, displacement, acceleration, velocity, period of motion, frequency, spring constant, string length, mass) associated with objects in oscillatory motion to use that data to determine the value of an unknown. (S.P. 2.2, 5.1)



The strings on this
guitar vibrate at
regular time intervals.
(credit: JAR)

When you pluck a guitar string, the resulting sound has a steady tone and lasts a long time. Each successive vibration of the string takes the same time as the previous one. We define **periodic motion** to be a motion that repeats itself at regular time intervals, such as exhibited by the guitar string or by an object on a spring moving up and down. The time to complete one oscillation remains constant and is called the **period** T . Its units are usually seconds, but may be any convenient unit of time. The word period refers to the time for some event whether repetitive or not; but we shall be primarily interested in periodic motion, which is by definition repetitive. A concept closely related to period is the frequency of an event. For example, if you get a paycheck twice a month, the frequency of payment is two per month and the period between checks is half a month. **Frequency** f is defined to be the number of events per unit time. For periodic motion, frequency is the number of oscillations per unit time. The relationship between frequency and period is

Equation:

$$f = \frac{1}{T}.$$

The SI unit for frequency is the *cycle per second*, which is defined to be a *hertz* (Hz):

Equation:

$$1 \text{ Hz} = 1 \frac{\text{cycle}}{\text{sec}} \text{ or } 1 \text{ Hz} = \frac{1}{\text{s}}$$

A cycle is one complete oscillation. Note that a vibration can be a single or multiple event, whereas oscillations are usually repetitive for a significant number of cycles.

Example:

Determine the Frequency of Two Oscillations: Medical Ultrasound and the Period of Middle C

We can use the formulas presented in this module to determine both the frequency based on known oscillations and the oscillation based on a known frequency. Let's try one example of each. (a) A medical imaging device produces ultrasound by oscillating with a period of $0.400\ \mu\text{s}$. What is the frequency of this oscillation? (b) The frequency of middle C on a typical musical instrument is 264 Hz. What is the time for one complete oscillation?

Strategy

Both questions (a) and (b) can be answered using the relationship between period and frequency. In question (a), the period T is given and we are asked to find frequency f . In question (b), the frequency f is given and we are asked to find the period T .

Solution a

1. Substitute $0.400\ \mu\text{s}$ for T in $f = \frac{1}{T}$:

Equation:

$$f = \frac{1}{T} = \frac{1}{0.400 \times 10^{-6}\ \text{s}}.$$

Solve to find

Equation:

$$f = 2.50 \times 10^6\ \text{Hz}.$$

Discussion a

The frequency of sound found in (a) is much higher than the highest frequency that humans can hear and, therefore, is called ultrasound. Appropriate oscillations at this frequency generate ultrasound used for noninvasive medical diagnoses, such as observations of a fetus in the womb.

Solution b

1. Identify the known values:

The time for one complete oscillation is the period T :

Equation:

$$f = \frac{1}{T}.$$

2. Solve for T :

Equation:

$$T = \frac{1}{f}.$$

3. Substitute the given value for the frequency into the resulting expression:

Equation:

$$T = \frac{1}{f} = \frac{1}{264 \text{ Hz}} = \frac{1}{264 \text{ cycles/s}} = 3.79 \times 10^{-3} \text{ s} = 3.79 \text{ ms}.$$

Discussion

The period found in (b) is the time per cycle, but this value is often quoted as simply the time in convenient units (ms or milliseconds in this case).

Exercise:

Check your Understanding

Problem:

Identify an event in your life (such as receiving a paycheck) that occurs regularly. Identify both the period and frequency of this event.

Solution:

I visit my parents for dinner every other Sunday. The frequency of my visits is 26 per calendar year. The period is two weeks.

Test Prep for AP Courses**Exercise:****Problem:**

A mass attached to a spring oscillates and completes 50 full cycles in 30 s. What is the time period and frequency of this system?

Solution:

The frequency is given by

$$f = \frac{1}{T} = \frac{50 \text{ cycles}}{30 \text{ s}} = 1.66 \text{ Hz}$$

Time period is:

$$T = \frac{1}{f} = \frac{1}{1.66} = 0.6 \text{ s}$$

Section Summary

- Periodic motion is a repetitious oscillation.
- The time for one oscillation is the period T .
- The number of oscillations per unit time is the frequency f .

- These quantities are related by
Equation:

$$f = \frac{1}{T}.$$

Problems & Exercises

Exercise:

Problem: What is the period of 60.0 Hz electrical power?

Solution:

16.7 ms

Exercise:

Problem:

If your heart rate is 150 beats per minute during strenuous exercise, what is the time per beat in units of seconds?

Solution:

0.400 s/beats

Exercise:

Problem:

Find the frequency of a tuning fork that takes 2.50×10^{-3} s to complete one oscillation.

Solution:

400 Hz

Exercise:

Problem:

A stroboscope is set to flash every 8.00×10^{-5} s. What is the frequency of the flashes?

Solution:

12,500 Hz

Exercise:**Problem:**

A tire has a tread pattern with a crevice every 2.00 cm. Each crevice makes a single vibration as the tire moves. What is the frequency of these vibrations if the car moves at 30.0 m/s?

Solution:

1.50 kHz

Exercise:**Problem: Engineering Application**

Each piston of an engine makes a sharp sound every other revolution of the engine. (a) How fast is a race car going if its eight-cylinder engine emits a sound of frequency 750 Hz, given that the engine makes 2000 revolutions per kilometer? (b) At how many revolutions per minute is the engine rotating?

Solution:

(a) 93.8 m/s

(b) 11.3×10^3 rev/min

Glossary

period

time it takes to complete one oscillation

periodic motion

motion that repeats itself at regular time intervals

frequency

number of events per unit of time

Simple Harmonic Motion: A Special Periodic Motion

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Describe a simple harmonic oscillator.
- Relate physical characteristics of a vibrating system to aspects of simple harmonic motion and any resulting waves.

The information presented in this section supports the following AP® learning objectives and science practices:

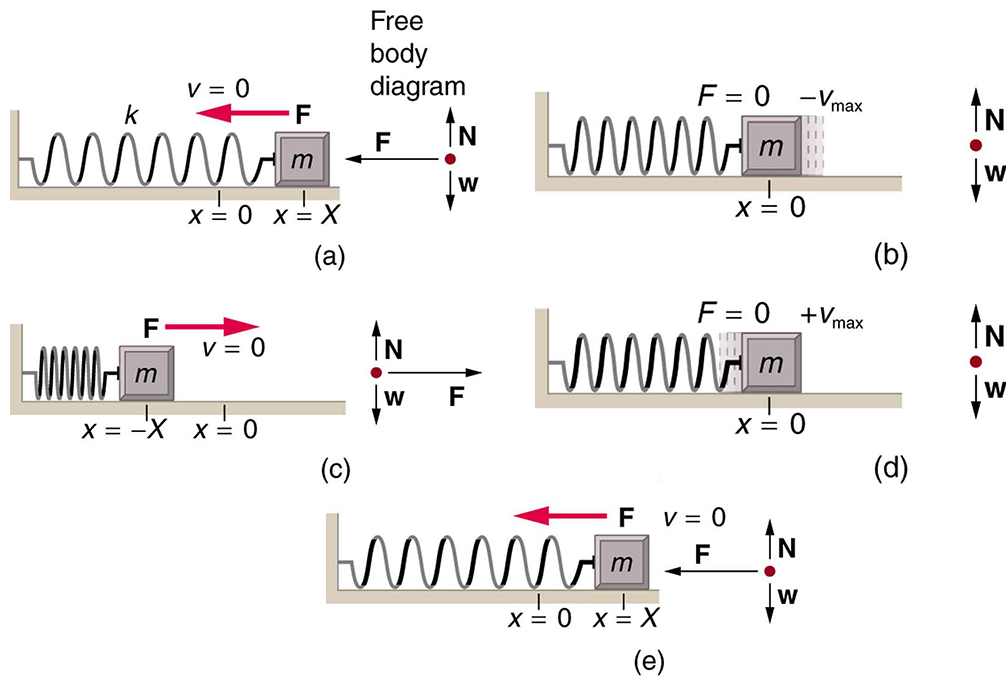
- **3.B.3.1** The student is able to predict which properties determine the motion of a simple harmonic oscillator and what the dependence of the motion is on those properties. (S.P. 6.4, 7.2)
- **3.B.3.4** The student is able to construct a qualitative and/or a quantitative explanation of oscillatory behavior given evidence of a restoring force. (S.P. 2.2, 6.2)
- **6.A.3.1** The student is able to use graphical representation of a periodic mechanical wave to determine the amplitude of the wave. (S.P. 1.4)
- **6.B.1.1** The student is able to use a graphical representation of a periodic mechanical wave (position versus time) to determine the period and frequency of the wave and describe how a change in the frequency would modify features of the representation. (S.P. 1.4, 2.2)

The oscillations of a system in which the net force can be described by Hooke's law are of special importance, because they are very common. They are also the simplest oscillatory systems. **Simple Harmonic Motion** (SHM) is the name given to oscillatory motion for a system where the net force can be described by Hooke's law, and such a system is called a **simple harmonic oscillator**. If the net force can be described by Hooke's law and there is no *damping* (by friction or other non-conservative forces), then a simple harmonic oscillator will oscillate with equal displacement on either side of the equilibrium position, as shown for an object on a spring in [\[link\]](#). The maximum displacement from equilibrium is called the **amplitude** X . The units for amplitude and displacement are the same, but depend on the type of oscillation. For the object on the spring, the units of amplitude and displacement are meters; whereas for sound oscillations, they have units of pressure (and other types of oscillations have yet other units). Because amplitude is the maximum displacement, it is related to the energy in the oscillation.

Note:

Take-Home Experiment: SHM and the Marble

Find a bowl or basin that is shaped like a hemisphere on the inside. Place a marble inside the bowl and tilt the bowl periodically so the marble rolls from the bottom of the bowl to equally high points on the sides of the bowl. Get a feel for the force required to maintain this periodic motion. What is the restoring force and what role does the force you apply play in the simple harmonic motion (SHM) of the marble?



An object attached to a spring sliding on a frictionless surface is an uncomplicated simple harmonic oscillator. When displaced from equilibrium, the object performs simple harmonic motion that has an amplitude X and a period T . The object's maximum speed occurs as it passes through equilibrium. The stiffer the spring is, the smaller the period T . The greater the mass of the object is, the greater the period T .

What is so significant about simple harmonic motion? One special thing is that the period T and frequency f of a simple harmonic oscillator are independent of amplitude. The string of a guitar, for example, will oscillate with the same frequency whether plucked gently or hard. Because the period is constant, a simple harmonic oscillator can be used as a clock.

Two important factors do affect the period of a simple harmonic oscillator. The period is related to how stiff the system is. A very stiff object has a large force constant k , which causes the system to have a smaller period. For example, you can adjust a diving board's stiffness—the stiffer it is, the

faster it vibrates, and the shorter its period. Period also depends on the mass of the oscillating system. The more massive the system is, the longer the period. For example, a heavy person on a diving board bounces up and down more slowly than a light one.

In fact, the mass m and the force constant k are the *only* factors that affect the period and frequency of simple harmonic motion.

Note:

Period of Simple Harmonic Oscillator

The *period of a simple harmonic oscillator* is given by

Equation:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

and, because $f = 1/T$, the *frequency of a simple harmonic oscillator* is

Equation:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}.$$

Note that neither T nor f has any dependence on amplitude.

Example:

Mechanical Waves

What do sound waves, water waves, and seismic waves have in common? They are all governed by Newton's laws and they can exist only when traveling in a medium, such as air, water, or rocks. Waves that require a medium to travel are collectively known as "mechanical waves."

Note:**Take-Home Experiment: Mass and Ruler Oscillations**

Find two identical wooden or plastic rulers. Tape one end of each ruler firmly to the edge of a table so that the length of each ruler that protrudes from the table is the same. On the free end of one ruler tape a heavy object such as a few large coins. Pluck the ends of the rulers at the same time and observe which one undergoes more cycles in a time period, and measure the period of oscillation of each of the rulers.

Example:**Calculate the Frequency and Period of Oscillations: Bad Shock Absorbers in a Car**

If the shock absorbers in a car go bad, then the car will oscillate at the least provocation, such as when going over bumps in the road and after stopping (See [\[link\]](#)). Calculate the frequency and period of these oscillations for such a car if the car's mass (including its load) is 900 kg and the force constant (k) of the suspension system is 6.53×10^4 N/m.

Strategy

The frequency of the car's oscillations will be that of a simple harmonic oscillator as given in the equation $f = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$. The mass and the force constant are both given.

Solution

1. Enter the known values of k and m :

Equation:

$$f = \frac{1}{2\pi} \sqrt{\frac{k}{m}} = \frac{1}{2\pi} \sqrt{\frac{6.53 \times 10^4 \text{ N/m}}{900 \text{ kg}}}.$$

2. Calculate the frequency:

Equation:

$$\frac{1}{2\pi} \sqrt{72.6/\text{s}^{-2}} = 1.3656/\text{s}^{-1} \approx 1.36/\text{s}^{-1} = 1.36 \text{ Hz}.$$

3. You could use $T = 2\pi\sqrt{\frac{m}{k}}$ to calculate the period, but it is simpler to use the relationship $T = 1/f$ and substitute the value just found for f :

Equation:

$$T = \frac{1}{f} = \frac{1}{1.356 \text{ Hz}} = 0.738 \text{ s.}$$

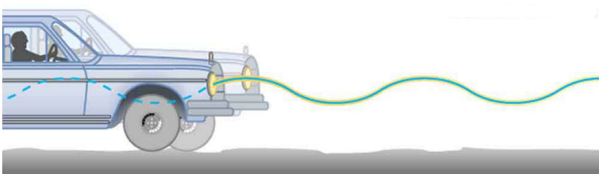
Discussion

The values of T and f both seem about right for a bouncing car. You can observe these oscillations if you push down hard on the end of a car and let go.

The Link between Simple Harmonic Motion and Waves

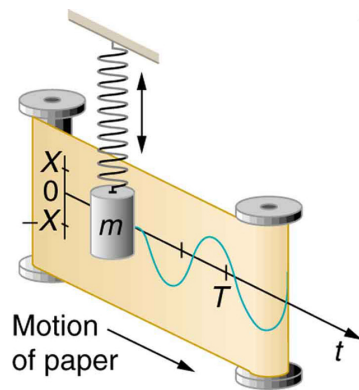
If a time-exposure photograph of the bouncing car were taken as it drove by, the headlight would make a wavelike streak, as shown in [\[link\]](#).

Similarly, [\[link\]](#) shows an object bouncing on a spring as it leaves a wavelike "trace of its position on a moving strip of paper. Both waves are sine functions. All simple harmonic motion is intimately related to sine and cosine waves.



The bouncing car makes a wavelike motion. If the restoring force in the suspension system can be described only by Hooke's law, then the wave is a sine function. (The wave is the

trace produced by the headlight
as the car moves to the right.)



The vertical
position of an
object bouncing
on a spring is
recorded on a
strip of moving
paper, leaving a
sine wave.

The displacement as a function of time t in any simple harmonic motion—that is, one in which the net restoring force can be described by Hooke's law, is given by

Equation:

$$x(t) = X \cos \frac{2\pi t}{T},$$

where X is amplitude. At $t = 0$, the initial position is $x_0 = X$, and the displacement oscillates back and forth with a period T . (When $t = T$, we get $x = X$ again because $\cos 2\pi = 1$.) Furthermore, from this expression for x , the velocity v as a function of time is given by:

Equation:

$$v(t) = -v_{\max} \sin \left(\frac{2\pi t}{T} \right),$$

where $v_{\max} = 2\pi X/T = X\sqrt{k/m}$. The object has zero velocity at maximum displacement—for example, $v = 0$ when $t = 0$, and at that time $x = X$. The minus sign in the first equation for $v(t)$ gives the correct direction for the velocity. Just after the start of the motion, for instance, the velocity is negative because the system is moving back toward the equilibrium point. Finally, we can get an expression for acceleration using Newton's second law. [Then we have $x(t)$, $v(t)$, t , and $a(t)$, the quantities needed for kinematics and a description of simple harmonic motion.]

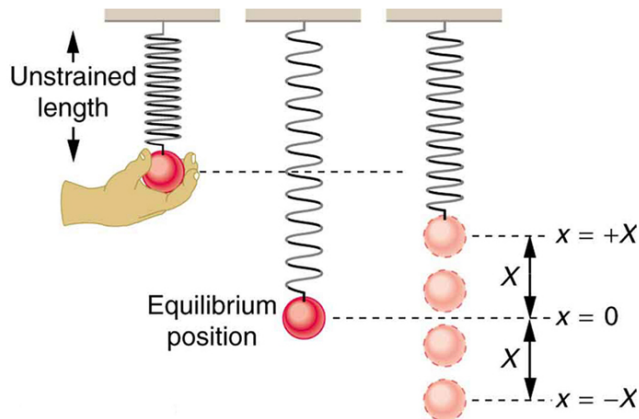
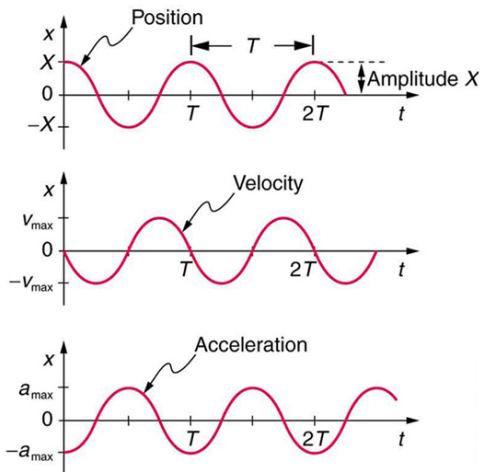
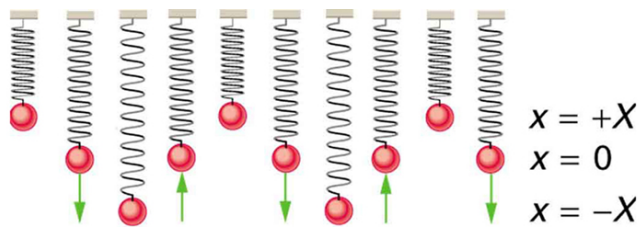
According to Newton's second law, the acceleration is $a = F/m = kx/m$. So, $a(t)$ is also a cosine function:

Equation:

$$a(t) = -\frac{kX}{m} \cos \frac{2\pi t}{T}.$$

Hence, $a(t)$ is directly proportional to and in the opposite direction to $x(t)$.

[\[link\]](#) shows the simple harmonic motion of an object on a spring and presents graphs of $x(t)$, $v(t)$, and $a(t)$ versus time.



Graphs of $x(t)$, $v(t)$, and $a(t)$ versus t for the motion of an object on a spring. The net force on the object can be described by Hooke's law, and so the object undergoes simple harmonic motion. Note that the initial position has the vertical displacement at its maximum value X ; v is initially zero and then negative as the object moves down; and the initial acceleration

is negative, back toward the equilibrium position and becomes zero at that point.

The most important point here is that these equations are mathematically straightforward and are valid for all simple harmonic motion. They are very useful in visualizing waves associated with simple harmonic motion, including visualizing how waves add with one another.

Exercise:

Check Your Understanding

Problem:

Suppose you pluck a banjo string. You hear a single note that starts out loud and slowly quiets over time. Describe what happens to the sound waves in terms of period, frequency and amplitude as the sound decreases in volume.

Solution:

Frequency and period remain essentially unchanged. Only amplitude decreases as volume decreases.

Exercise:

Check Your Understanding

Problem:

A babysitter is pushing a child on a swing. At the point where the swing reaches x , where would the corresponding point on a wave of this motion be located?

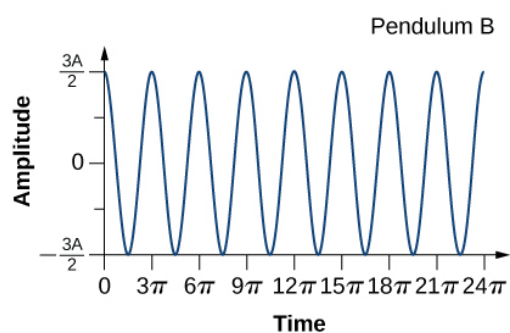
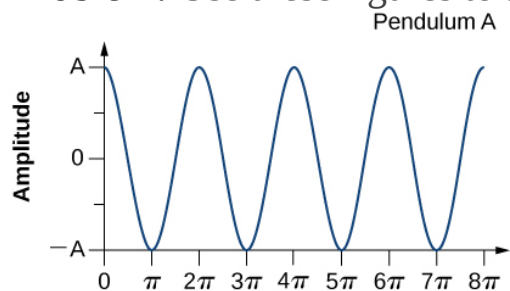
Solution:

x is the maximum deformation, which corresponds to the amplitude of the wave. The point on the wave would either be at the very top or the very bottom of the curve.

Test Prep for AP Courses

Exercise:

Problem: Use these figures to answer the following questions.



- Which of the two pendulums oscillates with larger amplitude?
- Which of the two pendulums oscillates at a higher frequency?

Exercise:

Problem:

A particle of mass 100 g undergoes a simple harmonic motion. The restoring force is provided by a spring with a spring constant of $40 \text{ N}\cdot\text{m}^{-1}$. What is the period of oscillation?

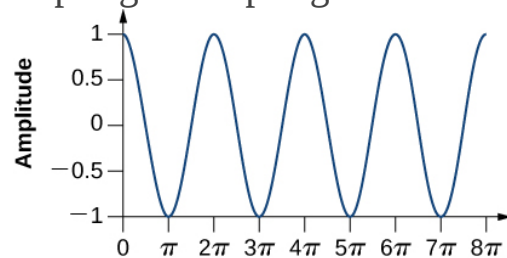
- 10π
- 0.5π
- 0.1π
- 1π

Solution:

(c)

Exercise:**Problem:**

The graph shows the simple harmonic motion of a mass m attached to a spring with spring constant k .



What is the displacement at time 8π ?

- a. 1 m
- b. 0 m
- c. Not defined
- d. -1 m

Exercise:**Problem:**

A pendulum of mass 200 g undergoes simple harmonic motion when acted upon by a force of 15 N. The pendulum crosses the point of equilibrium at a speed of $5 \text{ m}\cdot\text{s}^{-1}$. What is the energy of the pendulum at the center of the oscillation?

Solution:

The energy of the particle at the center of the oscillation is given by

$$E = \frac{1}{2}mv^2$$

$$= \frac{1}{2} \times 0.2 \text{ kg} \times (5 \text{ m}\cdot\text{s}^{-1})^2$$

$$= 2.5 \text{ J}$$

Note:

PhET Explorations: Masses and Springs

A realistic mass and spring laboratory. Hang masses from springs and adjust the spring stiffness and damping. You can even slow time. Transport the lab to different planets. A chart shows the kinetic, potential, and thermal energy for each spring.

[Masses
and
Spring
s](#)

Section Summary

- Simple harmonic motion is oscillatory motion for a system that can be described only by Hooke's law. Such a system is also called a simple harmonic oscillator.
- Maximum displacement is the amplitude X . The period T and frequency f of a simple harmonic oscillator are given by

$$T = 2\pi\sqrt{\frac{m}{k}} \text{ and } f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}, \text{ where } m \text{ is the mass of the system.}$$

- Displacement in simple harmonic motion as a function of time is given by $x(t) = X \cos \frac{2\pi t}{T}$.

- The velocity is given by $v(t) = -v_{\max} \sin \frac{2\pi t}{T}$, where $v_{\max} = \sqrt{k/m}X$.
- The acceleration is found to be $a(t) = -\frac{kX}{m} \cos \frac{2\pi t}{T}$.

Conceptual Questions

Exercise:

Problem:

What conditions must be met to produce simple harmonic motion?

Exercise:

Problem:

(a) If frequency is not constant for some oscillation, can the oscillation be simple harmonic motion?

(b) Can you think of any examples of harmonic motion where the frequency may depend on the amplitude?

Exercise:

Problem:

Give an example of a simple harmonic oscillator, specifically noting how its frequency is independent of amplitude.

Exercise:

Problem:

Explain why you expect an object made of a stiff material to vibrate at a higher frequency than a similar object made of a spongy material.

Exercise:

Problem:

As you pass a freight truck with a trailer on a highway, you notice that its trailer is bouncing up and down slowly. Is it more likely that the trailer is heavily loaded or nearly empty? Explain your answer.

Exercise:**Problem:**

Some people modify cars to be much closer to the ground than when manufactured. Should they install stiffer springs? Explain your answer.

Problems & Exercises**Exercise:****Problem:**

A type of cuckoo clock keeps time by having a mass bouncing on a spring, usually something cute like a cherub in a chair. What force constant is needed to produce a period of 0.500 s for a 0.0150-kg mass?

Solution:

2.37 N/m

Exercise:**Problem:**

If the spring constant of a simple harmonic oscillator is doubled, by what factor will the mass of the system need to change in order for the frequency of the motion to remain the same?

Exercise:

Problem:

A 0.500-kg mass suspended from a spring oscillates with a period of 1.50 s. How much mass must be added to the object to change the period to 2.00 s?

Solution:

0.389 kg

Exercise:**Problem:**

By how much leeway (both percentage and mass) would you have in the selection of the mass of the object in the previous problem if you did not wish the new period to be greater than 2.01 s or less than 1.99 s?

Exercise:**Problem:**

Suppose you attach the object with mass m to a vertical spring originally at rest, and let it bounce up and down. You release the object from rest at the spring's original rest length. (a) Show that the spring exerts an upward force of $2.00\ mg$ on the object at its lowest point. (b) If the spring has a force constant of $10.0\ \text{N/m}$ and a 0.25-kg -mass object is set in motion as described, find the amplitude of the oscillations. (c) Find the maximum velocity.

Exercise:**Problem:**

A diver on a diving board is undergoing simple harmonic motion. Her mass is $55.0\ \text{kg}$ and the period of her motion is $0.800\ \text{s}$. The next diver is a male whose period of simple harmonic oscillation is $1.05\ \text{s}$. What is his mass if the mass of the board is negligible?

Solution:

94.7 kg

Exercise:

Problem:

Suppose a diving board with no one on it bounces up and down in a simple harmonic motion with a frequency of 4.00 Hz. The board has an effective mass of 10.0 kg. What is the frequency of the simple harmonic motion of a 75.0-kg diver on the board?

Exercise:

Problem:



This child's toy
relies on springs to
keep infants
entertained. (credit:
By Humboldtthead,
Flickr)

The device pictured in [\[link\]](#) entertains infants while keeping them from wandering. The child bounces in a harness suspended from a door frame by a spring constant.

(a) If the spring stretches 0.250 m while supporting an 8.0-kg child, what is its spring constant?

(b) What is the time for one complete bounce of this child? (c) What is the child's maximum velocity if the amplitude of her bounce is 0.200 m?

Exercise:

Problem:

A 90.0-kg skydiver hanging from a parachute bounces up and down with a period of 1.50 s. What is the new period of oscillation when a second skydiver, whose mass is 60.0 kg, hangs from the legs of the first, as seen in [\[link\]](#).



The oscillations of one skydiver are about to be affected by a second skydiver. (credit: U.S. Army, www.army.mil)

Solution:

1.94 s

Glossary

amplitude

the maximum displacement from the equilibrium position of an object oscillating around the equilibrium position

simple harmonic motion

the oscillatory motion in a system where the net force can be described by Hooke's law

simple harmonic oscillator

a device that implements Hooke's law, such as a mass that is attached to a spring, with the other end of the spring being connected to a rigid support such as a wall

The Simple Pendulum

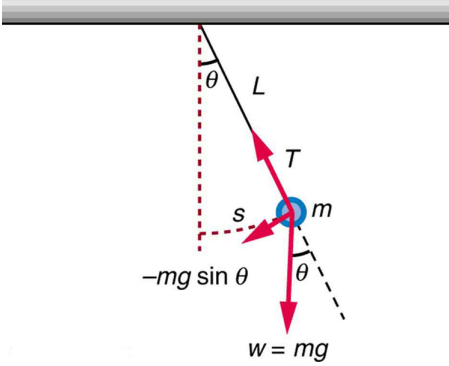
LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Determine the period of oscillation of a hanging pendulum.

The information presented in this section supports the following AP® learning objectives and science practices:

- **3.B.3.1** The student is able to predict which properties determine the motion of a simple harmonic oscillator and what the dependence of the motion is on those properties. **(S.P. 6.4, 7.2)**
- **3.B.3.2** The student is able to design a plan and collect data in order to ascertain the characteristics of the motion of a system undergoing oscillatory motion caused by a restoring force. **(S.P. 4.2)**
- **3.B.3.3** The student can analyze data to identify qualitative or quantitative relationships between given values and variables (i.e., force, displacement, acceleration, velocity, period of motion, frequency, spring constant, string length, mass) associated with objects in oscillatory motion to use that data to determine the value of an unknown. **(S.P. 2.2, 5.1)**
- **3.B.3.4** The student is able to construct a qualitative and/or a quantitative explanation of oscillatory behavior given evidence of a restoring force. **(S.P. 2.2, 6.2)**



A simple pendulum has a small-diameter bob and a string that has a very small mass but is strong enough not to stretch appreciably. The linear displacement from equilibrium is s , the length of the arc. Also shown are the forces on the bob, which result in a net force of $-mg \sin \theta$ toward the equilibrium position—that is, a restoring force.

Pendulums are in common usage. Some have crucial uses, such as in clocks; some are for fun, such as a child's swing; and some are just there, such as the sinker on a fishing line. For small displacements, a pendulum is a simple harmonic oscillator. A **simple pendulum** is defined to have an object that has a small mass, also known as the pendulum bob, which is suspended from a light wire or string, such as shown in [\[link\]](#). Exploring the simple pendulum a bit further, we can discover the conditions under which

it performs simple harmonic motion, and we can derive an interesting expression for its period.

We begin by defining the displacement to be the arc length s . We see from [\[link\]](#) that the net force on the bob is tangent to the arc and equals $-mg \sin \theta$. (The weight mg has components $mg \cos \theta$ along the string and $mg \sin \theta$ tangent to the arc.) Tension in the string exactly cancels the component $mg \cos \theta$ parallel to the string. This leaves a *net* restoring force back toward the equilibrium position at $\theta = 0$.

Now, if we can show that the restoring force is directly proportional to the displacement, then we have a simple harmonic oscillator. In trying to determine if we have a simple harmonic oscillator, we should note that for small angles (less than about 15°), $\sin \theta \approx \theta$ ($\sin \theta$ and θ differ by about 1% or less at smaller angles). Thus, for angles less than about 15° , the restoring force F is

Equation:

$$F \approx -mg\theta.$$

The displacement s is directly proportional to θ . When θ is expressed in radians, the arc length in a circle is related to its radius (L in this instance) by:

Equation:

$$s = L\theta,$$

so that

Equation:

$$\theta = \frac{s}{L}.$$

For small angles, then, the expression for the restoring force is:

Equation:

$$F \approx -\frac{mg}{L}s$$

This expression is of the form:

Equation:

$$F = -kx,$$

where the force constant is given by $k = mg/L$ and the displacement is given by $x = s$. For angles less than about 15° , the restoring force is directly proportional to the displacement, and the simple pendulum is a simple harmonic oscillator.

Using this equation, we can find the period of a pendulum for amplitudes less than about 15° . For the simple pendulum:

Equation:

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{mg/L}}.$$

Thus,

Equation:

$$T = 2\pi\sqrt{\frac{L}{g}}$$

for the period of a simple pendulum. This result is interesting because of its simplicity. The only things that affect the period of a simple pendulum are its length and the acceleration due to gravity. The period is completely independent of other factors, such as mass. As with simple harmonic oscillators, the period T for a pendulum is nearly independent of amplitude, especially if θ is less than about 15° . Even simple pendulum clocks can be finely adjusted and accurate.

Note the dependence of T on g . If the length of a pendulum is precisely known, it can actually be used to measure the acceleration due to gravity. Consider the following example.

Example:

Measuring Acceleration due to Gravity: The Period of a Pendulum

What is the acceleration due to gravity in a region where a simple pendulum having a length 75.000 cm has a period of 1.7357 s?

Strategy

We are asked to find g given the period T and the length L of a pendulum.

We can solve $T = 2\pi\sqrt{\frac{L}{g}}$ for g , assuming only that the angle of deflection is less than 15° .

Solution

1. Square $T = 2\pi\sqrt{\frac{L}{g}}$ and solve for g :

Equation:

$$g = 4\pi^2 \frac{L}{T^2}.$$

2. Substitute known values into the new equation:

Equation:

$$g = 4\pi^2 \frac{0.75000 \text{ m}}{(1.7357 \text{ s})^2}.$$

3. Calculate to find g :

Equation:

$$g = 9.8281 \text{ m/s}^2.$$

Discussion

This method for determining g can be very accurate. This is why length and period are given to five digits in this example. For the precision of the approximation $\sin \theta \approx \theta$ to be better than the precision of the pendulum length and period, the maximum displacement angle should be kept below about 0.5° .

Note:

Making Career Connections

Knowing g can be important in geological exploration; for example, a map of g over large geographical regions aids the study of plate tectonics and helps in the search for oil fields and large mineral deposits.

Note:

Take Home Experiment: Determining g

Use a simple pendulum to determine the acceleration due to gravity g in your own locale. Cut a piece of a string or dental floss so that it is about 1 m long. Attach a small object of high density to the end of the string (for example, a metal nut or a car key). Starting at an angle of less than 10° , allow the pendulum to swing and measure the pendulum's period for 10 oscillations using a stopwatch. Calculate g . How accurate is this measurement? How might it be improved?

Exercise:

Check Your Understanding

Problem:

An engineer builds two simple pendula. Both are suspended from small wires secured to the ceiling of a room. Each pendulum hovers 2 cm above the floor. Pendulum 1 has a bob with a mass of 10 kg. Pendulum 2 has a bob with a mass of 100 kg. Describe how the motion of the pendula will differ if the bobs are both displaced by 12° .

Solution:

The movement of the pendula will not differ at all because the mass of the bob has no effect on the motion of a simple pendulum. The pendula are only affected by the period (which is related to the pendulum's length) and by the acceleration due to gravity.

Note:**PhET Explorations: Pendulum Lab**

Play with one or two pendulums and discover how the period of a simple pendulum depends on the length of the string, the mass of the pendulum bob, and the amplitude of the swing. It's easy to measure the period using the photogate timer. You can vary friction and the strength of gravity. Use the pendulum to find the value of g on planet X. Notice the anharmonic behavior at large amplitude.

[Pendulum Lab](#)

Section Summary

- A mass m suspended by a wire of length L is a simple pendulum and undergoes simple harmonic motion for amplitudes less than about 15° .

The period of a simple pendulum is

Equation:

$$T = 2\pi\sqrt{\frac{L}{g}},$$

where L is the length of the string and g is the acceleration due to gravity.

Test Prep for AP Courses

Exercise:

Problem:

A ball is attached to a string of length 4 m to make a pendulum. The pendulum is placed at a location that is away from the Earth's surface by twice the radius of the Earth. What is the acceleration due to gravity at that height and what is the period of the oscillations?

Exercise:

Problem:

Which of the following gives the correct relation between the acceleration due to gravity and period of a pendulum?

- a. $g = \frac{2\pi L}{T^2}$
- b. $g = \frac{4\pi^2 L}{T^2}$
- c. $g = \frac{2\pi L}{T}$
- d. $g = \frac{2\pi^2 L}{T}$

Solution:

(b)

Exercise:

Problem:

Tom has two pendulums with him. Pendulum 1 has a ball of mass 0.1 kg attached to it and has a length of 5 m. Pendulum 2 has a ball of mass 0.5 kg attached to a string of length 1 m. How does mass of the ball affect the frequency of the pendulum? Which pendulum will have a higher frequency and why?

Conceptual Questions**Exercise:****Problem:**

Pendulum clocks are made to run at the correct rate by adjusting the pendulum's length. Suppose you move from one city to another where the acceleration due to gravity is slightly greater, taking your pendulum clock with you, will you have to lengthen or shorten the pendulum to keep the correct time, other factors remaining constant? Explain your answer.

Problems & Exercises

As usual, the acceleration due to gravity in these problems is taken to be $g = 9.80 \text{ m/s}^2$, unless otherwise specified.

Exercise:**Problem:**

What is the length of a pendulum that has a period of 0.500 s?

Solution:

6.21 cm

Exercise:

Problem:

Some people think a pendulum with a period of 1.00 s can be driven with “mental energy” or psycho kinetically, because its period is the same as an average heartbeat. True or not, what is the length of such a pendulum?

Exercise:

Problem: What is the period of a 1.00-m-long pendulum?

Solution:

2.01 s

Exercise:**Problem:**

How long does it take a child on a swing to complete one swing if her center of gravity is 4.00 m below the pivot?

Exercise:**Problem:**

The pendulum on a cuckoo clock is 5.00 cm long. What is its frequency?

Solution:

2.23 Hz

Exercise:**Problem:**

Two parakeets sit on a swing with their combined center of mass 10.0 cm below the pivot. At what frequency do they swing?

Exercise:

Problem:

(a) A pendulum that has a period of 3.00000 s and that is located where the acceleration due to gravity is 9.79 m/s^2 is moved to a location where the acceleration due to gravity is 9.82 m/s^2 . What is its new period? (b) Explain why so many digits are needed in the value for the period, based on the relation between the period and the acceleration due to gravity.

Solution:

(a) 2.99541 s

(b) Since the period is related to the square root of the acceleration of gravity, when the acceleration changes by 1% the period changes by $(0.01)^2 = 0.01\%$ so it is necessary to have at least 4 digits after the decimal to see the changes.

Exercise:**Problem:**

A pendulum with a period of 2.00000 s in one location ($g = 9.80 \text{ m/s}^2$) is moved to a new location where the period is now 1.99796 s. What is the acceleration due to gravity at its new location?

Exercise:**Problem:**

(a) What is the effect on the period of a pendulum if you double its length?

(b) What is the effect on the period of a pendulum if you decrease its length by 5.00%?

Solution:

(a) Period increases by a factor of 1.41 ($\sqrt{2}$)

(b) Period decreases to 97.5% of old period

Exercise:

Problem:

Find the ratio of the new/old periods of a pendulum if the pendulum were transported from Earth to the Moon, where the acceleration due to gravity is 1.63 m/s^2 .

Exercise:

Problem:

At what rate will a pendulum clock run on the Moon, where the acceleration due to gravity is 1.63 m/s^2 , if it keeps time accurately on Earth? That is, find the time (in hours) it takes the clock's hour hand to make one revolution on the Moon.

Solution:

Slow by a factor of 2.45

Exercise:

Problem:

Suppose the length of a clock's pendulum is changed by 1.000%, exactly at noon one day. What time will it read 24.00 hours later, assuming it the pendulum has kept perfect time before the change? Note that there are two answers, and perform the calculation to four-digit precision.

Exercise:

Problem:

If a pendulum-driven clock gains 5.00 s/day, what fractional change in pendulum length must be made for it to keep perfect time?

Solution:

length must increase by 0.0116%.

Glossary

simple pendulum

an object with a small mass suspended from a light wire or string

Energy and the Simple Harmonic Oscillator

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Describe the changes in energy that occur while a system undergoes simple harmonic motion.

To study the energy of a simple harmonic oscillator, we first consider all the forms of energy it can have. We know from [Hooke's Law: Stress and Strain Revisited](#) that the energy stored in the deformation of a simple harmonic oscillator is a form of potential energy given by:

Equation:

$$PE_{\text{el}} = \frac{1}{2}kx^2.$$

Because a simple harmonic oscillator has no dissipative forces, the other important form of energy is kinetic energy KE. Conservation of energy for these two forms is:

Equation:

$$KE + PE_{\text{el}} = \text{constant}$$

or

Equation:

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{constant}.$$

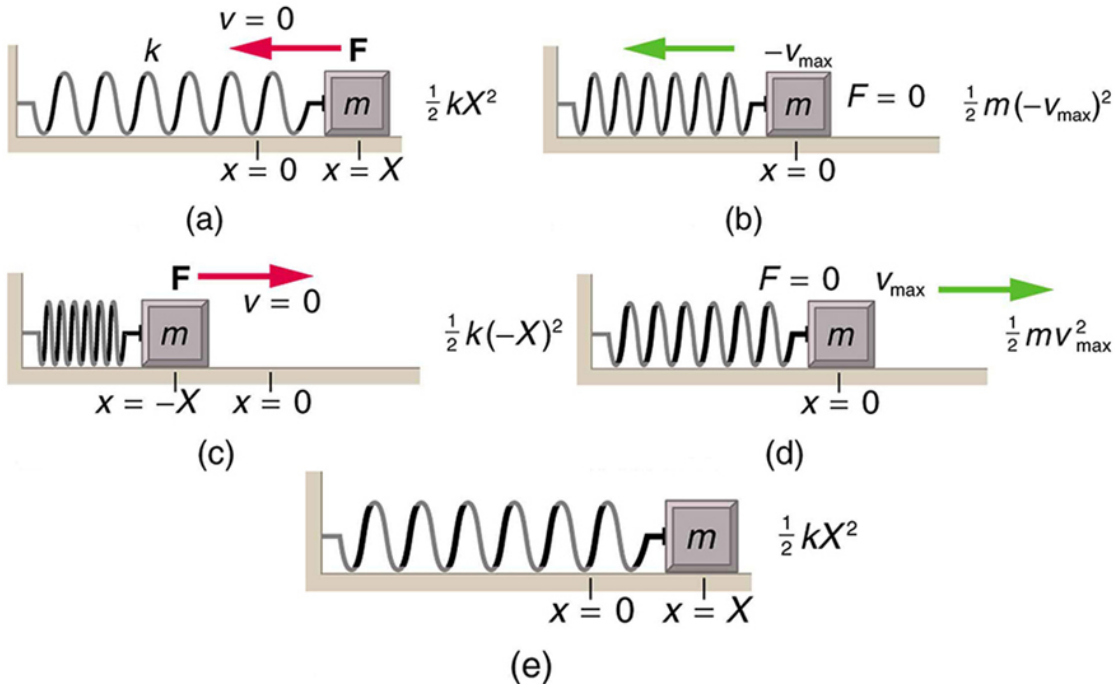
This statement of conservation of energy is valid for *all* simple harmonic oscillators, including ones where the gravitational force plays a role

Namely, for a simple pendulum we replace the velocity with $v = L\omega$, the spring constant with $k = mg/L$, and the displacement term with $x = L\theta$. Thus

Equation:

$$\frac{1}{2}mL^2\omega^2 + \frac{1}{2}mgL\theta^2 = \text{constant}.$$

In the case of undamped simple harmonic motion, the energy oscillates back and forth between kinetic and potential, going completely from one to the other as the system oscillates. So for the simple example of an object on a frictionless surface attached to a spring, as shown again in [\[link\]](#), the motion starts with all of the energy stored in the spring. As the object starts to move, the elastic potential energy is converted to kinetic energy, becoming entirely kinetic energy at the equilibrium position. It is then converted back into elastic potential energy by the spring, the velocity becomes zero when the kinetic energy is completely converted, and so on. This concept provides extra insight here and in later applications of simple harmonic motion, such as alternating current circuits.



The transformation of energy in simple harmonic motion is illustrated for an object attached to a spring on a frictionless surface.

The conservation of energy principle can be used to derive an expression for velocity v . If we start our simple harmonic motion with zero velocity and maximum displacement ($x = X$), then the total energy is

Equation:

$$\frac{1}{2}kX^2.$$

This total energy is constant and is shifted back and forth between kinetic energy and potential energy, at most times being shared by each. The conservation of energy for this system in equation form is thus:

Equation:

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kX^2.$$

Solving this equation for v yields:

Equation:

$$v = \pm \sqrt{\frac{k}{m}(X^2 - x^2)}.$$

Manipulating this expression algebraically gives:

Equation:

$$v = \pm \sqrt{\frac{k}{m}}X\sqrt{1 - \frac{x^2}{X^2}}$$

and so

Equation:

$$v = \pm v_{\max}\sqrt{1 - \frac{x^2}{X^2}},$$

where

Equation:

$$v_{\max} = \sqrt{\frac{k}{m}}X.$$

From this expression, we see that the velocity is a maximum (v_{\max}) at $x = 0$, as stated earlier in $v(t) = -v_{\max} \sin \frac{2\pi t}{T}$. Notice that the maximum velocity depends on three factors. Maximum velocity is directly proportional to amplitude. As you might guess, the greater the maximum displacement the greater the maximum velocity. Maximum velocity is also

greater for stiffer systems, because they exert greater force for the same displacement. This observation is seen in the expression for v_{\max} ; it is proportional to the square root of the force constant k . Finally, the maximum velocity is smaller for objects that have larger masses, because the maximum velocity is inversely proportional to the square root of m . For a given force, objects that have large masses accelerate more slowly.

A similar calculation for the simple pendulum produces a similar result, namely:

Equation:

$$\omega_{\max} = \sqrt{\frac{g}{L}} \theta_{\max}.$$

Note:

Making Connections: Mass Attached to a Spring

Consider a mass m attached to a spring, with spring constant k , fixed to a wall. When the mass is displaced from its equilibrium position and released, the mass undergoes simple harmonic motion. The spring exerts a force $F = -kv$ on the mass. The potential energy of the system is stored in the spring. It will be zero when the spring is in the equilibrium position. All the internal energy exists in the form of kinetic energy, given by $KE = \frac{1}{2}mv^2$. As the system oscillates, which means that the spring compresses and expands, there is a change in the structure of the system and a corresponding change in its internal energy. Its kinetic energy is converted to potential energy and vice versa. This occurs at an equal rate, which means that a loss of kinetic energy yields a gain in potential energy, thus preserving the work-energy theorem and the law of conservation of energy.

Example:

Determine the Maximum Speed of an Oscillating System: A Bumpy Road

Suppose that a car is 900 kg and has a suspension system that has a force constant $k = 6.53 \times 10^4$ N/m. The car hits a bump and bounces with an amplitude of 0.100 m. What is its maximum vertical velocity if you assume no damping occurs?

Strategy

We can use the expression for v_{\max} given in $v_{\max} = \sqrt{\frac{k}{m}} X$ to determine the maximum vertical velocity. The variables m and k are given in the problem statement, and the maximum displacement X is 0.100 m.

Solution

1. Identify known.
2. Substitute known values into $v_{\max} = \sqrt{\frac{k}{m}} X$:

Equation:

$$v_{\max} = \sqrt{\frac{6.53 \times 10^4 \text{ N/m}}{900 \text{ kg}}} (0.100 \text{ m}).$$

3. Calculate to find $v_{\max} = 0.852$ m/s.

Discussion

This answer seems reasonable for a bouncing car. There are other ways to use conservation of energy to find v_{\max} . We could use it directly, as was done in the example featured in [Hooke's Law: Stress and Strain Revisited](#).

The small vertical displacement y of an oscillating simple pendulum, starting from its equilibrium position, is given as

Equation:

$$y(t) = a \sin \omega t,$$

where a is the amplitude, ω is the angular velocity and t is the time taken.

Substituting $\omega = \frac{2\pi}{T}$, we have

Equation:

$$y(t) = a \sin\left(\frac{2\pi t}{T}\right).$$

Thus, the displacement of pendulum is a function of time as shown above. Also the velocity of the pendulum is given by

Equation:

$$v(t) = \frac{2a\pi}{T} \cos\left(\frac{2\pi t}{T}\right),$$

so the motion of the pendulum is a function of time.

Exercise:

Check Your Understanding

Problem:

Why does it hurt more if your hand is snapped with a ruler than with a loose spring, even if the displacement of each system is equal?

Solution:

The ruler is a stiffer system, which carries greater force for the same amount of displacement. The ruler snaps your hand with greater force, which hurts more.

Exercise:

Check Your Understanding

Problem:

You are observing a simple harmonic oscillator. Identify one way you could decrease the maximum velocity of the system.

Solution:

You could increase the mass of the object that is oscillating.

Test Prep for AP Courses

Exercise:

Problem:

A mass of 1 kg undergoes simple harmonic motion with amplitude of 1 m. If the period of the oscillation is 1 s, calculate the internal energy of the system.

Solution:

19.7 J

Section Summary

- Energy in the simple harmonic oscillator is shared between elastic potential energy and kinetic energy, with the total being constant:

Equation:

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{constant}.$$

- Maximum velocity depends on three factors: it is directly proportional to amplitude, it is greater for stiffer systems, and it is smaller for objects that have larger masses:

Equation:

$$v_{\text{max}} = \sqrt{\frac{k}{m}}X.$$

Conceptual Questions

Exercise:

Problem:

Explain in terms of energy how dissipative forces such as friction reduce the amplitude of a harmonic oscillator. Also explain how a driving mechanism can compensate. (A pendulum clock is such a system.)

Problems & Exercises**Exercise:****Problem:**

The length of nylon rope from which a mountain climber is suspended has a force constant of $1.40 \times 10^4 \text{ N/m}$.

- (a) What is the frequency at which he bounces, given his mass plus and the mass of his equipment are 90.0 kg?
 - (b) How much would this rope stretch to break the climber's fall if he free-falls 2.00 m before the rope runs out of slack? Hint: Use conservation of energy.
 - (c) Repeat both parts of this problem in the situation where twice this length of nylon rope is used.
-

Solution:

- (a) 1.99 Hz
- (b) 50.2 cm
- (c) 1.41 Hz, 0.710 m

Exercise:**Problem: Engineering Application**

Near the top of the Citigroup Center building in New York City, there is an object with mass of 4.00×10^5 kg on springs that have adjustable force constants. Its function is to dampen wind-driven oscillations of the building by oscillating at the same frequency as the building is being driven—the driving force is transferred to the object, which oscillates instead of the entire building. (a) What effective force constant should the springs have to make the object oscillate with a period of 2.00 s? (b) What energy is stored in the springs for a 2.00-m displacement from equilibrium?

Solution:

(a) 3.95×10^6 N/m

(b) 7.90×10^6 J

Uniform Circular Motion and Simple Harmonic Motion

LEARNING OBJECTIVES

By the end of this section, you will be able to:

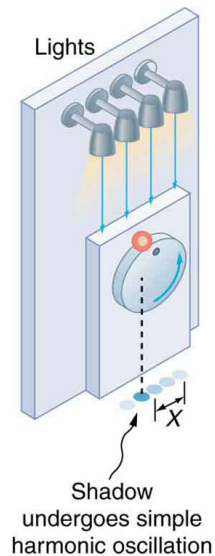
- Compare simple harmonic motion with uniform circular motion.



The horses on this merry-go-round exhibit uniform circular motion. (credit: Wonderlane, Flickr)

There is an easy way to produce simple harmonic motion by using uniform circular motion. [\[link\]](#) shows one way of using this method. A ball is attached to a uniformly rotating vertical turntable, and its shadow is projected on the floor as shown. The shadow undergoes simple harmonic

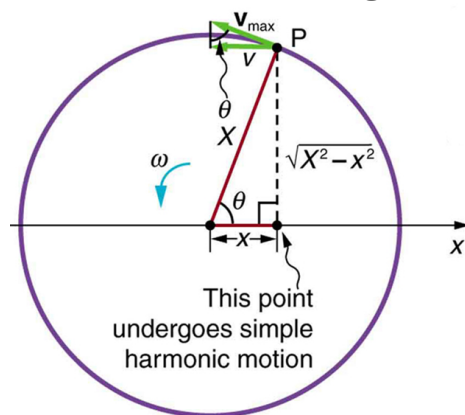
motion. Hooke's law usually describes uniform circular motions (ω constant) rather than systems that have large visible displacements. So observing the projection of uniform circular motion, as in [\[link\]](#), is often easier than observing a precise large-scale simple harmonic oscillator. If studied in sufficient depth, simple harmonic motion produced in this manner can give considerable insight into many aspects of oscillations and waves and is very useful mathematically. In our brief treatment, we shall indicate some of the major features of this relationship and how they might be useful.



The
shadow
of a ball
rotating
at
constant
angular
velocity
 ω on a
turntable
goes
back and
forth in
precise
simple

harmoni
c
motion.

[\[link\]](#) shows the basic relationship between uniform circular motion and simple harmonic motion. The point P travels around the circle at constant angular velocity ω . The point P is analogous to an object on the merry-go-round. The projection of the position of P onto a fixed axis undergoes simple harmonic motion and is analogous to the shadow of the object. At the time shown in the figure, the projection has position x and moves to the left with velocity v . The velocity of the point P around the circle equals v_{\max} . The projection of v_{\max} on the x -axis is the velocity v of the simple harmonic motion along the x -axis.



A point P moving on a circular path with a constant angular velocity ω is undergoing uniform circular motion. Its projection on the x -axis undergoes simple harmonic motion. Also shown is the velocity of this point around the circle, v_{\max} , and its

projection, which is v .

Note that these velocities form a similar triangle to the displacement triangle.

To see that the projection undergoes simple harmonic motion, note that its position x is given by

Equation:

$$x = X \cos \theta,$$

where $\theta = \omega t$, ω is the constant angular velocity, and X is the radius of the circular path. Thus,

Equation:

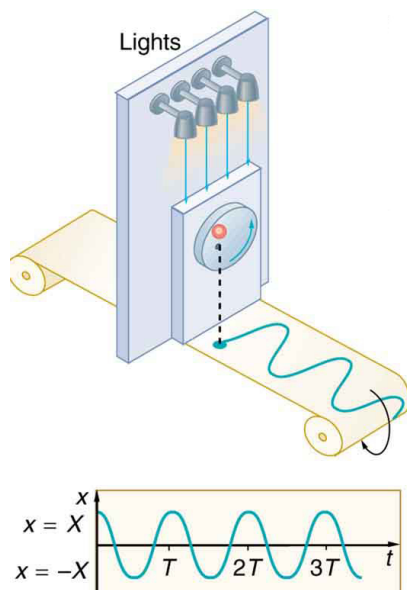
$$x = X \cos \omega t.$$

The angular velocity ω is in radians per unit time; in this case 2π radians is the time for one revolution T . That is, $\omega = 2\pi/T$. Substituting this expression for ω , we see that the position x is given by:

Equation:

$$x(t) = \cos\left(\frac{2\pi t}{T}\right).$$

This expression is the same one we had for the position of a simple harmonic oscillator in [Simple Harmonic Motion: A Special Periodic Motion](#). If we make a graph of position versus time as in [\[link\]](#), we see again the wavelike character (typical of simple harmonic motion) of the projection of uniform circular motion onto the x -axis.



The position of the projection of uniform circular motion performs simple harmonic motion, as this wavelike graph of x versus t indicates.

Now let us use [\[link\]](#) to do some further analysis of uniform circular motion as it relates to simple harmonic motion. The triangle formed by the velocities in the figure and the triangle formed by the displacements (X , x , and $\sqrt{X^2 - x^2}$) are similar right triangles. Taking ratios of similar sides, we see that

Equation:

$$\frac{v}{v_{\max}} = \frac{\sqrt{X^2 - x^2}}{X} = \sqrt{1 - \frac{x^2}{X^2}}.$$

We can solve this equation for the speed v or

Equation:

$$v = v_{\max} \sqrt{1 - \frac{x^2}{X^2}}.$$

This expression for the speed of a simple harmonic oscillator is exactly the same as the equation obtained from conservation of energy considerations in [Energy and the Simple Harmonic Oscillator](#). You can begin to see that it is possible to get all of the characteristics of simple harmonic motion from an analysis of the projection of uniform circular motion.

Finally, let us consider the period T of the motion of the projection. This period is the time it takes the point P to complete one revolution. That time is the circumference of the circle $2\pi X$ divided by the velocity around the circle, v_{\max} . Thus, the period T is

Equation:

$$T = \frac{2\pi X}{v_{\max}}.$$

We know from conservation of energy considerations that

Equation:

$$v_{\max} = \sqrt{\frac{k}{m}} X.$$

Solving this equation for X/v_{\max} gives

Equation:

$$\frac{X}{v_{\max}} = \sqrt{\frac{m}{k}}.$$

Substituting this expression into the equation for T yields

Equation:

$$T = 2\pi\sqrt{\frac{m}{k}}.$$

Thus, the period of the motion is the same as for a simple harmonic oscillator. We have determined the period for any simple harmonic oscillator using the relationship between uniform circular motion and simple harmonic motion.

Some modules occasionally refer to the connection between uniform circular motion and simple harmonic motion. Moreover, if you carry your study of physics and its applications to greater depths, you will find this relationship useful. It can, for example, help to analyze how waves add when they are superimposed.

Exercise:**Check Your Understanding****Problem:**

Identify an object that undergoes uniform circular motion. Describe how you could trace the simple harmonic motion of this object as a wave.

Solution:

A record player undergoes uniform circular motion. You could attach a dowel rod to one point on the outside edge of the turntable and attach a pen to the other end of the dowel. As the record player turns, the pen will move. You can drag a long piece of paper under the pen, capturing its motion as a wave.

Test Prep for AP Courses**Exercise:**

Problem:

In the equation $x = A \sin \omega t$, what values can the position x take?

- a. -1 to $+1$
- b. $-A$ to $+A$
- c. 0
- d. $-t$ to t

Section Summary

A projection of uniform circular motion undergoes simple harmonic oscillation.

Problems & Exercises**Exercise:****Problem:**

(a) What is the maximum velocity of an 85.0-kg person bouncing on a bathroom scale having a force constant of $1.50 \times 10^6 \text{ N/m}$, if the amplitude of the bounce is 0.200 cm? (b) What is the maximum energy stored in the spring?

Solution:

- a). 0.266 m/s
- b). 3.00 J

Exercise:

Problem:

A novelty clock has a 0.0100-kg mass object bouncing on a spring that has a force constant of 1.25 N/m. What is the maximum velocity of the object if the object bounces 3.00 cm above and below its equilibrium position? (b) How many joules of kinetic energy does the object have at its maximum velocity?

Exercise:**Problem:**

At what positions is the speed of a simple harmonic oscillator half its maximum? That is, what values of x/X give $v = \pm v_{\max}/2$, where X is the amplitude of the motion?

Solution:

$$\pm \frac{\sqrt{3}}{2}$$

Exercise:**Problem:**

A ladybug sits 12.0 cm from the center of a Beatles music album spinning at 33.33 rpm. What is the maximum velocity of its shadow on the wall behind the turntable, if illuminated parallel to the record by the parallel rays of the setting Sun?

Damped Harmonic Motion

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Compare and discuss underdamped and overdamped oscillating systems.
- Explain critically damped systems.



In order to counteract dampening forces, this mom needs to keep pushing the swing. (credit: Mohd Fazlin Mohd Effendy Ooi, Flickr)

A guitar string stops oscillating a few seconds after being plucked. To keep a child happy on a swing, you must keep pushing. Although we can often make friction and other non-conservative forces negligibly small, completely

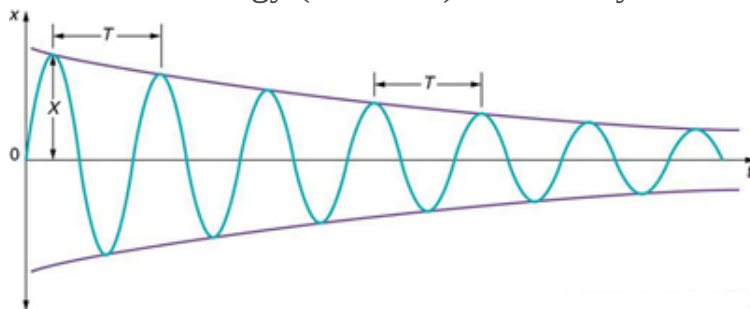
undamped motion is rare. In fact, we may even want to damp oscillations, such as with car shock absorbers.

For a system that has a small amount of damping, the period and frequency are nearly the same as for simple harmonic motion, but the amplitude gradually decreases as shown in [\[link\]](#). This occurs because the non-conservative damping force removes energy from the system, usually in the form of thermal energy. In general, energy removal by non-conservative forces is described as

Equation:

$$W_{nc} = \Delta(\text{KE} + \text{PE}),$$

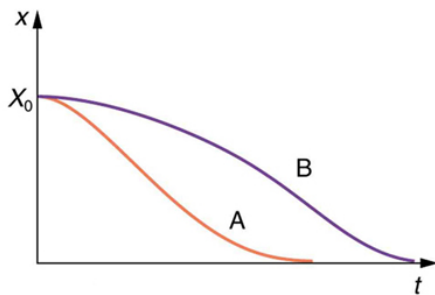
where W_{nc} is work done by a non-conservative force (here the damping force). For a damped harmonic oscillator, W_{nc} is negative because it removes mechanical energy ($\text{KE} + \text{PE}$) from the system.



In this graph of displacement versus time for a harmonic oscillator with a small amount of damping, the amplitude slowly decreases, but the period and frequency are nearly the same as if the system were completely undamped.

If you gradually *increase* the amount of damping in a system, the period and frequency begin to be affected, because damping opposes and hence slows the back and forth motion. (The net force is smaller in both directions.) If there is very large damping, the system does not even oscillate—it slowly moves toward equilibrium. [\[link\]](#) shows the displacement of a harmonic oscillator for different amounts of damping. When we want to damp out oscillations, such as in the suspension of a car, we may want the system to return to equilibrium as quickly

as possible **Critical damping** is defined as the condition in which the damping of an oscillator results in it returning as quickly as possible to its equilibrium position. The critically damped system may overshoot the equilibrium position, but if it does, it will do so only once. Critical damping is represented by Curve A in [\[link\]](#). With less-than critical damping, the system will return to equilibrium faster but will overshoot and cross over one or more times. Such a system is **underdamped**; its displacement is represented by the curve in [\[link\]](#). Curve B in [\[link\]](#) represents an **overdamped** system. As with critical damping, it too may overshoot the equilibrium position, but will reach equilibrium over a longer period of time.



Displacement versus time
for a critically damped
harmonic oscillator (A)
and an overdamped
harmonic oscillator (B).
The critically damped
oscillator returns to
equilibrium at $X = 0$ in
the smallest time possible
without overshooting.

Critical damping is often desired, because such a system returns to equilibrium rapidly and remains at equilibrium as well. In addition, a constant force applied to a critically damped system moves the system to a new equilibrium position in the shortest time possible without overshooting or oscillating about the new position. For example, when you stand on bathroom scales that have a needle gauge, the needle moves to its equilibrium position without oscillating. It would be quite inconvenient if the needle oscillated about the new equilibrium position for a long time before settling. Damping forces can vary greatly in character. Friction, for example, is sometimes independent of velocity (as assumed in most

places in this text). But many damping forces depend on velocity—sometimes in complex ways, sometimes simply being proportional to velocity.

Note:

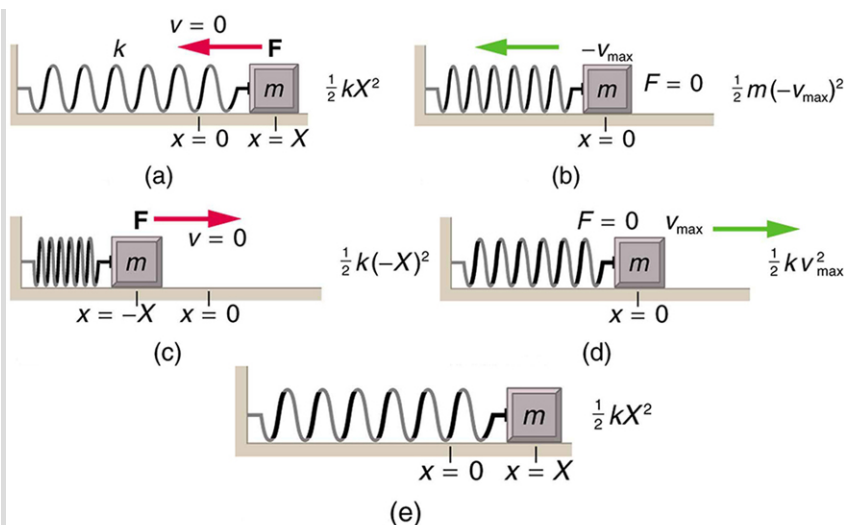
Making Connections: Damped Oscillator

Consider a mass attached to a spring. This system oscillates when in air because air exerts almost no force on the spring. Now put this system in a liquid, say, water. You will see that the system hardly oscillates when in water. When the system is submerged in water an external force acts on the oscillator. This force is exerted by the liquid against the motion of the spring-mass oscillator and is responsible for the inhibition of oscillations. A force that inhibits oscillations is called a “damping force,” and the system that experiences it is called a “damped oscillator.” A damped oscillator sees a change in its energy. With time the total energy of the oscillator, which would be its mechanical energy, decreases. Since energy has to be conserved, the energy gets converted into thermal energy, which is stored in the water and the spring.

Example:

Damping an Oscillatory Motion: Friction on an Object Connected to a Spring

Damping oscillatory motion is important in many systems, and the ability to control the damping is even more so. This is generally attained using non-conservative forces such as the friction between surfaces, and viscosity for objects moving through fluids. The following example considers friction. Suppose a 0.200-kg object is connected to a spring as shown in [\[link\]](#), but there is simple friction between the object and the surface, and the coefficient of friction μ_k is equal to 0.0800. (a) What is the frictional force between the surfaces? (b) What total distance does the object travel if it is released 0.100 m from equilibrium, starting at $v = 0$? The force constant of the spring is $k = 50.0 \text{ N/m}$.



The transformation of energy in simple harmonic motion is illustrated for an object attached to a spring on a frictionless surface.

Strategy

This problem requires you to integrate your knowledge of various concepts regarding waves, oscillations, and damping. To solve an integrated concept problem, you must first identify the physical principles involved. Part (a) is about the frictional force. This is a topic involving the application of Newton's Laws. Part (b) requires an understanding of work and conservation of energy, as well as some understanding of horizontal oscillatory systems.

Now that we have identified the principles we must apply in order to solve the problems, we need to identify the knowns and unknowns for each part of the question, as well as the quantity that is constant in Part (a) and Part (b) of the question.

Solution a

1. Choose the proper equation: Friction is $f = \mu_k mg$.
2. Identify the known values.
3. Enter the known values into the equation:

Equation:

$$f = (0.0800)(0.200 \text{ kg})(9.80 \text{ m/s}^2).$$

4. Calculate and convert units: $f = 0.157 \text{ N}$.

Discussion a

The force here is small because the system and the coefficients are small.

Solution b

Identify the known:

- The system involves elastic potential energy as the spring compresses and expands, friction that is related to the work done, and the kinetic energy as the body speeds up and slows down.
- Energy is not conserved as the mass oscillates because friction is a non-conservative force.
- The motion is horizontal, so gravitational potential energy does not need to be considered.
- Because the motion starts from rest, the energy in the system is initially $PE_{el,i} = (1/2)kX^2$. This energy is removed by work done by friction $W_{nc} = -fd$, where d is the total distance traveled and $f = \mu_k mg$ is the force of friction. When the system stops moving, the friction force will balance the force exerted by the spring, so $PE_{el,f} = (1/2)kx^2$ where x is the final position and is given by

Equation:

$$\begin{aligned} F_{el} &= f \\ kx &= \mu_k mg. \\ x &= \frac{\mu_k mg}{k} \end{aligned}$$

1. By equating the work done to the energy removed, solve for the distance d .
2. The work done by the non-conservative forces equals the initial, stored elastic potential energy. Identify the correct equation to use:

Equation:

$$W_{nc} = \Delta(K.E + P.E) = PE_{el,f} - PE_{el,i} = \frac{1}{2}k\left(\left(\frac{\mu_k mg}{k}\right)^2 - X^2\right).$$

3. Recall that $W_{nc} = -fd$.
4. Enter the friction as $f = \mu_k mg$ into $W_{nc} = -fd$, thus

Equation:

$$W_{nc} = -\mu_k mgd.$$

5. Combine these two equations to find

Equation:

$$\frac{1}{2}k\left(\left(\frac{\mu_k mg}{k}\right)^2 - X^2\right) = -\mu_k mgd.$$

6. Solve the equation for d :

Equation:

$$d = \frac{k}{2\mu_k mg} \left(X^2 - \left(\frac{\mu_k mg}{k} \right)^2 \right).$$

7. Enter the known values into the resulting equation:

Equation:

mtd

mtd

8. Calculate d and convert units:

Equation:

$$d = 1.59 \text{ m.}$$

Discussion b

This is the total distance traveled back and forth across $x = 0$, which is the undamped equilibrium position. The number of oscillations about the equilibrium position will be more than $d/X = (1.59 \text{ m})/(0.100 \text{ m}) = 15.9$ because the amplitude of the oscillations is decreasing with time. At the end of the motion, this system will not return to $x = 0$ for this type of damping force, because static friction will exceed the restoring force. This system is underdamped. In contrast, an overdamped system with a simple constant damping force would not cross the equilibrium position $x = 0$ a single time. For example, if this system had a damping force 20 times greater, it would only move 0.0484 m toward the equilibrium position from its original 0.100-m position.

This worked example illustrates how to apply problem-solving strategies to situations that integrate the different concepts you have learned. The first step is to identify the physical principles involved in the problem. The second step is to solve for the unknowns using familiar problem-solving strategies. These are found throughout the text, and many worked examples show how to use them

for single topics. In this integrated concepts example, you can see how to apply them across several topics. You will find these techniques useful in applications of physics outside a physics course, such as in your profession, in other science disciplines, and in everyday life.

Exercise:

Check Your Understanding

Problem: Why are completely undamped harmonic oscillators so rare?

Solution:

Friction often comes into play whenever an object is moving. Friction causes damping in a harmonic oscillator.

Exercise:

Check Your Understanding

Problem:

Describe the difference between overdamping, underdamping, and critical damping.

Solution:

An overdamped system moves slowly toward equilibrium. An underdamped system moves quickly to equilibrium, but will oscillate about the equilibrium point as it does so. A critically damped system moves as quickly as possible toward equilibrium without oscillating about the equilibrium.

Test Prep for AP Courses

Exercise:

Problem:

The non-conservative damping force removes energy from a system in which form?

- a. Mechanical energy
- b. Electrical energy
- c. Thermal energy
- d. None of the above

Solution:

(c)

Exercise:

Problem:

The time rate of change of mechanical energy for a damped oscillator is always:

- a. 0
- b. Negative
- c. Positive
- d. Undefined

Exercise:

Problem:

A 0.5-kg object is connected to a spring that undergoes oscillatory motion. There is friction between the object and the surface it is kept on given by coefficient of friction $\mu_k = 0.06$. If the object is released 0.2 m from equilibrium, what is the distance that the object travels? Given that the force constant of the spring is 50 N m^{-1} and the frictional force between the objects is 0.294 N.

Solution:

$$d = \frac{k}{2\mu_k mg} \left(X^2 - \frac{\mu_k mg}{k} \right)^2$$

$$\text{where } k = 50 \text{ N} \cdot \text{m}^{-1}$$

$$\mu_k = 0.06$$

$$m = 0.5 \text{ kg}$$

$$d = \frac{50\text{N} \cdot \text{m}^{-1}}{2 \times 0.06 \times 9.8\text{m} \cdot \text{s}^{-2}} \left((0.2)^2 - \left(\frac{(0.06 \times 0.5\text{kg} \times 9.8\text{m} \cdot \text{s}^{-2})^2}{(50\text{N} \cdot \text{m}^{-1})^2} \right) \right)$$

$$= 1.698 \text{ m}$$

Section Summary

- Damped harmonic oscillators have non-conservative forces that dissipate their energy.
- Critical damping returns the system to equilibrium as fast as possible without overshooting.
- An underdamped system will oscillate through the equilibrium position.
- An overdamped system moves more slowly toward equilibrium than one that is critically damped.

Conceptual Questions

Exercise:

Problem:

Give an example of a damped harmonic oscillator. (They are more common than undamped or simple harmonic oscillators.)

Exercise:

Problem:

How would a car bounce after a bump under each of these conditions?

- overdamping
- underdamping
- critical damping

Exercise:

Problem:

Most harmonic oscillators are damped and, if undriven, eventually come to a stop. How is this observation related to the second law of thermodynamics?

Problems & Exercises

Exercise:

Problem:

The amplitude of a lightly damped oscillator decreases by 3.0% during each cycle. What percentage of the mechanical energy of the oscillator is lost in each cycle?

Glossary

critical damping

the condition in which the damping of an oscillator causes it to return as quickly as possible to its equilibrium position without oscillating back and forth about this position

over damping

the condition in which damping of an oscillator causes it to return to equilibrium without oscillating; oscillator moves more slowly toward equilibrium than in the critically damped system

under damping

the condition in which damping of an oscillator causes it to return to equilibrium with the amplitude gradually decreasing to zero; system returns to equilibrium faster but overshoots and crosses the equilibrium position one or more times

Forced Oscillations and Resonance

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Differentiate between different types of damping.
- Explain resonance.

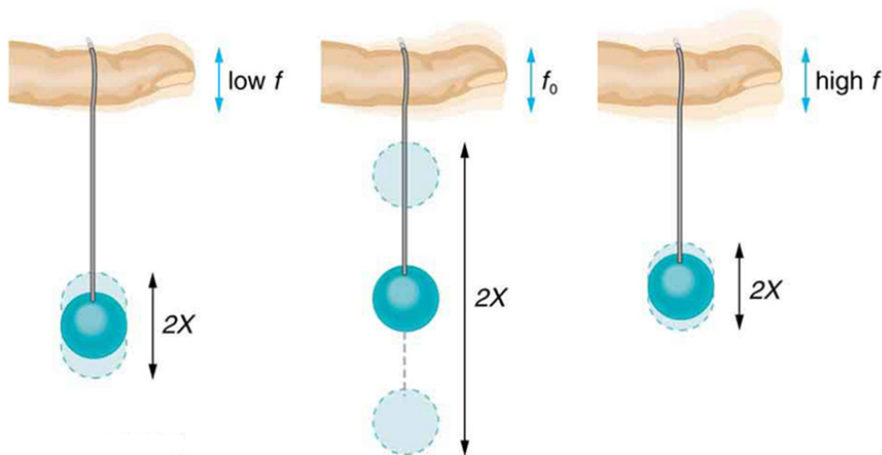


You can cause the strings in a piano to vibrate simply by producing sound waves from your voice. (credit: Matt Billings, Flickr)

Sit in front of a piano sometime and sing a loud brief note at it with the dampers off its strings. It will sing the same note back at you—the strings, having the same frequencies as your voice, are resonating in response to the

forces from the sound waves that you sent to them. Your voice and a piano's strings is a good example of the fact that objects—in this case, piano strings—can be forced to oscillate but oscillate best at their natural frequency. In this section, we shall briefly explore applying a *periodic driving force* acting on a simple harmonic oscillator. The driving force puts energy into the system at a certain frequency, not necessarily the same as the natural frequency of the system. The **natural frequency** is the frequency at which a system would oscillate if there were no driving and no damping force.

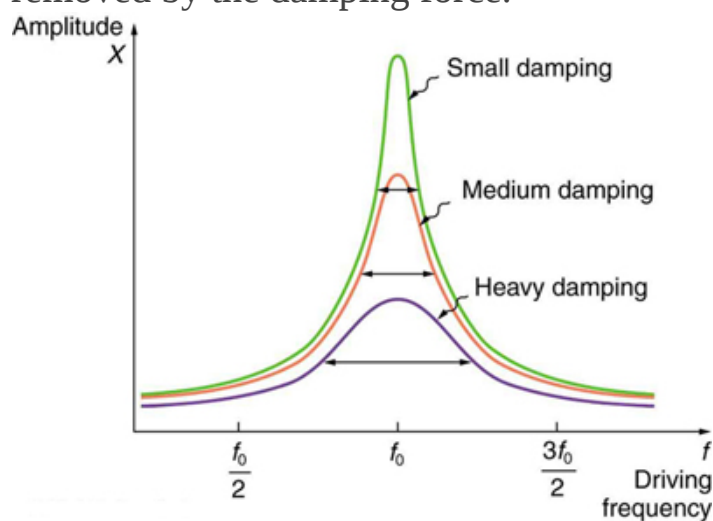
Most of us have played with toys involving an object supported on an elastic band, something like the paddle ball suspended from a finger in [\[link\]](#). Imagine the finger in the figure is your finger. At first you hold your finger steady, and the ball bounces up and down with a small amount of damping. If you move your finger up and down slowly, the ball will follow along without bouncing much on its own. As you increase the frequency at which you move your finger up and down, the ball will respond by oscillating with increasing amplitude. When you drive the ball at its natural frequency, the ball's oscillations increase in amplitude with each oscillation for as long as you drive it. The phenomenon of driving a system with a frequency equal to its natural frequency is called **resonance**. A system being driven at its natural frequency is said to **resonate**. As the driving frequency gets progressively higher than the resonant or natural frequency, the amplitude of the oscillations becomes smaller, until the oscillations nearly disappear and your finger simply moves up and down with little effect on the ball.



The paddle ball on its rubber band moves in

response to the finger supporting it. If the finger moves with the natural frequency f_0 of the ball on the rubber band, then a resonance is achieved, and the amplitude of the ball's oscillations increases dramatically. At higher and lower driving frequencies, energy is transferred to the ball less efficiently, and it responds with lower-amplitude oscillations.

[\[link\]](#) shows a graph of the amplitude of a damped harmonic oscillator as a function of the frequency of the periodic force driving it. There are three curves on the graph, each representing a different amount of damping. All three curves peak at the point where the frequency of the driving force equals the natural frequency of the harmonic oscillator. The highest peak, or greatest response, is for the least amount of damping, because less energy is removed by the damping force.



Amplitude of a harmonic oscillator as a function of the frequency of the driving force. The curves represent the same oscillator with the same natural frequency but with different amounts of damping. Resonance occurs when the driving frequency equals the

natural frequency, and the greatest response is for the least amount of damping. The narrowest response is also for the least damping.

It is interesting that the widths of the resonance curves shown in [\[link\]](#) depend on damping: the less the damping, the narrower the resonance. The message is that if you want a driven oscillator to resonate at a very specific frequency, you need as little damping as possible. Little damping is the case for piano strings and many other musical instruments. Conversely, if you want small-amplitude oscillations, such as in a car's suspension system, then you want heavy damping. Heavy damping reduces the amplitude, but the tradeoff is that the system responds at more frequencies.

These features of driven harmonic oscillators apply to a huge variety of systems. When you tune a radio, for example, you are adjusting its resonant frequency so that it only oscillates to the desired station's broadcast (driving) frequency. The more selective the radio is in discriminating between stations, the smaller its damping. Magnetic resonance imaging (MRI) is a widely used medical diagnostic tool in which atomic nuclei (mostly hydrogen nuclei) are made to resonate by incoming radio waves (on the order of 100 MHz). A child on a swing is driven by a parent at the swing's natural frequency to achieve maximum amplitude. In all of these cases, the efficiency of energy transfer from the driving force into the oscillator is best at resonance. Speed bumps and gravel roads prove that even a car's suspension system is not immune to resonance. In spite of finely engineered shock absorbers, which ordinarily convert mechanical energy to thermal energy almost as fast as it comes in, speed bumps still cause a large-amplitude oscillation. On gravel roads that are corrugated, you may have noticed that if you travel at the "wrong" speed, the bumps are very noticeable whereas at other speeds you may hardly feel the bumps at all. [\[link\]](#) shows a photograph of a famous example (the Tacoma Narrows Bridge) of the destructive effects of a driven harmonic oscillation. The Millennium Bridge in London was closed for a short period of time for the same reason while inspections were carried out.

In our bodies, the chest cavity is a clear example of a system at resonance. The diaphragm and chest wall drive the oscillations of the chest cavity which result in the lungs inflating and deflating. The system is critically damped and the muscular diaphragm oscillates at the resonant value for the system, making it highly efficient.



In 1940, the Tacoma Narrows Bridge in Washington state collapsed. Heavy cross winds drove the bridge into oscillations at its resonant frequency. Damping decreased when support cables broke loose and started to slip over the towers, allowing increasingly greater amplitudes until the structure failed (credit: PRI's *Studio 360*, via Flickr)

Exercise:

Check Your Understanding

Problem:

A famous magic trick involves a performer singing a note toward a crystal glass until the glass shatters. Explain why the trick works in terms of resonance and natural frequency.

Solution:

The performer must be singing a note that corresponds to the natural frequency of the glass. As the sound wave is directed at the glass, the glass responds by resonating at the same frequency as the sound wave. With enough energy introduced into the system, the glass begins to vibrate and eventually shatters.

Test Prep for AP Courses**Exercise:**

Problem: How is constant amplitude sustained in forced oscillations?

Section Summary

- A system's natural frequency is the frequency at which the system will oscillate if not affected by driving or damping forces.
- A periodic force driving a harmonic oscillator at its natural frequency produces resonance. The system is said to resonate.
- The less damping a system has, the higher the amplitude of the forced oscillations near resonance. The more damping a system has, the broader response it has to varying driving frequencies.

Conceptual Questions**Exercise:****Problem:**

Why are soldiers in general ordered to “route step” (walk out of step) across a bridge?

Problems & Exercises

Exercise:**Problem:**

How much energy must the shock absorbers of a 1200-kg car dissipate in order to damp a bounce that initially has a velocity of 0.800 m/s at the equilibrium position? Assume the car returns to its original vertical position.

Solution:

384 J

Exercise:**Problem:**

If a car has a suspension system with a force constant of 5.00×10^4 N/m, how much energy must the car's shocks remove to dampen an oscillation starting with a maximum displacement of 0.0750 m?

Exercise:**Problem:**

(a) How much will a spring that has a force constant of 40.0 N/m be stretched by an object with a mass of 0.500 kg when hung motionless from the spring? (b) Calculate the decrease in gravitational potential energy of the 0.500-kg object when it descends this distance. (c) Part of this gravitational energy goes into the spring. Calculate the energy stored in the spring by this stretch, and compare it with the gravitational potential energy. Explain where the rest of the energy might go.

Solution:

(a). 0.123 m

(b). -0.600 J

(c). 0.300 J. The rest of the energy may go into heat caused by friction and other damping forces.

Exercise:

Problem:

Suppose you have a 0.750-kg object on a horizontal surface connected to a spring that has a force constant of 150 N/m. There is simple friction between the object and surface with a static coefficient of friction $\mu_s = 0.100$. (a) How far can the spring be stretched without moving the mass? (b) If the object is set into oscillation with an amplitude twice the distance found in part (a), and the kinetic coefficient of friction is $\mu_k = 0.0850$, what total distance does it travel before stopping? Assume it starts at the maximum amplitude.

Exercise:

Problem:

Engineering Application: A suspension bridge oscillates with an effective force constant of 1.00×10^8 N/m. (a) How much energy is needed to make it oscillate with an amplitude of 0.100 m? (b) If soldiers march across the bridge with a cadence equal to the bridge's natural frequency and impart 1.00×10^4 J of energy each second, how long does it take for the bridge's oscillations to go from 0.100 m to 0.500 m amplitude?

Solution:

(a) 5.00×10^5 J

(b) 1.20×10^3 s

Glossary

natural frequency

the frequency at which a system would oscillate if there were no driving and no damping forces

resonance

the phenomenon of driving a system with a frequency equal to the system's natural frequency

resonate

a system being driven at its natural frequency

Waves

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Describe various characteristics associated with a wave.
- Differentiate between transverse and longitudinal waves.



Waves in the ocean behave similarly to all other types of waves. (credit: Steve Jurveston, Flickr)

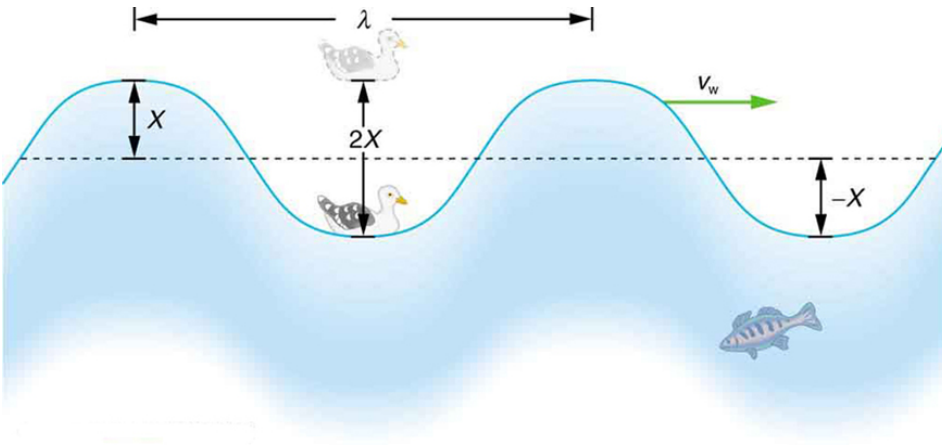
What do we mean when we say something is a wave? The most intuitive and easiest wave to imagine is the familiar water wave. More precisely, a **wave** is a disturbance that propagates, or moves from the place it was created. For water waves, the disturbance is in the surface of the water,

perhaps created by a rock thrown into a pond or by a swimmer splashing the surface repeatedly. For sound waves, the disturbance is a change in air pressure, perhaps created by the oscillating cone inside a speaker. For earthquakes, there are several types of disturbances, including disturbance of Earth's surface and pressure disturbances under the surface. Even radio waves are most easily understood using an analogy with water waves. Visualizing water waves is useful because there is more to it than just a mental image. Water waves exhibit characteristics common to all waves, such as amplitude, period, frequency and energy. All wave characteristics can be described by a small set of underlying principles.

A wave is a disturbance that propagates, or moves from the place it was created. The simplest waves repeat themselves for several cycles and are associated with simple harmonic motion. Let us start by considering the simplified water wave in [\[link\]](#). The wave is an up and down disturbance of the water surface. It causes a sea gull to move up and down in simple harmonic motion as the wave crests and troughs (peaks and valleys) pass under the bird. The time for one complete up and down motion is the wave's period T . The wave's frequency is $f = 1/T$, as usual. The wave itself moves to the right in the figure. This movement of the wave is actually the disturbance moving to the right, not the water itself (or the bird would move to the right). We define **wave velocity** v_w to be the speed at which the disturbance moves. Wave velocity is sometimes also called the *propagation velocity* or *propagation speed*, because the disturbance propagates from one location to another.

Note:**Misconception Alert**

Many people think that water waves push water from one direction to another. In fact, the particles of water tend to stay in one location, save for moving up and down due to the energy in the wave. The energy moves forward through the water, but the water stays in one place. If you feel yourself pushed in an ocean, what you feel is the energy of the wave, not a rush of water.



An idealized ocean wave passes under a sea gull that bobs up and down in simple harmonic motion. The wave has a wavelength λ , which is the distance between adjacent identical parts of the wave. The up and down disturbance of the surface propagates parallel to the surface at a speed v_w .

The water wave in the figure also has a length associated with it, called its **wavelength** λ , the distance between adjacent identical parts of a wave. (λ is the distance parallel to the direction of propagation.) The speed of propagation v_w is the distance the wave travels in a given time, which is one wavelength in the time of one period. In equation form, that is

Equation:

$$v_w = \frac{\lambda}{T}$$

or

Equation:

$$v_w = f\lambda.$$

This fundamental relationship holds for all types of waves. For water waves, v_w is the speed of a surface wave; for sound, v_w is the speed of sound; and for visible light, v_w is the speed of light, for example.

Note:

Applying the Science Practices: Different Types of Waves

Consider a spring fixed to a wall with a mass connected to its end. This fixed point on the wall exerts a force on the complete spring-and-mass system, and this implies that the momentum of the complete system is not conserved. Now, consider energy. Since the system is fixed to a point on the wall, it does not do any work; hence, the total work done is conserved, which means that the energy is conserved. Consequently, we have an oscillator in which energy is conserved but momentum is not. Now, consider a system of two masses connected to each other by a spring. This type of system also forms an oscillator. Since there is no fixed point, momentum is conserved as the forces acting on the two masses are equal and opposite. Energy for such a system will be conserved, because there are no external forces acting on the spring-two-masses system. It is clear from above that, for momentum to be conserved, momentum needs to be carried by waves. This is a typical example of a mechanical oscillator producing mechanical waves that need a medium in which to propagate. Sound waves are also examples of mechanical waves. There are some waves that can travel in the absence of a medium of propagation. Such waves are called “electromagnetic waves.” Light waves are examples of electromagnetic waves. Electromagnetic waves are created by the vibration of electric charge. This vibration creates a wave with both electric and magnetic field components.

Note:

Take-Home Experiment: Waves in a Bowl

Fill a large bowl or basin with water and wait for the water to settle so there are no ripples. Gently drop a cork into the middle of the bowl. Estimate the wavelength and period of oscillation of the water wave that

propagates away from the cork. Remove the cork from the bowl and wait for the water to settle again. Gently drop the cork at a height that is different from the first drop. Does the wavelength depend upon how high above the water the cork is dropped?

Example:

Calculate the Velocity of Wave Propagation: Gull in the Ocean

Calculate the wave velocity of the ocean wave in [\[link\]](#) if the distance between wave crests is 10.0 m and the time for a sea gull to bob up and down is 5.00 s.

Strategy

We are asked to find v_w . The given information tells us that $\lambda = 10.0$ m and $T = 5.00$ s. Therefore, we can use $v_w = \frac{\lambda}{T}$ to find the wave velocity.

Solution

1. Enter the known values into $v_w = \frac{\lambda}{T}$:

Equation:

$$v_w = \frac{10.0 \text{ m}}{5.00 \text{ s}}.$$

2. Solve for v_w to find $v_w = 2.00$ m/s.

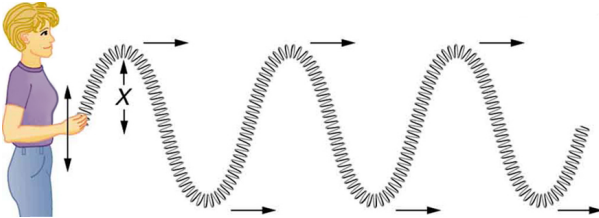
Discussion

This slow speed seems reasonable for an ocean wave. Note that the wave moves to the right in the figure at this speed, not the varying speed at which the sea gull moves up and down.

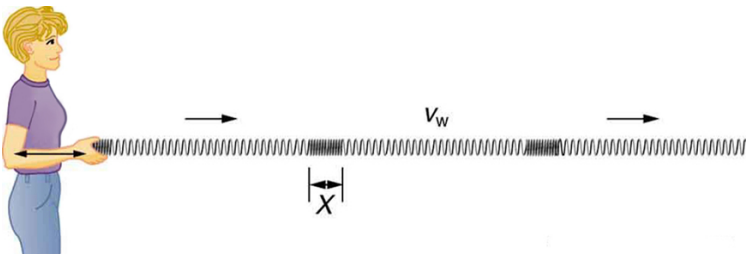
Transverse and Longitudinal Waves

A simple wave consists of a periodic disturbance that propagates from one place to another. The wave in [\[link\]](#) propagates in the horizontal direction while the surface is disturbed in the vertical direction. Such a wave is called a **transverse wave** or shear wave; in such a wave, the disturbance is

perpendicular to the direction of propagation. In contrast, in a **longitudinal wave** or compressional wave, the disturbance is parallel to the direction of propagation. [\[link\]](#) shows an example of a longitudinal wave. The size of the disturbance is its amplitude X and is completely independent of the speed of propagation v_w .



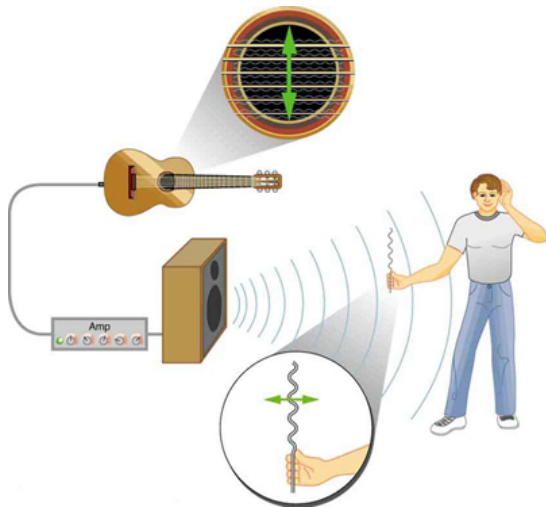
In this example of a transverse wave, the wave propagates horizontally, and the disturbance in the cord is in the vertical direction.



In this example of a longitudinal wave, the wave propagates horizontally, and the disturbance in the cord is also in the horizontal direction.

Waves may be transverse, longitudinal, or *a combination of the two*. (Water waves are actually a combination of transverse and longitudinal. The simplified water wave illustrated in [\[link\]](#) shows no longitudinal motion of the bird.) The waves on the strings of musical instruments are transverse—so are electromagnetic waves, such as visible light.

Sound waves in air and water are longitudinal. Their disturbances are periodic variations in pressure that are transmitted in fluids. Fluids do not have appreciable shear strength, and thus the sound waves in them must be longitudinal or compressional. Sound in solids can be both longitudinal and transverse.



The wave on a guitar string is transverse. The sound wave rattles a sheet of paper in a direction that shows the sound wave is longitudinal.

Earthquake waves under Earth's surface also have both longitudinal and transverse components (called compressional or P-waves and shear or S-waves, respectively). These components have important individual characteristics—they propagate at different speeds, for example.

Earthquakes also have surface waves that are similar to surface waves on water.

Note:

Applying the Science Practices: Electricity in Your Home

The source of electricity is of a sinusoidal nature. If we appropriately probe using an oscilloscope (an instrument used to display and analyze electronic signals), we can precisely determine the frequency and wavelength of the waveform. Inquire about the maximum voltage current that you get in your house and plot a sinusoidal waveform representing the frequency, wavelength, and period for it.

Exercise:

Check Your Understanding

Problem:

Why is it important to differentiate between longitudinal and transverse waves?

Solution:

In the different types of waves, energy can propagate in a different direction relative to the motion of the wave. This is important to understand how different types of waves affect the materials around them.

Note:

PhET Explorations: Wave on a String

Watch a string vibrate in slow motion. Wiggle the end of the string and make waves, or adjust the frequency and amplitude of an oscillator. Adjust the damping and tension. The end can be fixed, loose, or open.

[Wave](#)
[on a](#)
[String](#)
[g](#)

Test Prep for AP Courses

Exercise:

Problem:

What is the difference between the waves coming from a tuning fork and electromagnetic waves?

Solution:

The waves coming from a tuning fork are mechanical waves that are longitudinal in nature, whereas electromagnetic waves are transverse in nature.

Exercise:

Problem:

Represent longitudinal and transverse waves in a graphical form.

Exercise:

Problem:

Why is the sound produced by a tambourine different from that produced by drums?

Solution:

The sound energy coming out of an instrument depends on its size. The sound waves produced are relative to the size of the musical instrument. A smaller instrument such as a tambourine will produce a high-pitched sound (higher frequency, shorter wavelength), whereas a larger instrument such as a drum will produce a deeper sound (lower frequency, longer wavelength).

Exercise:

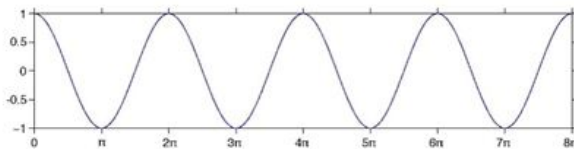
Problem:

A transverse wave is traveling left to right. Which of the following is correct about the motion of particles in the wave?

- a. The particles move up and down when the wave travels in a vacuum.
- b. The particles move left and right when the wave travels in a medium.
- c. The particles move up and down when the wave travels in a medium.
- d. The particles move right and left when the wave travels in a vacuum.

Exercise:

Problem:



The graph shows propagation of a mechanical wave. What is the wavelength of this wave?

Solution:

2π m

Section Summary

- A wave is a disturbance that moves from the point of creation with a wave velocity v_w .
- A wave has a wavelength λ , which is the distance between adjacent identical parts of the wave.
- Wave velocity and wavelength are related to the wave's frequency and period by $v_w = \frac{\lambda}{T}$ or $v_w = f\lambda$.
- A transverse wave has a disturbance perpendicular to its direction of propagation, whereas a longitudinal wave has a disturbance parallel to its direction of propagation.

Conceptual Questions

Exercise:

Problem:

Give one example of a transverse wave and another of a longitudinal wave, being careful to note the relative directions of the disturbance and wave propagation in each.

Exercise:

Problem:

What is the difference between propagation speed and the frequency of a wave? Does one or both affect wavelength? If so, how?

Problems & Exercises

Exercise:

Problem:

Storms in the South Pacific can create waves that travel all the way to the California coast, which are 12,000 km away. How long does it take them if they travel at 15.0 m/s?

Solution:

$$t = 9.26 \text{ d}$$

Exercise:**Problem:**

Waves on a swimming pool propagate at 0.750 m/s. You splash the water at one end of the pool and observe the wave go to the opposite end, reflect, and return in 30.0 s. How far away is the other end of the pool?

Exercise:**Problem:**

Wind gusts create ripples on the ocean that have a wavelength of 5.00 cm and propagate at 2.00 m/s. What is their frequency?

Solution:

$$f = 40.0 \text{ Hz}$$

Exercise:**Problem:**

How many times a minute does a boat bob up and down on ocean waves that have a wavelength of 40.0 m and a propagation speed of 5.00 m/s?

Exercise:**Problem:**

Scouts at a camp shake the rope bridge they have just crossed and observe the wave crests to be 8.00 m apart. If they shake it the bridge twice per second, what is the propagation speed of the waves?

Solution:

$$v_w = 16.0 \text{ m/s}$$

Exercise:

Problem:

What is the wavelength of the waves you create in a swimming pool if you splash your hand at a rate of 2.00 Hz and the waves propagate at 0.800 m/s?

Exercise:

Problem:

What is the wavelength of an earthquake that shakes you with a frequency of 10.0 Hz and gets to another city 84.0 km away in 12.0 s?

Solution:

$$\lambda = 700 \text{ m}$$

Exercise:

Problem:

Radio waves transmitted through space at $3.00 \times 10^8 \text{ m/s}$ by the Voyager spacecraft have a wavelength of 0.120 m. What is their frequency?

Exercise:

Problem:

Your ear is capable of differentiating sounds that arrive at the ear just 1.00 ms apart. What is the minimum distance between two speakers that produce sounds that arrive at noticeably different times on a day when the speed of sound is 340 m/s?

Solution:

$$d = 34.0 \text{ cm}$$

Exercise:

Problem:

(a) Seismographs measure the arrival times of earthquakes with a precision of 0.100 s. To get the distance to the epicenter of the quake, they compare the arrival times of S- and P-waves, which travel at different speeds. [\[link\]](#)) If S- and P-waves travel at 4.00 and 7.20 km/s, respectively, in the region considered, how precisely can the distance to the source of the earthquake be determined? (b) Seismic waves from underground detonations of nuclear bombs can be used to locate the test site and detect violations of test bans. Discuss whether your answer to (a) implies a serious limit to such detection. (Note also that the uncertainty is greater if there is an uncertainty in the propagation speeds of the S- and P-waves.)



A seismograph as described in above problem.(credit: Oleg Alexandrov)

Glossary

longitudinal wave

a wave in which the disturbance is parallel to the direction of propagation

transverse wave

a wave in which the disturbance is perpendicular to the direction of propagation

wave velocity

the speed at which the disturbance moves. Also called the propagation velocity or propagation speed

wavelength

the distance between adjacent identical parts of a wave

Superposition and Interference

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Determine the resultant waveform when two waves act in superposition relative to each other.
- Explain standing waves.
- Describe the mathematical representation of overtones and beat frequency.



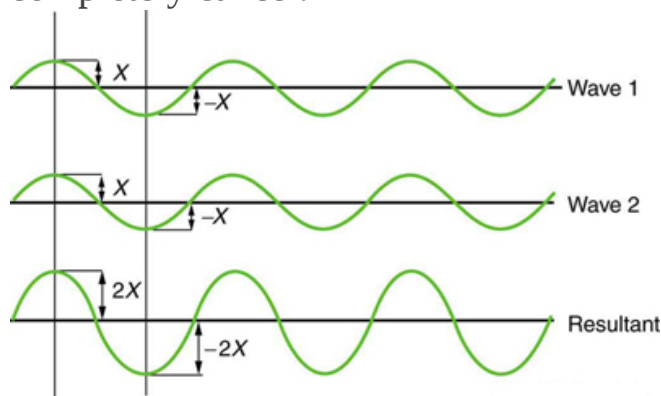
These waves result from the superposition of several waves from different sources, producing a complex pattern.
(credit: waterborough, Wikimedia Commons)

Most waves do not look very simple. They look more like the waves in [\[link\]](#) than like the simple water wave considered in [Waves](#). (Simple waves may be created by a simple harmonic oscillation, and thus have a sinusoidal shape). Complex waves are more interesting, even beautiful, but they look formidable. Most waves appear complex because they result from several simple waves adding together. Luckily, the rules for adding waves are quite simple.

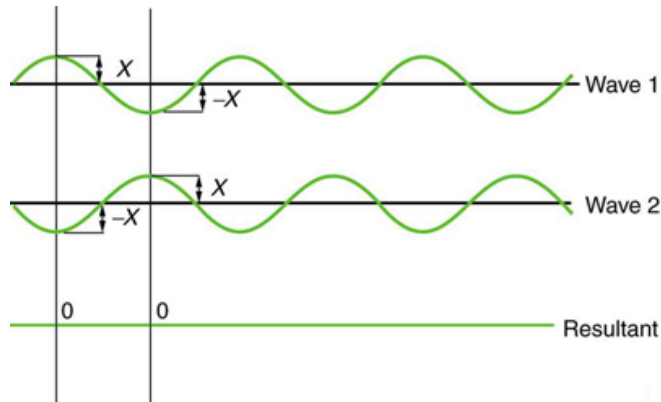
When two or more waves arrive at the same point, they superimpose themselves on one another. More specifically, the disturbances of waves are superimposed when they come together—a phenomenon called **superposition**. Each disturbance corresponds to a force, and forces add. If the disturbances are along the same line, then the resulting wave is a simple addition of the disturbances of the individual waves—that is, their amplitudes add. [\[link\]](#) and [\[link\]](#) illustrate superposition in two special cases, both of which produce simple results.

[\[link\]](#) shows two identical waves that arrive at the same point exactly in phase. The crests of the two waves are precisely aligned, as are the troughs. This superposition produces pure **constructive interference**. Because the disturbances add, pure constructive interference produces a wave that has twice the amplitude of the individual waves, but has the same wavelength.

[\[link\]](#) shows two identical waves that arrive exactly out of phase—that is, precisely aligned crest to trough—producing pure **destructive interference**. Because the disturbances are in the opposite direction for this superposition, the resulting amplitude is zero for pure destructive interference—the waves completely cancel.



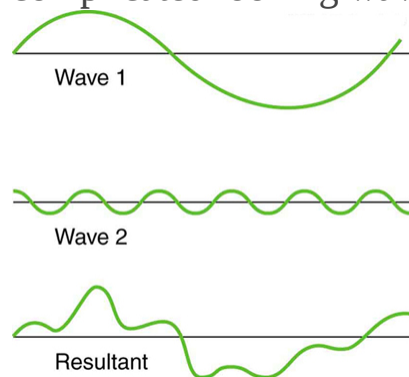
Pure constructive interference of two identical waves produces one with twice the amplitude, but the same wavelength.



Pure destructive interference of two identical waves produces zero amplitude, or complete cancellation.

While pure constructive and pure destructive interference do occur, they require precisely aligned identical waves. The superposition of most waves produces a combination of constructive and destructive interference and can vary from place to place and time to time. Sound from a stereo, for example, can be loud in one spot and quiet in another. Varying loudness means the sound waves add partially constructively and partially destructively at different locations. A stereo has at least two speakers creating sound waves, and waves can reflect from walls. All these waves superimpose. An example of sounds that vary over time from constructive to destructive is found in the combined whine of airplane jets heard by a stationary passenger. The combined sound can fluctuate up and down in volume as the sound from the two engines varies in time from constructive to destructive. These examples are of waves that are similar.

An example of the superposition of two dissimilar waves is shown in [\[link\]](#). Here again, the disturbances add and subtract, producing a more complicated looking wave.



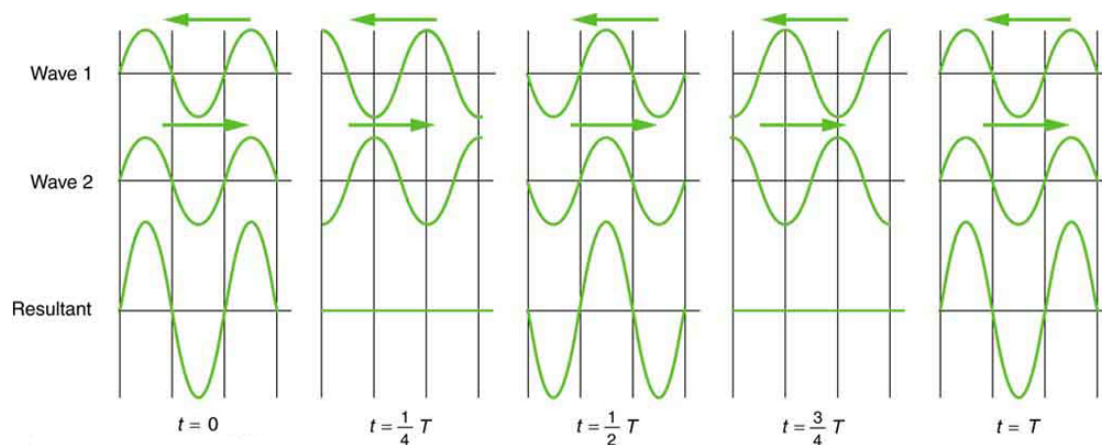
Superposition of
non-identical
waves exhibits both
constructive and
destructive
interference.

Standing Waves

Sometimes waves do not seem to move; rather, they just vibrate in place. Unmoving waves can be seen on the surface of a glass of milk in a refrigerator, for example. Vibrations from the refrigerator motor create waves on the milk that oscillate up and down but do not seem to move across the surface. These waves are formed by the superposition of two or more moving waves, such as illustrated in [\[link\]](#) for two identical waves moving in opposite directions. The waves move through each other with their disturbances adding as they go by. If the two waves have the same amplitude and wavelength, then they alternate between constructive and destructive interference. The resultant looks like a wave standing in place and, thus, is called a **standing wave**. Waves on the glass of milk are one example of standing waves. There are other standing waves, such as on guitar strings and in organ pipes. With the glass of milk, the two waves that

produce standing waves may come from reflections from the side of the glass.

A closer look at earthquakes provides evidence for conditions appropriate for resonance, standing waves, and constructive and destructive interference. A building may be vibrated for several seconds with a driving frequency matching that of the natural frequency of vibration of the building—producing a resonance resulting in one building collapsing while neighboring buildings do not. Often buildings of a certain height are devastated while other taller buildings remain intact. The building height matches the condition for setting up a standing wave for that particular height. As the earthquake waves travel along the surface of Earth and reflect off denser rocks, constructive interference occurs at certain points. Often areas closer to the epicenter are not damaged while areas farther away are damaged.

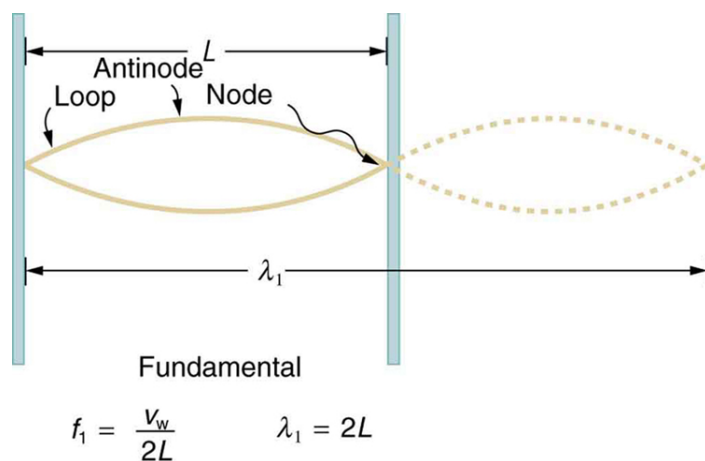


Standing wave created by the superposition of two identical waves moving in opposite directions. The oscillations are at fixed locations in space and result from alternately constructive and destructive interference.

Standing waves are also found on the strings of musical instruments and are due to reflections of waves from the ends of the string. [\[link\]](#) and [\[link\]](#)

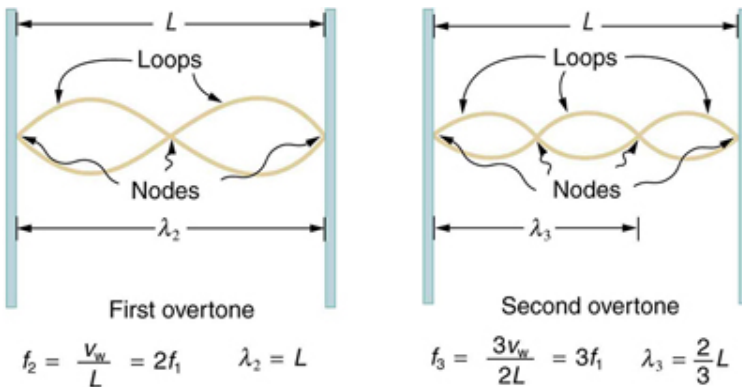
show three standing waves that can be created on a string that is fixed at both ends. **Nodes** are the points where the string does not move; more generally, nodes are where the wave disturbance is zero in a standing wave. The fixed ends of strings must be nodes, too, because the string cannot move there. The word **antinode** is used to denote the location of maximum amplitude in standing waves. Standing waves on strings have a frequency that is related to the propagation speed v_w of the disturbance on the string. The wavelength λ is determined by the distance between the points where the string is fixed in place.

The lowest frequency, called the **fundamental frequency**, is thus for the longest wavelength, which is seen to be $\lambda_1 = 2L$. Therefore, the fundamental frequency is $f_1 = v_w/\lambda_1 = v_w/2L$. In this case, the **overtone**s or harmonics are multiples of the fundamental frequency. As seen in [\[link\]](#), the first harmonic can easily be calculated since $\lambda_2 = L$. Thus, $f_2 = v_w/\lambda_2 = v_w/L = 2f_1$. Similarly, $f_3 = 3f_1$, and so on. All of these frequencies can be changed by adjusting the tension in the string. The greater the tension, the greater v_w is and the higher the frequencies. This observation is familiar to anyone who has ever observed a string instrument being tuned. We will see in later chapters that standing waves are crucial to many resonance phenomena, such as in sounding boxes on string instruments.



The figure shows a string oscillating

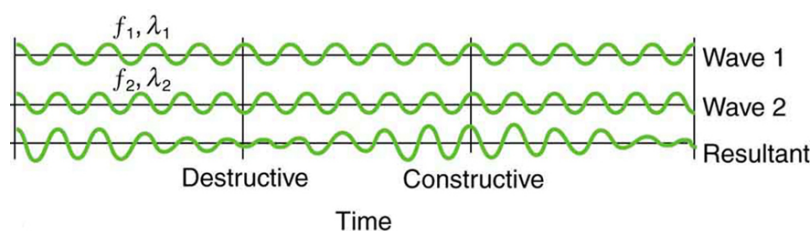
at its fundamental frequency.



First and second harmonic frequencies
are shown.

Beats

Striking two adjacent keys on a piano produces a warbling combination usually considered to be unpleasant. The superposition of two waves of similar but not identical frequencies is the culprit. Another example is often noticeable in jet aircraft, particularly the two-engine variety, while taxiing. The combined sound of the engines goes up and down in loudness. This varying loudness happens because the sound waves have similar but not identical frequencies. The discordant warbling of the piano and the fluctuating loudness of the jet engine noise are both due to alternately constructive and destructive interference as the two waves go in and out of phase. [\[link\]](#) illustrates this graphically.



Beats are produced by the superposition of two waves of slightly different frequencies but identical amplitudes. The waves alternate in time between constructive interference and destructive interference, giving the resulting wave a time-varying amplitude.

The wave resulting from the superposition of two similar-frequency waves has a frequency that is the average of the two. This wave fluctuates in amplitude, or *beats*, with a frequency called the **beat frequency**. We can determine the beat frequency by adding two waves together mathematically. Note that a wave can be represented at one point in space as

Equation:

$$x = X \cos\left(\frac{2\pi t}{T}\right) = X \cos(2\pi ft),$$

where $f = 1/T$ is the frequency of the wave. Adding two waves that have different frequencies but identical amplitudes produces a resultant

Equation:

$$x = x_1 + x_2.$$

More specifically,

Equation:

$$x = X \cos(2\pi f_1 t) + X \cos(2\pi f_2 t).$$

Using a trigonometric identity, it can be shown that

Equation:

$$x = 2X \cos(\pi f_B t) \cos(2\pi f_{\text{ave}} t),$$

where

Equation:

$$f_B = |f_1 - f_2|$$

is the beat frequency, and f_{ave} is the average of f_1 and f_2 . These results mean that the resultant wave has twice the amplitude and the average frequency of the two superimposed waves, but it also fluctuates in overall amplitude at the beat frequency f_B . The first cosine term in the expression effectively causes the amplitude to go up and down. The second cosine term is the wave with frequency f_{ave} . This result is valid for all types of waves. However, if it is a sound wave, providing the two frequencies are similar, then what we hear is an average frequency that gets louder and softer (or warbles) at the beat frequency.

Note:

Real World Connections: Tuning Forks

The [MIT physics demo](#) entitled “Tuning Forks: Resonance and Beat Frequency” provides a qualitative picture of how wave interference produces beats.

Description: Two identical forks and sounding boxes are placed next to each other. Striking one tuning fork will cause the other to resonate at the same frequency. When a weight is attached to one tuning fork, they are no longer identical. Thus, one will not cause the other to resonate. When two different forks are struck at the same time, the interference of their pitches produces beats.

Note:**Real World Connections: Jump Rope**

This is a fun activity with which to learn about interference and superposition. Take a jump rope and hold it at the two ends with one of your friends. While each of you is holding the rope, snap your hands to produce a wave from each side. Record your observations and see if they match with the following:

- a. One wave starts from the right end and travels to the left end of the rope.
- b. Another wave starts at the left end and travels to the right end of the rope.
- c. The waves travel at the same speed.
- d. The shape of the waves depends on the way the person snaps his or her hands.
- e. There is a region of overlap.
- f. The shapes of the waves are identical to their original shapes after they overlap.

Now, snap the rope up and down and ask your friend to snap his or her end of the rope sideways. The resultant that one sees here is the vector sum of two individual displacements.

This activity illustrates superposition and interference. When two or more waves interact with each other at a point, the disturbance at that point is given by the sum of the disturbances each wave will produce in the absence of the other. This is the principle of superposition. Interference is a result of superposition of two or more waves to form a resultant wave of greater or lower amplitude.

While beats may sometimes be annoying in audible sounds, we will find that beats have many applications. Observing beats is a very useful way to compare similar frequencies. There are applications of beats as apparently disparate as in ultrasonic imaging and radar speed traps.

Exercise:**Check Your Understanding**

Problem:

Imagine you are holding one end of a jump rope, and your friend holds the other. If your friend holds her end still, you can move your end up and down, creating a transverse wave. If your friend then begins to move her end up and down, generating a wave in the opposite direction, what resultant wave forms would you expect to see in the jump rope?

Solution:

The rope would alternate between having waves with amplitudes two times the original amplitude and reaching equilibrium with no amplitude at all. The wavelengths will result in both constructive and destructive interference

Exercise:**Check Your Understanding**

Problem: Define nodes and antinodes.

Solution:

Nodes are areas of wave interference where there is no motion. Antinodes are areas of wave interference where the motion is at its maximum point.

Exercise:**Check Your Understanding****Problem:**

You hook up a stereo system. When you test the system, you notice that in one corner of the room, the sounds seem dull. In another area, the sounds seem excessively loud. Describe how the sound moving about the room could result in these effects.

Solution:

With multiple speakers putting out sounds into the room, and these sounds bouncing off walls, there is bound to be some wave interference. In the dull areas, the interference is probably mostly destructive. In the louder areas, the interference is probably mostly constructive.

Note:

PhET Explorations: Wave Interference

Make waves with a dripping faucet, audio speaker, or laser! Add a second source or a pair of slits to create an interference pattern.

[Wave
Interference](#)

Test Prep for AP Courses

Exercise:

Problem:

A guitar string has a number of frequencies at which it vibrates naturally. Which of the following is true in this context?

- a. The resonant frequencies of the string are integer multiples of fundamental frequencies.
- b. The resonant frequencies of the string are not integer multiples of fundamental frequencies.
- c. They have harmonic overtones.
- d. None of the above

Exercise:

Problem:

Explain the principle of superposition with figures that show the changes in the wave amplitude.

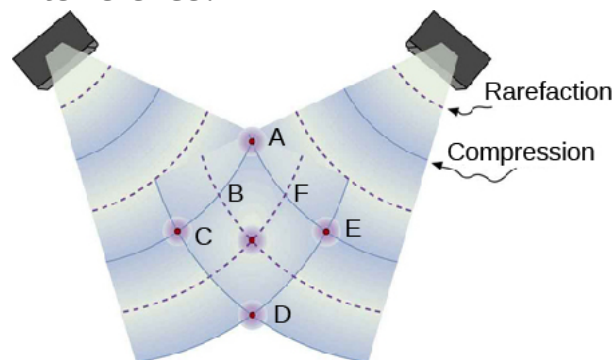
Solution:

The student explains the principle of superposition and then shows two waves adding up to form a bigger wave when a crest adds with a crest and a trough with another trough. Also the student shows a wave getting cancelled out when a crest meets a trough and vice versa.

Exercise:

Problem:

In this figure which points represent the points of constructive interference?



- a. A, B, F
- b. A, B, C, D, E, F
- c. A, C, D, E
- d. A, B, D

Exercise:

Problem:

A string is fixed on both sides. It is snapped from both ends at the same time by applying an equal force. What happens to the shape of the waves generated in the string? Also, will you observe an overlap of waves?

Solution:

The student must note that the shape of the wave remains the same and there is first an overlap and then receding of the waves.

Exercise:**Problem:**

In the preceding question, what would happen to the amplitude of the waves generated in this way? Also, consider another scenario where the string is snapped up from one end and down from the other end. What will happen in this situation?

Exercise:**Problem:**

Two sine waves travel in the same direction in a medium. The amplitude of each wave is A , and the phase difference between the two is 180° . What is the resultant amplitude?

- a. $2A$
- b. $3A$
- c. 0
- d. $9A$

Solution:

(c)

Exercise:

Problem:

Standing wave patterns consist of nodes and antinodes formed by repeated interference between two waves of the same frequency traveling in opposite directions. What are nodes and antinodes and how are they produced?

Section Summary

- Superposition is the combination of two waves at the same location.
- Constructive interference occurs when two identical waves are superimposed in phase.
- Destructive interference occurs when two identical waves are superimposed exactly out of phase.
- A standing wave is one in which two waves superimpose to produce a wave that varies in amplitude but does not propagate.
- Nodes are points of no motion in standing waves.
- An antinode is the location of maximum amplitude of a standing wave.
- Waves on a string are resonant standing waves with a fundamental frequency and can occur at higher multiples of the fundamental, called overtones or harmonics.
- Beats occur when waves of similar frequencies f_1 and f_2 are superimposed. The resulting amplitude oscillates with a beat frequency given by

Equation:

$$f_B = |f_1 - f_2|.$$

Conceptual Questions

Exercise:

Problem:

Speakers in stereo systems have two color-coded terminals to indicate how to hook up the wires. If the wires are reversed, the speaker moves in a direction opposite that of a properly connected speaker. Explain why it is important to have both speakers connected the same way.

Problems & Exercises**Exercise:****Problem:**

A car has two horns, one emitting a frequency of 199 Hz and the other emitting a frequency of 203 Hz. What beat frequency do they produce?

Solution:

$$f = 4 \text{ Hz}$$

Exercise:**Problem:**

The middle-C hammer of a piano hits two strings, producing beats of 1.50 Hz. One of the strings is tuned to 260.00 Hz. What frequencies could the other string have?

Exercise:**Problem:**

Two tuning forks having frequencies of 460 and 464 Hz are struck simultaneously. What average frequency will you hear, and what will the beat frequency be?

Solution:

462 Hz,

4 Hz

Exercise:

Problem:

Twin jet engines on an airplane are producing an average sound frequency of 4100 Hz with a beat frequency of 0.500 Hz. What are their individual frequencies?

Exercise:

Problem:

A wave traveling on a Slinky® that is stretched to 4 m takes 2.4 s to travel the length of the Slinky and back again. (a) What is the speed of the wave? (b) Using the same Slinky stretched to the same length, a standing wave is created which consists of three antinodes and four nodes. At what frequency must the Slinky be oscillating?

Solution:

(a) 3.33 m/s

(b) 1.25 Hz

Exercise:

Problem:

Three adjacent keys on a piano (F, F-sharp, and G) are struck simultaneously, producing frequencies of 349, 370, and 392 Hz. What beat frequencies are produced by this discordant combination?

Glossary

antinode

the location of maximum amplitude in standing waves

beat frequency

the frequency of the amplitude fluctuations of a wave

constructive interference

when two waves arrive at the same point exactly in phase; that is, the crests of the two waves are precisely aligned, as are the troughs

destructive interference

when two identical waves arrive at the same point exactly out of phase; that is, precisely aligned crest to trough

fundamental frequency

the lowest frequency of a periodic waveform

nodes

the points where the string does not move; more generally, nodes are where the wave disturbance is zero in a standing wave

overtones

multiples of the fundamental frequency of a sound

superposition

the phenomenon that occurs when two or more waves arrive at the same point

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Calculate the intensity and the power of rays and waves.



The destructive effect of an earthquake is palpable evidence of the energy carried in these waves. The Richter scale rating of earthquakes is related to both their amplitude and the energy they carry.

(credit: Petty Officer 2nd Class Candice Villarreal, U.S. Navy)

All waves carry energy. The energy of some waves can be directly observed. Earthquakes can shake whole cities to the ground, performing the work of thousands of wrecking balls.

Loud sounds pulverize nerve cells in the inner ear, causing permanent hearing loss. Ultrasound is used for deep-heat treatment of muscle strains. A laser beam can burn away a malignancy. Water waves chew up beaches.

The amount of energy in a wave is related to its amplitude. Large-amplitude earthquakes produce large ground displacements. Loud sounds have higher pressure amplitudes and come from larger-amplitude source vibrations than soft sounds. Large ocean breakers churn up the shore more than small ones. More quantitatively, a wave is a displacement that is resisted by a restoring force. The larger the displacement x , the larger the force $F = kx$ needed to create it. Because work W is related to force multiplied by distance (Fx) and energy is put into the wave by the work done to create it, the energy in a wave is related to amplitude. In fact, a wave's energy is directly proportional to its amplitude squared because

Equation:

$$W \propto Fx = kx^2.$$

The energy effects of a wave depend on time as well as amplitude. For example, the longer deep-heat ultrasound is applied, the more energy it transfers. Waves can also be concentrated or spread out. Sunlight, for example, can be focused to burn wood. Earthquakes spread out, so they do less damage the farther they get from the source. In both cases, changing the area the waves cover has important effects. All these pertinent factors are included in the definition of **intensity** I as power per unit area:

Equation:

$$I = \frac{P}{A}$$

where P is the power carried by the wave through area A . The definition of intensity is valid for any energy in transit, including that carried by waves. The SI unit for intensity is watts per square meter (W/m^2). For example, infrared and visible energy from the Sun impinge on Earth at an intensity of $1300 \text{ W}/\text{m}^2$ just above the atmosphere. There are other intensity-related units in use, too. The most common is the decibel. For example, a 90 decibel sound level corresponds to an intensity of $10^{-3} \text{ W}/\text{m}^2$. (This quantity is not much power per unit area considering that 90 decibels is a relatively high sound level. Decibels will be discussed in some detail in a later chapter.

Example:

Calculating intensity and power: How much energy is in a ray of sunlight?

The average intensity of sunlight on Earth's surface is about $700 \text{ W}/\text{m}^2$.

(a) Calculate the amount of energy that falls on a solar collector having an area of 0.500 m^2 in 4.00 h.

(b) What intensity would such sunlight have if concentrated by a magnifying glass onto an area 200 times smaller than its own?

Strategy a

Because power is energy per unit time or $P = \frac{E}{t}$, the definition of intensity can be written as $I = \frac{P}{A} = \frac{E/t}{A}$, and this equation can be solved for E with the given information.

Solution a

1. Begin with the equation that states the definition of intensity:

Equation:

$$I = \frac{P}{A}.$$

2. Replace P with its equivalent E/t :

Equation:

$$I = \frac{E/t}{A}.$$

3. Solve for E :

Equation:

$$E = IAt.$$

4. Substitute known values into the equation:

Equation:

$$E = (700 \text{ W/m}^2)(0.500 \text{ m}^2)[(4.00 \text{ h})(3600 \text{ s/h})].$$

5. Calculate to find E and convert units:

Equation:

$$5.04 \times 10^6 \text{ J},$$

Discussion a

The energy falling on the solar collector in 4 h in part is enough to be useful—for example, for heating a significant amount of water.

Strategy b

Taking a ratio of new intensity to old intensity and using primes for the new quantities, we will find that it depends on the ratio of the areas. All other quantities will cancel.

Solution b

1. Take the ratio of intensities, which yields:

Equation:

$$\frac{I'}{I} = \frac{P'/A'}{P/A} = \frac{A}{A'} \left(\text{The powers cancel because } P' = P \right).$$

2. Identify the knowns:

Equation:

$$A = 200A',$$

Equation:

$$\frac{I'}{I} = 200.$$

3. Substitute known quantities:

Equation:

$$I' = 200I = 200(700 \text{ W/m}^2).$$

4. Calculate to find I' :

Equation:

$$I' = 1.40 \times 10^5 \text{ W/m}^2.$$

Discussion b

Decreasing the area increases the intensity considerably. The intensity of the concentrated sunlight could even start a fire.

Example:

Determine the combined intensity of two waves: Perfect constructive interference

If two identical waves, each having an intensity of 1.00 W/m^2 , interfere perfectly constructively, what is the intensity of the resulting wave?

Strategy

We know from [Superposition and Interference](#) that when two identical waves, which have equal amplitudes X , interfere perfectly constructively, the resulting wave has an amplitude of $2X$. Because a wave's intensity is proportional to amplitude squared, the intensity of the resulting wave is four times as great as in the individual waves.

Solution

1. Recall that intensity is proportional to amplitude squared.
2. Calculate the new amplitude:

Equation:

$$I' \propto (X')^2 = (2X)^2 = 4X^2.$$

3. Recall that the intensity of the old amplitude was:

Equation:

$$I \propto X^2.$$

4. Take the ratio of new intensity to the old intensity. This gives:

Equation:

$$\frac{I'}{I} = 4.$$

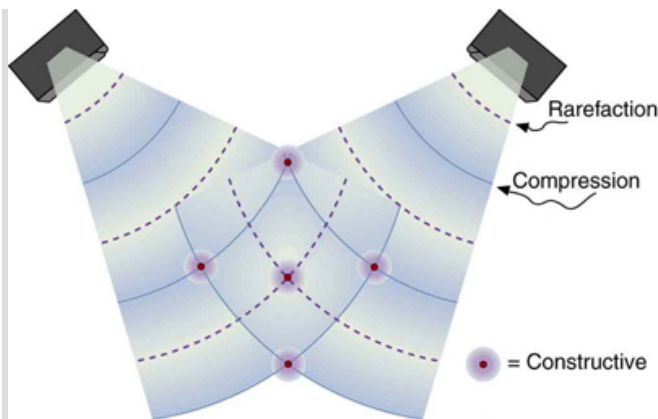
5. Calculate to find I' :

Equation:

$$I' = 4I = 4.00 \text{ W/m}^2.$$

Discussion

The intensity goes up by a factor of 4 when the amplitude doubles. This answer is a little disquieting. The two individual waves each have intensities of 1.00 W/m^2 , yet their sum has an intensity of 4.00 W/m^2 , which may appear to violate conservation of energy. This violation, of course, cannot happen. What does happen is intriguing. The area over which the intensity is 4.00 W/m^2 is much less than the area covered by the two waves before they interfered. There are other areas where the intensity is zero. The addition of waves is not as simple as our first look in [Superposition and Interference](#) suggested. We actually get a pattern of both constructive interference and destructive interference whenever two waves are added. For example, if we have two stereo speakers putting out 1.00 W/m^2 each, there will be places in the room where the intensity is 4.00 W/m^2 , other places where the intensity is zero, and others in between. [\[link\]](#) shows what this interference might look like. We will pursue interference patterns elsewhere in this text.



These stereo speakers produce both constructive interference and destructive interference in the room, a property common to the superposition of all types of waves. The shading is proportional to intensity.

Exercise:
Check Your Understanding

Problem:

Which measurement of a wave is most important when determining the wave's intensity?

Solution:

Amplitude, because a wave's energy is directly proportional to its amplitude squared.

Section Summary

Intensity is defined to be the power per unit area:

$$I = \frac{P}{A} \text{ and has units of } \text{W}/\text{m}^2.$$

Conceptual Questions

Exercise:

Problem:

Two identical waves undergo pure constructive interference. Is the resultant intensity twice that of the individual waves? Explain your answer.

Exercise:

Problem:

Circular water waves decrease in amplitude as they move away from where a rock is dropped. Explain why.

Problems & Exercises

Exercise:

Problem: Medical Application

Ultrasound of intensity $1.50 \times 10^2 \text{ W}/\text{m}^2$ is produced by the rectangular head of a medical imaging device measuring 3.00 by 5.00 cm. What is its power output?

Solution:

0.225 W

Exercise:

Problem:

The low-frequency speaker of a stereo set has a surface area of 0.05 m^2 and produces 1 W of acoustical power. What is the intensity at the speaker? If the speaker projects sound uniformly in all directions, at what distance from the speaker is the intensity 0.1 W/m^2 ?

Exercise:**Problem:**

To increase intensity of a wave by a factor of 50, by what factor should the amplitude be increased?

Solution:

7.07

Exercise:**Problem: Engineering Application**

A device called an insolation meter is used to measure the intensity of sunlight has an area of 100 cm^2 and registers 6.50 W . What is the intensity in W/m^2 ?

Exercise:**Problem: Astronomy Application**

Energy from the Sun arrives at the top of the Earth's atmosphere with an intensity of 1.30 kW/m^2 . How long does it take for $1.8 \times 10^9 \text{ J}$ to arrive on an area of 1.00 m^2 ?

Solution:

16.0 d

Exercise:

Problem:

Suppose you have a device that extracts energy from ocean breakers in direct proportion to their intensity. If the device produces 10.0 kW of power on a day when the breakers are 1.20 m high, how much will it produce when they are 0.600 m high?

Solution:

2.50 kW

Exercise:**Problem: Engineering Application**

(a) A photovoltaic array of (solar cells) is 10.0% efficient in gathering solar energy and converting it to electricity. If the average intensity of sunlight on one day is 700 W/m^2 , what area should your array have to gather energy at the rate of 100 W? (b) What is the maximum cost of the array if it must pay for itself in two years of operation averaging 10.0 hours per day? Assume that it earns money at the rate of 9.00 ¢ per kilowatt-hour.

Exercise:**Problem:**

A microphone receiving a pure sound tone feeds an oscilloscope, producing a wave on its screen. If the sound intensity is originally $2.00 \times 10^{-5} \text{ W/m}^2$, but is turned up until the amplitude increases by 30.0%, what is the new intensity?

Solution:

$3.38 \times 10^{-5} \text{ W/m}^2$

Exercise:

Problem: Medical Application

(a) What is the intensity in W/m^2 of a laser beam used to burn away cancerous tissue that, when 90.0% absorbed, puts 500 J of energy into a circular spot 2.00 mm in diameter in 4.00 s? (b) Discuss how this intensity compares to the average intensity of sunlight (about $700 \text{ W}/\text{m}^2$) and the implications that would have if the laser beam entered your eye. Note how your answer depends on the time duration of the exposure.

Glossary

intensity

power per unit area

Connection for AP® Courses
class="introduction"

This tree fell
some time
ago. When it
fell, atoms in
the air were
disturbed.
Physicists
would call
this
disturbance
sound
whether
someone was
around to
hear it or not.
(credit: B.A.
Bowen
Photography
)



In this chapter, the concept of waves is specifically applied to the phenomena of sound. As such, Big Idea 6 continues to be supported, as sound waves carry energy and momentum from one location to another without the permanent transfer of mass. This energy is carried through vibrations caused by disturbances in air pressure (Enduring Understanding 6.A). As air pressure increases, amplitudes of vibration and energy transfer do as well. This idea (Enduring Understanding 6.A.4) explains why a very loud sound can break glass.

The chapter continues the fundamental analysis of waves addressed in Chapter 16. Sound waves are periodic, and can therefore be expressed as a function of position and time. Furthermore, sound waves are described by amplitude, frequency, wavelength, and speed (Enduring Understanding 6.B). The relationship between speed and frequency is analyzed further in [Section 17.4](#), as the frequency of sound depends upon the relative motion between the source and observer. This concept, known as the Doppler effect, supports Essential Knowledge 6.B.5.

Like all other waves, sound waves can overlap. When they do so, their interaction will produce an amplitude variation within the resultant wave. This amplitude can be determined by adding the displacement of the two

pulses, through a process called superposition. This process, covered in [Section 17.5](#), reinforces the content in Enduring Understanding 6.D.1.

In situations where the interfering waves are confined, such as on a fixed length of string or in a tube, standing waves can result. These waves are the result of interference between the incident and reflecting wave. Standing waves are described using nodes and antinodes, and their wavelengths are determined by the size of the region to which they are confined. This chapter's description of both standing waves and the concept of beats strongly support Enduring Understanding 6.D, as well as Essential Knowledge 6.D.1, 6.D.3, and 6.D.4.

The concepts in this chapter support:

Big Idea 6 Waves can transfer energy and momentum from one location to another without the permanent transfer of mass and serve as a mathematical model for the description of other phenomena.

Enduring Understanding 6.B A periodic wave is one that repeats as a function of both time and position and can be described by its amplitude, frequency, wavelength, speed, and energy.

Essential Knowledge 6.B.5 The observed frequency of a wave depends on the relative motion of the source and the observer. This is a qualitative measurement only.

Enduring Understanding 6.D Interference and superposition lead to standing waves and beats.

Essential Knowledge 6.D.1 Two or more wave pulses can interact in such a way as to produce amplitude variations in the resultant wave. When two pulses cross, they travel through each other; they do not bounce off each other. Where the pulses overlap, the resulting displacement can be determined by adding the displacements of the two pulses. This is called superposition.

Essential Knowledge 6.D.3 Standing waves are the result of the addition of incident and reflected waves that are confined to a region and have nodes

and antinodes. Examples should include waves on a fixed length of string, and sound waves in both closed and open tubes.

Essential Knowledge 6.D.4 The possible wavelengths of a standing wave are determined by the size of the region in which it is confined.

Sound

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define sound and hearing.
- Describe sound as a longitudinal wave.



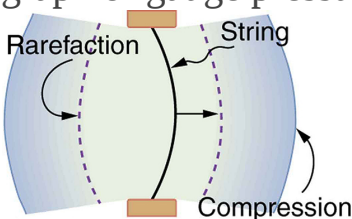
This glass has been shattered by a high-intensity sound wave of the same frequency as the resonant frequency of the glass. While the sound is not visible, the effects of the sound prove its existence.

(credit: ||read||,
Flickr)

Sound can be used as a familiar illustration of waves. Because hearing is one of our most important senses, it is interesting to see how the physical properties of sound correspond to our perceptions of it. **Hearing** is the perception of sound, just as vision is the perception of visible light. But sound has important applications beyond hearing. Ultrasound, for example, is not heard but can be employed to form medical images and is also used in treatment.

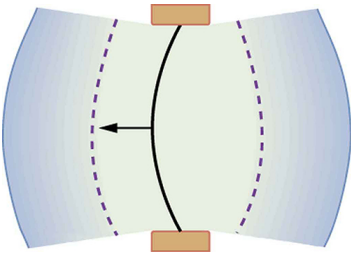
The physical phenomenon of **sound** is defined to be a disturbance of matter that is transmitted from its source outward. Sound is a wave. On the atomic scale, it is a disturbance of atoms that is far more ordered than their thermal motions. In many instances, sound is a periodic wave, and the atoms undergo simple harmonic motion. In this text, we shall explore such periodic sound waves.

A vibrating string produces a sound wave as illustrated in [\[link\]](#), [\[link\]](#), and [\[link\]](#). As the string oscillates back and forth, it transfers energy to the air, mostly as thermal energy created by turbulence. But a small part of the string's energy goes into compressing and expanding the surrounding air, creating slightly higher and lower local pressures. These compressions (high pressure regions) and rarefactions (low pressure regions) move out as longitudinal pressure waves having the same frequency as the string—they are the disturbance that is a sound wave. (Sound waves in air and most fluids are longitudinal, because fluids have almost no shear strength. In solids, sound waves can be both transverse and longitudinal.) [\[link\]](#) shows a graph of gauge pressure versus distance from the vibrating string.

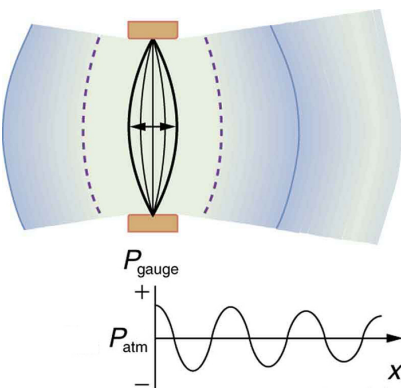


A vibrating
string moving to

the right
compresses the
air in front of it
and expands the
air behind it.



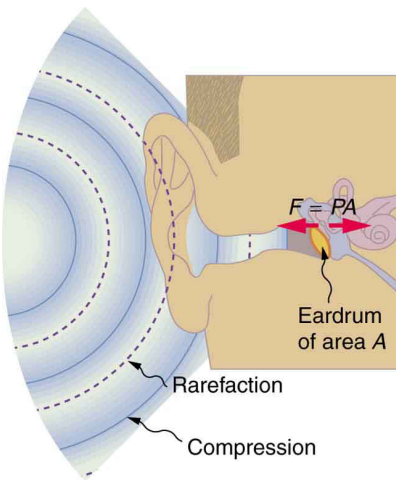
As the string
moves to the
left, it creates
another
compression and
rarefaction as
the ones on the
right move away
from the string.



After many
vibrations, there are
a series of

compressions and rarefactions moving out from the string as a sound wave. The graph shows gauge pressure versus distance from the source. Pressures vary only slightly from atmospheric for ordinary sounds.

The amplitude of a sound wave decreases with distance from its source, because the energy of the wave is spread over a larger and larger area. But it is also absorbed by objects, such as the eardrum in [\[link\]](#), and converted to thermal energy by the viscosity of air. In addition, during each compression a little heat transfers to the air and during each rarefaction even less heat transfers from the air, so that the heat transfer reduces the organized disturbance into random thermal motions. (These processes can be viewed as a manifestation of the second law of thermodynamics presented in [Introduction to the Second Law of Thermodynamics: Heat Engines and Their Efficiency](#).) Whether the heat transfer from compression to rarefaction is significant depends on how far apart they are—that is, it depends on wavelength. Wavelength, frequency, amplitude, and speed of propagation are important for sound, as they are for all waves.



Sound wave compressions and rarefactions travel up the ear canal and force the eardrum to vibrate. There is a net force on the eardrum, since the sound wave pressures differ from the atmospheric pressure found behind the eardrum. A complicated mechanism converts the vibrations to nerve impulses, which are perceived by the person.

Note:**PhET Explorations: Wave Interference**

Make waves with a dripping faucet, audio speaker, or laser! Add a second source or a pair of slits to create an interference pattern.



Make waves with a dripping faucet, audio speaker, or laser! Add a second source or a pair of slits to create an interference pattern.

[Wave Interference](#)

Section Summary

- Sound is a disturbance of matter that is transmitted from its source outward.
- Sound is one type of wave.
- Hearing is the perception of sound.

Glossary

sound

a disturbance of matter that is transmitted from its source outward

hearing

the perception of sound

Speed of Sound, Frequency, and Wavelength

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define pitch.
- Describe the relationship between the speed of sound, its frequency, and its wavelength.
- Describe the effects on the speed of sound as it travels through various media.
- Describe the effects of temperature on the speed of sound.

The information presented in this section supports the following AP® learning objectives and science practices:

- **6.B.4.1** The student is able to design an experiment to determine the relationship between periodic wave speed, wavelength, and frequency, and relate these concepts to everyday examples. (S.P. 4.2, 5.1, 7.2)



When a firework

explodes, the light
energy is perceived
before the sound
energy. Sound
travels more slowly
than light does.
(credit: Dominic
Alves, Flickr)

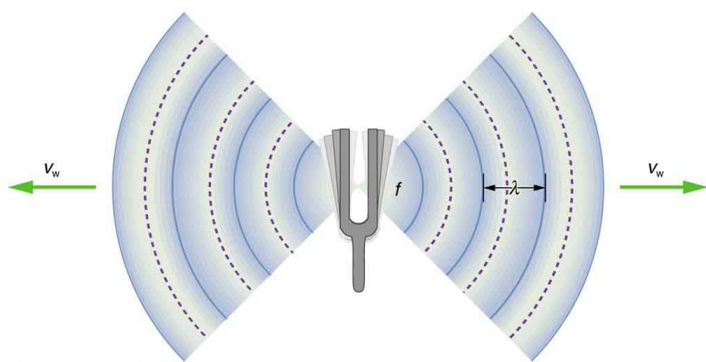
Sound, like all waves, travels at a certain speed and has the properties of frequency and wavelength. You can observe direct evidence of the speed of sound while watching a fireworks display. The flash of an explosion is seen well before its sound is heard, implying both that sound travels at a finite speed and that it is much slower than light. You can also directly sense the frequency of a sound. Perception of frequency is called **pitch**. The wavelength of sound is not directly sensed, but indirect evidence is found in the correlation of the size of musical instruments with their pitch. Small instruments, such as a piccolo, typically make high-pitch sounds, while large instruments, such as a tuba, typically make low-pitch sounds. High pitch means small wavelength, and the size of a musical instrument is directly related to the wavelengths of sound it produces. So a small instrument creates short-wavelength sounds. Similar arguments hold that a large instrument creates long-wavelength sounds.

The relationship of the speed of sound, its frequency, and wavelength is the same as for all waves:

Equation:

$$v_w = f\lambda,$$

where v_w is the speed of sound, f is its frequency, and λ is its wavelength. The wavelength of a sound is the distance between adjacent identical parts of a wave—for example, between adjacent compressions as illustrated in [\[link\]](#). The frequency is the same as that of the source and is the number of waves that pass a point per unit time.



A sound wave emanates from a source vibrating at a frequency f , propagates at v_w , and has a wavelength λ .

[\[link\]](#) makes it apparent that the speed of sound varies greatly in different media. The speed of sound in a medium is determined by a combination of the medium's rigidity (or compressibility in gases) and its density. The more rigid (or less compressible) the medium, the faster the speed of sound. This observation is analogous to the fact that the frequency of a simple harmonic motion is directly proportional to the stiffness of the oscillating object. The greater the density of a medium, the slower the speed of sound. This observation is analogous to the fact that the frequency of a simple harmonic motion is inversely proportional to the mass of the oscillating object. The speed of sound in air is low, because air is compressible. Because liquids and solids are relatively rigid and very difficult to compress, the speed of sound in such media is generally greater than in gases.

Note:**Applying the Science Practices: Bottle Music**

When liquid is poured into a small-necked container like a soda bottle, it can make for a fun musical experience! Find a small-necked bottle and pour water into it. When you blow across the surface of the bottle, a musical pitch should be created. This pitch, which corresponds to the

resonant frequency of the air remaining in the bottle, can be determined using [\[link\]](#). Your task is to design an experiment and collect data to confirm this relationship between the frequency created by blowing into the bottle and the depth of air remaining.

1. Use the explanation above to design an experiment that will yield data on depth of air column and frequency of pitch. Use the data table below to record your data.

Depth of air column (λ)	Frequency of pitch generated (f)

2. Construct a graph using the information collected above. The graph should include all five data points and should display frequency on the dependent axis.
3. What type of relationship is displayed on your graph? (direct, inverse, quadratic, etc.)
4. Does your graph align with equation 17.1, given earlier in this section? Explain.

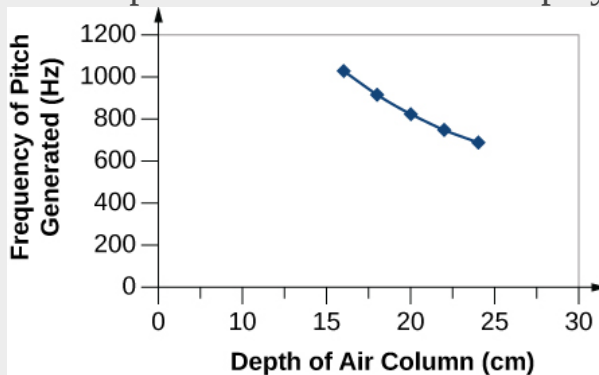
Note: For an explanation of why a frequency is created when you blow across a small-necked container, explore [Section 17.5](#) later in this chapter.

Note:
Answer

1. As the depth of the air column increases, the frequency values must decrease. A sample set of data is displayed below.

Depth of air column (λ)	Frequency of pitch generated (f)
24 cm	689.6 Hz
22 cm	752.3 Hz
20 cm	827.5 Hz
18 cm	919.4 Hz
16 cm	1034.4 Hz

2. The graph drawn should have frequency on the vertical axis, contain five data points, and trend downward and to the right. A graph using the sample data from above is displayed below.



A graph of the depth of air column versus the frequency of pitch generated.

3. Inverse relationship.

Depth of air column (λ)	Frequency of pitch generated (f)	Product of wavelength and frequency
24 cm	689.6 Hz	165.5
22 cm	752.3 Hz	165.5
20 cm	827.5 Hz	165.5
18 cm	919.4 Hz	165.5
16 cm	1034.4 Hz	165.5

4. The graph does align with the equation $v = f\lambda$. As the wavelength decreases, the frequency of the pitch generated increases. This relationship is validated by both the sample data table and the sample graph. Additionally, as [\[link\]](#) demonstrates, the product of λ and f is constant across all five data points.

In addition to these explanations, the student may use the formula as given in the problem statement to show that the product $f \times \text{air column height}$ is consistently 165.5.

Medium	$v_w(\text{m/s})$
<i>Gases at 0°C</i>	
Air	331
Carbon dioxide	259
Oxygen	316
Helium	965
Hydrogen	1290
<i>Liquids at 20°C</i>	
Ethanol	1160
Mercury	1450
Water, fresh	1480
Sea water	1540
Human tissue	1540
<i>Solids (longitudinal or bulk)</i>	
Vulcanized rubber	54
Polyethylene	920
Marble	3810
Glass, Pyrex	5640

Medium	$v_w(\text{m/s})$
Lead	1960
Aluminum	5120
Steel	5960

Speed of Sound in Various Media

Earthquakes, essentially sound waves in Earth's crust, are an interesting example of how the speed of sound depends on the rigidity of the medium. Earthquakes have both longitudinal and transverse components, and these travel at different speeds. The bulk modulus of granite is greater than its shear modulus. For that reason, the speed of longitudinal or pressure waves (P-waves) in earthquakes in granite is significantly higher than the speed of transverse or shear waves (S-waves). Both components of earthquakes travel slower in less rigid material, such as sediments. P-waves have speeds of 4 to 7 km/s, and S-waves correspondingly range in speed from 2 to 5 km/s, both being faster in more rigid material. The P-wave gets progressively farther ahead of the S-wave as they travel through Earth's crust. The time between the P- and S-waves is routinely used to determine the distance to their source, the epicenter of the earthquake.

The speed of sound is affected by temperature in a given medium. For air at sea level, the speed of sound is given by

Equation:

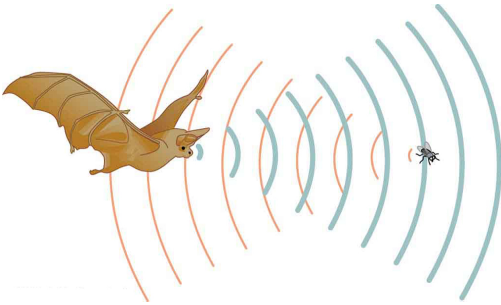
$$v_w = (331 \text{ m/s}) \sqrt{\frac{T}{273 \text{ K}}},$$

where the temperature (denoted as T) is in units of kelvin. The speed of sound in gases is related to the average speed of particles in the gas, v_{rms} , and that

Equation:

$$v_{\text{rms}} = \sqrt{\frac{3 kT}{m}},$$

where k is the Boltzmann constant (1.38×10^{-23} J/K) and m is the mass of each (identical) particle in the gas. So, it is reasonable that the speed of sound in air and other gases should depend on the square root of temperature. While not negligible, this is not a strong dependence. At 0°C , the speed of sound is 331 m/s, whereas at 20.0°C it is 343 m/s, less than a 4% increase. [\[link\]](#) shows a use of the speed of sound by a bat to sense distances. Echoes are also used in medical imaging.



A bat uses sound echoes to find its way about and to catch prey. The time for the echo to return is directly proportional to the distance.

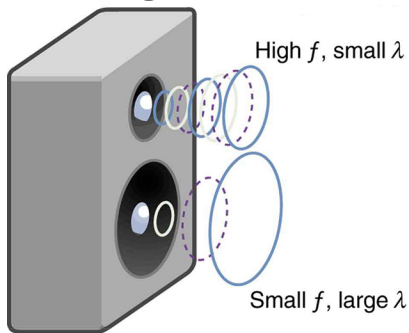
One of the more important properties of sound is that its speed is nearly independent of frequency. This independence is certainly true in open air for sounds in the audible range of 20 to 20,000 Hz. If this independence were not true, you would certainly notice it for music played by a marching band in a football stadium, for example. Suppose that high-frequency sounds traveled faster—then the farther you were from the band, the more the sound from the low-pitch instruments would lag that from the high-pitch ones. But the music from all instruments arrives in cadence independent of

distance, and so all frequencies must travel at nearly the same speed. Recall that

Equation:

$$v_w = f\lambda.$$

In a given medium under fixed conditions, v_w is constant, so that there is a relationship between f and λ ; the higher the frequency, the smaller the wavelength. See [\[link\]](#) and consider the following example.



Because they travel
at the same speed
in a given medium,
low-frequency
sounds must have a
greater wavelength
than high-
frequency sounds.

Here, the lower-
frequency sounds
are emitted by the
large speaker,
called a woofer,
while the higher-
frequency sounds
are emitted by the
small speaker,
called a tweeter.

Example:**Calculating Wavelengths: What Are the Wavelengths of Audible Sounds?**

Calculate the wavelengths of sounds at the extremes of the audible range, 20 and 20,000 Hz, in 30.0°C air. (Assume that the frequency values are accurate to two significant figures.)

Strategy

To find wavelength from frequency, we can use $v_w = f\lambda$.

Solution

1. Identify knowns. The value for v_w , is given by

Equation:

$$v_w = (331 \text{ m/s}) \sqrt{\frac{T}{273 \text{ K}}}.$$

2. Convert the temperature into kelvin and then enter the temperature into the equation

Equation:

$$v_w = (331 \text{ m/s}) \sqrt{\frac{303 \text{ K}}{273 \text{ K}}} = 348.7 \text{ m/s}.$$

3. Solve the relationship between speed and wavelength for λ :

Equation:

$$\lambda = \frac{v_w}{f}.$$

4. Enter the speed and the minimum frequency to give the maximum wavelength:

Equation:

$$\lambda_{\max} = \frac{348.7 \text{ m/s}}{20 \text{ Hz}} = 17 \text{ m.}$$

5. Enter the speed and the maximum frequency to give the minimum wavelength:

Equation:

$$\lambda_{\min} = \frac{348.7 \text{ m/s}}{20,000 \text{ Hz}} = 0.017 \text{ m} = 1.7 \text{ cm.}$$

Discussion

Because the product of f multiplied by λ equals a constant, the smaller f is, the larger λ must be, and vice versa.

The speed of sound can change when sound travels from one medium to another. However, the frequency usually remains the same because it is like a driven oscillation and has the frequency of the original source. If v_w changes and f remains the same, then the wavelength λ must change. That is, because $v_w = f\lambda$, the higher the speed of a sound, the greater its wavelength for a given frequency.

Note:

Making Connections: Take-Home Investigation—Voice as a Sound Wave
Suspend a sheet of paper so that the top edge of the paper is fixed and the bottom edge is free to move. You could tape the top edge of the paper to the edge of a table. Gently blow near the edge of the bottom of the sheet and note how the sheet moves. Speak softly and then louder such that the sounds hit the edge of the bottom of the paper, and note how the sheet moves. Explain the effects.

Exercise:

Check Your Understanding

Problem:

Imagine you observe two fireworks explode. You hear the explosion of one as soon as you see it. However, you see the other firework for several milliseconds before you hear the explosion. Explain why this is so.

Solution:

Sound and light both travel at definite speeds. The speed of sound is slower than the speed of light. The first firework is probably very close by, so the speed difference is not noticeable. The second firework is farther away, so the light arrives at your eyes noticeably sooner than the sound wave arrives at your ears.

Exercise:

Check Your Understanding

Problem:

You observe two musical instruments that you cannot identify. One plays high-pitch sounds and the other plays low-pitch sounds. How could you determine which is which without hearing either of them play?

Solution:

Compare their sizes. High-pitch instruments are generally smaller than low-pitch instruments because they generate a smaller wavelength.

Test Prep for AP Courses

Exercise:

Problem:

A teacher wants to demonstrate that the speed of sound is not a constant value. Considering her regular classroom voice as the control, which of the following will increase the speed of sound leaving her mouth?

- I. Submerge her mouth underwater and speak at the same volume.
 - II. Increase the temperature of the room and speak at the same volume.
 - III. Increase the pitch of her voice and speak at the same volume.
- a. I only
 - b. I and II only
 - c. I, II and III
 - d. II and III
 - e. III only

Solution:

(b)

Exercise:**Problem:**

All members of an orchestra begin tuning their instruments at the same time. While some woodwind instruments play high frequency notes, other stringed instruments play notes of lower frequency. Yet an audience member will hear all notes simultaneously, in apparent contrast to the equation.

Explain how a student could demonstrate the flaw in the above logic, using a slinky, stopwatch, and meter stick. Make sure to explain what relationship is truly demonstrated in the above equation, in addition to what would be necessary to get the speed of the slinky to actually change. You may include diagrams and equations as part of your explanation.

Section Summary

The relationship of the speed of sound v_w , its frequency f , and its wavelength λ is given by

Equation:

$$v_w = f\lambda,$$

which is the same relationship given for all waves.

In air, the speed of sound is related to air temperature T by

Equation:

$$v_w = (331 \text{ m/s}) \sqrt{\frac{T}{273 \text{ K}}}.$$

v_w is the same for all frequencies and wavelengths.

Conceptual Questions

Exercise:

Problem:

How do sound vibrations of atoms differ from thermal motion?

Exercise:

Problem:

When sound passes from one medium to another where its propagation speed is different, does its frequency or wavelength change? Explain your answer briefly.

Problems & Exercises

Exercise:**Problem:**

When poked by a spear, an operatic soprano lets out a 1200-Hz shriek. What is its wavelength if the speed of sound is 345 m/s?

Solution:

0.288 m

Exercise:**Problem:**

What frequency sound has a 0.10-m wavelength when the speed of sound is 340 m/s?

Exercise:**Problem:**

Calculate the speed of sound on a day when a 1500 Hz frequency has a wavelength of 0.221 m.

Solution:

332 m/s

Exercise:**Problem:**

(a) What is the speed of sound in a medium where a 100-kHz frequency produces a 5.96-cm wavelength? (b) Which substance in [\[link\]](#) is this likely to be?

Exercise:**Problem:**

Show that the speed of sound in 20.0°C air is 343 m/s, as claimed in the text.

Solution:

$$\begin{aligned}v_w &= (331 \text{ m/s})\sqrt{\frac{T}{273 \text{ K}}} = (331 \text{ m/s})\sqrt{\frac{293 \text{ K}}{273 \text{ K}}} \\&= 343 \text{ m/s}\end{aligned}$$

Exercise:**Problem:**

Air temperature in the Sahara Desert can reach 56.0°C (about 134°F). What is the speed of sound in air at that temperature?

Exercise:**Problem:**

Dolphins make sounds in air and water. What is the ratio of the wavelength of a sound in air to its wavelength in seawater? Assume air temperature is 20.0°C .

Solution:

0.223

Exercise:**Problem:**

A sonar echo returns to a submarine 1.20 s after being emitted. What is the distance to the object creating the echo? (Assume that the submarine is in the ocean, not in fresh water.)

Exercise:**Problem:**

(a) If a submarine's sonar can measure echo times with a precision of 0.0100 s , what is the smallest difference in distances it can detect? (Assume that the submarine is in the ocean, not in fresh water.)

(b) Discuss the limits this time resolution imposes on the ability of the sonar system to detect the size and shape of the object creating the echo.

Solution:

(a) 7.70 m

(b) This means that sonar is good for spotting and locating large objects, but it isn't able to resolve smaller objects, or detect the detailed shapes of objects. Objects like ships or large pieces of airplanes can be found by sonar, while smaller pieces must be found by other means.

Exercise:

Problem:

A physicist at a fireworks display times the lag between seeing an explosion and hearing its sound, and finds it to be 0.400 s. (a) How far away is the explosion if air temperature is 24.0°C and if you neglect the time taken for light to reach the physicist? (b) Calculate the distance to the explosion taking the speed of light into account. Note that this distance is negligibly greater.

Exercise:

Problem:

Suppose a bat uses sound echoes to locate its insect prey, 3.00 m away. (See [link](#).) (a) Calculate the echo times for temperatures of 5.00°C and 35.0°C . (b) What percent uncertainty does this cause for the bat in locating the insect? (c) Discuss the significance of this uncertainty and whether it could cause difficulties for the bat. (In practice, the bat continues to use sound as it closes in, eliminating most of any difficulties imposed by this and other effects, such as motion of the prey.)

Solution:

(a) 18.0 ms, 17.1 ms

(b) 5.00%

(c) This uncertainty could definitely cause difficulties for the bat, if it didn't continue to use sound as it closed in on its prey. A 5% uncertainty could be the difference between catching the prey around the neck or around the chest, which means that it could miss grabbing its prey.

Glossary

pitch

the perception of the frequency of a sound

Sound Intensity and Sound Level

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define intensity, sound intensity, and sound pressure level.
- Calculate sound intensity levels in decibels (dB).

The information presented in this section supports the following AP® learning objectives and science practices:

- **6.A.4.1** The student is able to explain and/or predict qualitatively how the energy carried by a sound wave relates to the amplitude of the wave, and/or apply this concept to a real-world example. **(S.P. 6.4)**



Noise on crowded
roadways like this one in
Delhi makes it hard to

hear others unless they
shout. (credit: Lingaraj G
J, Flickr)

In a quiet forest, you can sometimes hear a single leaf fall to the ground. After settling into bed, you may hear your blood pulsing through your ears. But when a passing motorist has his stereo turned up, you cannot even hear what the person next to you in your car is saying. We are all very familiar with the loudness of sounds and aware that they are related to how energetically the source is vibrating. In cartoons depicting a screaming person (or an animal making a loud noise), the cartoonist often shows an open mouth with a vibrating uvula, the hanging tissue at the back of the mouth, to suggest a loud sound coming from the throat [\[link\]](#). High noise exposure is hazardous to hearing, and it is common for musicians to have hearing losses that are sufficiently severe that they interfere with the musicians' abilities to perform. The relevant physical quantity is sound intensity, a concept that is valid for all sounds whether or not they are in the audible range.

Intensity is defined to be the power per unit area carried by a wave. Power is the rate at which energy is transferred by the wave. In equation form, **intensity** I is

Equation:

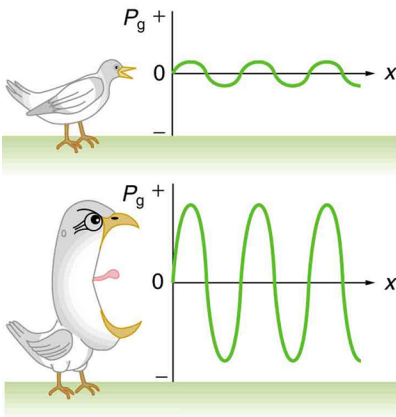
$$I = \frac{P}{A},$$

where P is the power through an area A . The SI unit for I is W/m^2 . The intensity of a sound wave is related to its amplitude squared by the following relationship:

Equation:

$$I = \frac{(\Delta p)^2}{2\rho v_w}.$$

Here Δp is the pressure variation or pressure amplitude (half the difference between the maximum and minimum pressure in the sound wave) in units of pascals (Pa) or N/m^2 . (We are using a lower case p for pressure to distinguish it from power, denoted by P above.) The energy (as kinetic energy $\frac{mv^2}{2}$) of an oscillating element of air due to a traveling sound wave is proportional to its amplitude squared. In this equation, ρ is the density of the material in which the sound wave travels, in units of kg/m^3 , and v_w is the speed of sound in the medium, in units of m/s. The pressure variation is proportional to the amplitude of the oscillation, and so I varies as $(\Delta p)^2$ ([link](#)). This relationship is consistent with the fact that the sound wave is produced by some vibration; the greater its pressure amplitude, the more the air is compressed in the sound it creates.



Graphs of the gauge pressures in two sound waves of different intensities.

The more intense sound is produced by a source that has larger-amplitude oscillations and has greater pressure maxima and minima. Because pressures are higher

in the greater-intensity sound, it can exert larger forces on the objects it encounters.

Sound intensity levels are quoted in decibels (dB) much more often than sound intensities in watts per meter squared. Decibels are the unit of choice in the scientific literature as well as in the popular media. The reasons for this choice of units are related to how we perceive sounds. How our ears perceive sound can be more accurately described by the logarithm of the intensity rather than directly to the intensity. The **sound intensity level** β in decibels of a sound having an intensity I in watts per meter squared is defined to be

Equation:

$$\beta \text{ (dB)} = 10 \log_{10} \left(\frac{I}{I_0} \right),$$

where $I_0 = 10^{-12} \text{ W/m}^2$ is a reference intensity. In particular, I_0 is the lowest or threshold intensity of sound a person with normal hearing can perceive at a frequency of 1000 Hz. Sound intensity level is not the same as intensity. Because β is defined in terms of a ratio, it is a unitless quantity telling you the *level* of the sound relative to a fixed standard (10^{-12} W/m^2 , in this case). The units of decibels (dB) are used to indicate this ratio is multiplied by 10 in its definition. The bel, upon which the decibel is based, is named for Alexander Graham Bell, the inventor of the telephone.

Sound Intensity level β (dB)	Intensity $I(\text{W/m}^2)$	Example/effect
0	1×10^{-12}	Threshold of hearing at 1000 Hz
10	1×10^{-11}	Rustle of leaves
20	1×10^{-10}	Whisper at 1 m distance
30	1×10^{-9}	Quiet home
40	1×10^{-8}	Average home
50	1×10^{-7}	Average office, soft music
60	1×10^{-6}	Normal conversation
70	1×10^{-5}	Noisy office, busy traffic
80	1×10^{-4}	Loud radio, classroom lecture
90	1×10^{-3}	Inside a heavy truck; damage from prolonged exposure [footnote] Several government agencies and health-related professional associations recommend that 85 dB not be exceeded for 8-hour daily exposures in the absence of hearing protection.
100	1×10^{-2}	Noisy factory, siren at 30 m; damage from 8 h per day exposure
110	1×10^{-1}	Damage from 30 min per day exposure

Sound intensity level β (dB)	Intensity $I(\text{W/m}^2)$	Example/effect
120	1	Loud rock concert, pneumatic chipper at 2 m; threshold of pain
140	1×10^2	Jet airplane at 30 m; severe pain, damage in seconds
160	1×10^4	Bursting of eardrums

Sound Intensity Levels and Intensities

The decibel level of a sound having the threshold intensity of 10^{-12} W/m^2 is $\beta = 0 \text{ dB}$, because $\log_{10} 1 = 0$. That is, the threshold of hearing is 0 decibels. [\[link\]](#) gives levels in decibels and intensities in watts per meter squared for some familiar sounds.

One of the more striking things about the intensities in [\[link\]](#) is that the intensity in watts per meter squared is quite small for most sounds. The ear is sensitive to as little as a trillionth of a watt per meter squared—even more impressive when you realize that the area of the eardrum is only about 1 cm^2 , so that only 10^{-16} W falls on it at the threshold of hearing! Air molecules in a sound wave of this intensity vibrate over a distance of less than one molecular diameter, and the gauge pressures involved are less than 10^{-9} atm .

Another impressive feature of the sounds in [\[link\]](#) is their numerical range. Sound intensity varies by a factor of 10^{12} from threshold to a sound that causes damage in seconds. You are unaware of this tremendous range in sound intensity because how your ears respond can be described approximately as the logarithm of intensity. Thus, sound intensity levels in decibels fit your experience better than intensities in watts per meter squared. The decibel scale is also easier to relate to because most people are

more accustomed to dealing with numbers such as 0, 53, or 120 than numbers such as 1.00×10^{-11} .

One more observation readily verified by examining [\[link\]](#) or using $I = \frac{(\Delta p)^2}{2\rho v_w}$ is that each factor of 10 in intensity corresponds to 10 dB. For example, a 90 dB sound compared with a 60 dB sound is 30 dB greater, or three factors of 10 (that is, 10^3 times) as intense. Another example is that if one sound is 10^7 as intense as another, it is 70 dB higher. See [\[link\]](#).

I_2/I_1	$\beta_2-\beta_1$
2.0	3.0 dB
5.0	7.0 dB
10.0	10.0 dB

Ratios of Intensities and Corresponding Differences in Sound Intensity Levels

Example:
Calculating Sound Intensity Levels: Sound Waves
 Calculate the sound intensity level in decibels for a sound wave traveling in air at 0°C and having a pressure amplitude of 0.656 Pa.
Strategy
 We are given Δp , so we can calculate I using the equation $I = (\Delta p)^2/(2\rho v_w)^2$. Using I , we can calculate β straight from its definition in $\beta \text{ (dB)} = 10 \log_{10}(I/I_0)$.
Solution
 (1) Identify knowns:

Sound travels at 331 m/s in air at 0°C.

Air has a density of 1.29 kg/m³ at atmospheric pressure and 0°C.

(2) Enter these values and the pressure amplitude into $I = (\Delta p)^2 / (2\rho v_w)$:

Equation:

$$I = \frac{(\Delta p)^2}{2\rho v_w} = \frac{(0.656 \text{ Pa})^2}{2(1.29 \text{ kg/m}^3)(331 \text{ m/s})} = 5.04 \times 10^{-4} \text{ W/m}^2.$$

(3) Enter the value for I and the known value for I_0 into

$\beta \text{ (dB)} = 10 \log_{10}(I/I_0)$. Calculate to find the sound intensity level in decibels:

Equation:

$$10 \log_{10} 5.04 \times 10^8 = 10 \quad 8.70 \quad \text{dB} = 87 \text{ dB}.$$

Discussion

This 87 dB sound has an intensity five times as great as an 80 dB sound.

So a factor of five in intensity corresponds to a difference of 7 dB in sound intensity level. This value is true for any intensities differing by a factor of five.

Example:

Change Intensity Levels of a Sound: What Happens to the Decibel Level?

Show that if one sound is twice as intense as another, it has a sound level about 3 dB higher.

Strategy

You are given that the ratio of two intensities is 2 to 1, and are then asked to find the difference in their sound levels in decibels. You can solve this problem using the properties of logarithms.

Solution

(1) Identify knowns:

The ratio of the two intensities is 2 to 1, or:

Equation:

$$\frac{I_2}{I_1} = 2.00.$$

We wish to show that the difference in sound levels is about 3 dB. That is, we want to show:

Equation:

$$\beta_2 - \beta_1 = 3 \text{ dB}.$$

Note that:

Equation:

$$\log_{10} b - \log_{10} a = \log_{10} \left(\frac{b}{a} \right).$$

(2) Use the definition of β to get:

Equation:

$$\beta_2 - \beta_1 = 10 \log_{10} \left(\frac{I_2}{I_1} \right) = 10 \log_{10} 2.00 = 10 (0.301) \text{ dB}.$$

Thus,

Equation:

$$\beta_2 - \beta_1 = 3.01 \text{ dB}.$$

Discussion

This means that the two sound intensity levels differ by 3.01 dB, or about 3 dB, as advertised. Note that because only the ratio I_2/I_1 is given (and not the actual intensities), this result is true for any intensities that differ by a factor of two. For example, a 56.0 dB sound is twice as intense as a 53.0 dB sound, a 97.0 dB sound is half as intense as a 100 dB sound, and so on.

It should be noted at this point that there is another decibel scale in use, called the **sound pressure level**, based on the ratio of the pressure amplitude to a reference pressure. This scale is used particularly in

applications where sound travels in water. It is beyond the scope of most introductory texts to treat this scale because it is not commonly used for sounds in air, but it is important to note that very different decibel levels may be encountered when sound pressure levels are quoted. For example, ocean noise pollution produced by ships may be as great as 200 dB expressed in the sound pressure level, where the more familiar sound intensity level we use here would be something under 140 dB for the same sound.

Note:**Take-Home Investigation: Feeling Sound**

Find a CD player and a CD that has rock music. Place the player on a light table, insert the CD into the player, and start playing the CD. Place your hand gently on the table next to the speakers. Increase the volume and note the level when the table just begins to vibrate as the rock music plays. Increase the reading on the volume control until it doubles. What has happened to the vibrations?

Exercise:**Check Your Understanding****Problem:**

Describe how amplitude is related to the loudness of a sound.

Solution:

Amplitude is directly proportional to the experience of loudness. As amplitude increases, loudness increases.

Exercise:**Check Your Understanding****Problem:**

Identify common sounds at the levels of 10 dB, 50 dB, and 100 dB.

Solution:

10 dB: Running fingers through your hair.

50 dB: Inside a quiet home with no television or radio.

100 dB: Take-off of a jet plane.

Test Prep for AP Courses**Exercise:****Problem:**

In order to waken a sleeping child, the volume on an alarm clock is tripled. Under this new scenario, how much more energy will be striking the child's ear drums each second?

- a. twice as much
- b. three times as much
- c. approximately 4.8 times as much
- d. six times as much
- e. nine times as much

Solution:

(e)

Exercise:**Problem:**

A musician strikes the strings of a guitar such that they vibrate with twice the amplitude.

- a. Explain why this requires an energy input greater than twice the original value.

- b. Explain why the sound leaving the string will not result in a decibel level that is twice as great.

Section Summary

- Intensity is the same for a sound wave as was defined for all waves; it is

Equation:

$$I = \frac{P}{A},$$

where P is the power crossing area A . The SI unit for I is watts per meter squared. The intensity of a sound wave is also related to the pressure amplitude Δp

Equation:

$$I = \frac{(\Delta p)^2}{2\rho v_w},$$

where ρ is the density of the medium in which the sound wave travels and v_w is the speed of sound in the medium.

- Sound intensity level in units of decibels (dB) is

Equation:

$$\beta \text{ (dB)} = 10 \log_{10} \left(\frac{I}{I_0} \right),$$

where $I_0 = 10^{-12} \text{ W/m}^2$ is the threshold intensity of hearing.

Conceptual Questions

Exercise:**Problem:**

Six members of a synchronized swim team wear earplugs to protect themselves against water pressure at depths, but they can still hear the music and perform the combinations in the water perfectly. One day, they were asked to leave the pool so the dive team could practice a few dives, and they tried to practice on a mat, but seemed to have a lot more difficulty. Why might this be?

Exercise:**Problem:**

A community is concerned about a plan to bring train service to their downtown from the town's outskirts. The current sound intensity level, even though the rail yard is blocks away, is 70 dB downtown. The mayor assures the public that there will be a difference of only 30 dB in sound in the downtown area. Should the townspeople be concerned? Why?

Problems & Exercises**Exercise:****Problem:**

What is the intensity in watts per meter squared of 85.0-dB sound?

Solution:

$$3.16 \times 10^{-4} \text{ W/m}^2$$

Exercise:**Problem:**

The warning tag on a lawn mower states that it produces noise at a level of 91.0 dB. What is this in watts per meter squared?

Exercise:**Problem:**

A sound wave traveling in 20°C air has a pressure amplitude of 0.5 Pa. What is the intensity of the wave?

Solution:

$$3.04 \times 10^{-4} \text{ W/m}^2$$

Exercise:**Problem:**

What intensity level does the sound in the preceding problem correspond to?

Exercise:**Problem:**

What sound intensity level in dB is produced by earphones that create an intensity of $4.00 \times 10^{-2} \text{ W/m}^2$?

Solution:

$$106 \text{ dB}$$

Exercise:**Problem:**

Show that an intensity of 10^{-12} W/m^2 is the same as 10^{-16} W/cm^2 .

Exercise:**Problem:**

(a) What is the decibel level of a sound that is twice as intense as a 90.0-dB sound? (b) What is the decibel level of a sound that is one-fifth as intense as a 90.0-dB sound?

Solution:

(a) 93 dB

(b) 83 dB

Exercise:**Problem:**

(a) What is the intensity of a sound that has a level 7.00 dB lower than a $4.00 \times 10^{-9} \text{ W/m}^2$ sound? (b) What is the intensity of a sound that is 3.00 dB higher than a $4.00 \times 10^{-9} \text{ W/m}^2$ sound?

Exercise:**Problem:**

(a) How much more intense is a sound that has a level 17.0 dB higher than another? (b) If one sound has a level 23.0 dB less than another, what is the ratio of their intensities?

Solution:

(a) 50.1

(b) 5.01×10^{-3} or $\frac{1}{200}$

Exercise:**Problem:**

People with good hearing can perceive sounds as low in level as -8.00 dB at a frequency of 3000 Hz. What is the intensity of this sound in watts per meter squared?

Exercise:

Problem:

If a large housefly 3.0 m away from you makes a noise of 40.0 dB, what is the noise level of 1000 flies at that distance, assuming interference has a negligible effect?

Solution:

70.0 dB

Exercise:**Problem:**

Ten cars in a circle at a boom box competition produce a 120-dB sound intensity level at the center of the circle. What is the average sound intensity level produced there by each stereo, assuming interference effects can be neglected?

Exercise:**Problem:**

The amplitude of a sound wave is measured in terms of its maximum gauge pressure. By what factor does the amplitude of a sound wave increase if the sound intensity level goes up by 40.0 dB?

Solution:

100

Exercise:**Problem:**

If a sound intensity level of 0 dB at 1000 Hz corresponds to a maximum gauge pressure (sound amplitude) of 10^{-9} atm, what is the maximum gauge pressure in a 60-dB sound? What is the maximum gauge pressure in a 120-dB sound?

Exercise:

Problem:

An 8-hour exposure to a sound intensity level of 90.0 dB may cause hearing damage. What energy in joules falls on a 0.800-cm-diameter eardrum so exposed?

Solution:

$$1.45 \times 10^{-3} \text{ J}$$

Exercise:**Problem:**

(a) Ear trumpets were never very common, but they did aid people with hearing losses by gathering sound over a large area and concentrating it on the smaller area of the eardrum. What decibel increase does an ear trumpet produce if its sound gathering area is 900 cm^2 and the area of the eardrum is 0.500 cm^2 , but the trumpet only has an efficiency of 5.00% in transmitting the sound to the eardrum? (b) Comment on the usefulness of the decibel increase found in part (a).

Exercise:**Problem:**

Sound is more effectively transmitted into a stethoscope by direct contact than through the air, and it is further intensified by being concentrated on the smaller area of the eardrum. It is reasonable to assume that sound is transmitted into a stethoscope 100 times as effectively compared with transmission through the air. What, then, is the gain in decibels produced by a stethoscope that has a sound gathering area of 15.0 cm^2 , and concentrates the sound onto two eardrums with a total area of 0.900 cm^2 with an efficiency of 40.0%?

Solution:

28.2 dB

Exercise:**Problem:**

Loudspeakers can produce intense sounds with surprisingly small energy input in spite of their low efficiencies. Calculate the power input needed to produce a 90.0-dB sound intensity level for a 12.0-cm-diameter speaker that has an efficiency of 1.00%. (This value is the sound intensity level right at the speaker.)

Glossary

intensity

the power per unit area carried by a wave

sound intensity level

a unitless quantity telling you the level of the sound relative to a fixed standard

sound pressure level

the ratio of the pressure amplitude to a reference pressure

Doppler Effect and Sonic Booms

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define Doppler effect, Doppler shift, and sonic boom.
- Calculate the frequency of a sound heard by someone observing Doppler shift.
- Describe the sounds produced by objects moving faster than the speed of sound.

The information presented in this section supports the following AP® learning objectives and science practices:

- **6.B.5.1** The student is able to create or use a wave front diagram to demonstrate or interpret qualitatively the observed frequency of a wave, dependent upon relative motions of source and observer. **(S.P. 1.4)**

The characteristic sound of a motorcycle buzzing by is an example of the **Doppler effect**. The high-pitch scream shifts dramatically to a lower-pitch roar as the motorcycle passes by a stationary observer. The closer the motorcycle brushes by, the more abrupt the shift. The faster the motorcycle moves, the greater the shift. We also hear this characteristic shift in frequency for passing race cars, airplanes, and trains. It is so familiar that it is used to imply motion and children often mimic it in play.

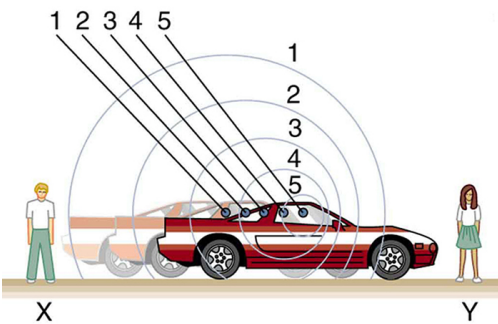
The Doppler effect is an alteration in the observed frequency of a sound due to motion of either the source or the observer. Although less familiar, this

effect is easily noticed for a stationary source and moving observer. For example, if you ride a train past a stationary warning bell, you will hear the bell's frequency shift from high to low as you pass by. The actual change in frequency due to relative motion of source and observer is called a **Doppler shift**. The Doppler effect and Doppler shift are named for the Austrian physicist and mathematician Christian Johann Doppler (1803–1853), who did experiments with both moving sources and moving observers. Doppler, for example, had musicians play on a moving open train car and also play standing next to the train tracks as a train passed by. Their music was observed both on and off the train, and changes in frequency were measured.

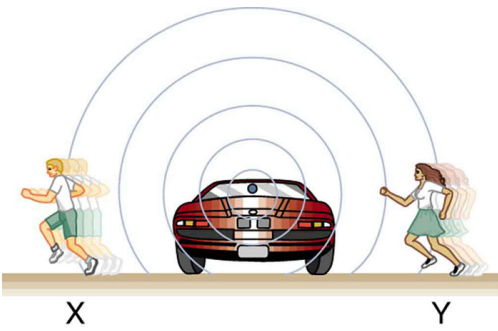
What causes the Doppler shift? [\[link\]](#), [\[link\]](#), and [\[link\]](#) compare sound waves emitted by stationary and moving sources in a stationary air mass. Each disturbance spreads out spherically from the point where the sound was emitted. If the source is stationary, then all of the spheres representing the air compressions in the sound wave centered on the same point, and the stationary observers on either side see the same wavelength and frequency as emitted by the source, as in [\[link\]](#). If the source is moving, as in [\[link\]](#), then the situation is different. Each compression of the air moves out in a sphere from the point where it was emitted, but the point of emission moves. This moving emission point causes the air compressions to be closer together on one side and farther apart on the other. Thus, the wavelength is shorter in the direction the source is moving (on the right in [\[link\]](#)), and longer in the opposite direction (on the left in [\[link\]](#)). Finally, if the observers move, as in [\[link\]](#), the frequency at which they receive the compressions changes. The observer moving toward the source receives them at a higher frequency, and the person moving away from the source receives them at a lower frequency.



Sounds emitted by a source spread out in spherical waves. Because the source, observers, and air are stationary, the wavelength and frequency are the same in all directions and to all observers.



Sounds emitted by a source moving to the right spread out from the points at which they were emitted. The wavelength is reduced and, consequently, the frequency is increased in the direction of motion, so that the observer on the right hears a higher-pitch sound. The opposite is true for the observer on the left, where the wavelength is increased and the frequency is reduced.



The same effect is produced when the observers move relative to the source. Motion toward the source increases frequency as the observer on the right passes through more wave crests than she would if stationary. Motion away from the source decreases frequency as the observer on the left passes through fewer wave crests than he would if stationary.

We know that wavelength and frequency are related by $v_w = f\lambda$, where v_w is the fixed speed of sound. The sound moves in a medium and has the same speed v_w in that medium whether the source is moving or not. Thus f multiplied by λ is a constant. Because the observer on the right in [\[link\]](#) receives a shorter wavelength, the frequency she receives must be higher. Similarly, the observer on the left receives a longer wavelength, and hence he hears a lower frequency. The same thing happens in [\[link\]](#). A higher frequency is received by the observer moving toward the source, and a lower frequency is received by an observer moving away from the source.

In general, then, relative motion of source and observer toward one another increases the received frequency. Relative motion apart decreases frequency. The greater the relative speed is, the greater the effect.

Note:

The Doppler Effect

The Doppler effect occurs not only for sound but for any wave when there is relative motion between the observer and the source. There are Doppler shifts in the frequency of sound, light, and water waves, for example.

Doppler shifts can be used to determine velocity, such as when ultrasound is reflected from blood in a medical diagnostic. The recession of galaxies is determined by the shift in the frequencies of light received from them and has implied much about the origins of the universe. Modern physics has been profoundly affected by observations of Doppler shifts.

For a stationary observer and a moving source, the frequency f_{obs} received by the observer can be shown to be

Equation:

$$f_{\text{obs}} = f_s \left(\frac{v_w}{v_w \pm v_s} \right),$$

where f_s is the frequency of the source, v_s is the speed of the source along a line joining the source and observer, and v_w is the speed of sound. The minus sign is used for motion toward the observer and the plus sign for motion away from the observer, producing the appropriate shifts up and down in frequency. Note that the greater the speed of the source, the greater the effect. Similarly, for a stationary source and moving observer, the frequency received by the observer f_{obs} is given by

Equation:

$$f_{\text{obs}} = f_s \left(\frac{v_w \pm v_{\text{obs}}}{v_w} \right),$$

where v_{obs} is the speed of the observer along a line joining the source and observer. Here the plus sign is for motion toward the source, and the minus is for motion away from the source.

Example:

Calculate Doppler Shift: A Train Horn

Suppose a train that has a 150-Hz horn is moving at 35.0 m/s in still air on a day when the speed of sound is 340 m/s.

(a) What frequencies are observed by a stationary person at the side of the tracks as the train approaches and after it passes?

(b) What frequency is observed by the train's engineer traveling on the train?

Strategy

To find the observed frequency in (a), $f_{\text{obs}} = f_s \left(\frac{v_w}{v_w \pm v_s} \right)$, must be used because the source is moving. The minus sign is used for the approaching train, and the plus sign for the receding train. In (b), there are two Doppler shifts—one for a moving source and the other for a moving observer.

Solution for (a)

(1) Enter known values into $f_{\text{obs}} = f_s \left(\frac{v_w}{v_w - v_s} \right)$.

Equation:

$$f_{\text{obs}} = f_s \left(\frac{v_w}{v_w - v_s} \right) = (150 \text{ Hz}) \left(\frac{340 \text{ m/s}}{340 \text{ m/s} - 35.0 \text{ m/s}} \right)$$

(2) Calculate the frequency observed by a stationary person as the train approaches.

Equation:

$$f_{\text{obs}} = (150 \text{ Hz})(1.11) = 167 \text{ Hz}$$

(3) Use the same equation with the plus sign to find the frequency heard by a stationary person as the train recedes.

Equation:

$$f_{\text{obs}} = f_s \left(\frac{v_w}{v_w + v_s} \right) = (150 \text{ Hz}) \left(\frac{340 \text{ m/s}}{340 \text{ m/s} + 35.0 \text{ m/s}} \right)$$

(4) Calculate the second frequency.

Equation:

$$f_{\text{obs}} = (150 \text{ Hz})(0.907) = 136 \text{ Hz}$$

Discussion on (a)

The numbers calculated are valid when the train is far enough away that the motion is nearly along the line joining train and observer. In both cases, the shift is significant and easily noticed. Note that the shift is 17.0 Hz for motion toward and 14.0 Hz for motion away. The shifts are not symmetric.

Solution for (b)

(1) Identify knowns:

- It seems reasonable that the engineer would receive the same frequency as emitted by the horn, because the relative velocity between them is zero.
- Relative to the medium (air), the speeds are $v_s = v_{\text{obs}} = 35.0 \text{ m/s}$.
- The first Doppler shift is for the moving observer; the second is for the moving source.

(2) Use the following equation:

Equation:

$$f_{\text{obs}} = \left[f_s \left(\frac{v_w \pm v_{\text{obs}}}{v_w} \right) \right] \left(\frac{v_w}{v_w \pm v_s} \right).$$

The quantity in the square brackets is the Doppler-shifted frequency due to a moving observer. The factor on the right is the effect of the moving source.

(3) Because the train engineer is moving in the direction toward the horn, we must use the plus sign for v_{obs} ; however, because the horn is also moving in the direction away from the engineer, we also use the plus sign for v_s . But the train is carrying both the engineer and the horn at the same velocity, so $v_s = v_{\text{obs}}$. As a result, everything but f_s cancels, yielding

Equation:

$$f_{\text{obs}} = f_s.$$

Discussion for (b)

We may expect that there is no change in frequency when source and observer move together because it fits your experience. For example, there is no Doppler shift in the frequency of conversations between driver and passenger on a motorcycle. People talking when a wind moves the air between them also observe no Doppler shift in their conversation. The crucial point is that source and observer are not moving relative to each other.

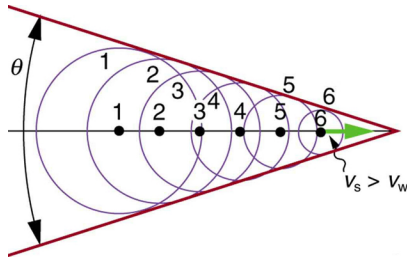
Sonic Booms to Bow Wakes

What happens to the sound produced by a moving source, such as a jet airplane, that approaches or even exceeds the speed of sound? The answer to this question applies not only to sound but to all other waves as well.

Suppose a jet airplane is coming nearly straight at you, emitting a sound of frequency f_s . The greater the plane's speed v_s , the greater the Doppler shift and the greater the value observed for f_{obs} . Now, as v_s approaches the speed of sound, f_{obs} approaches infinity, because the denominator in

$f_{\text{obs}} = f_s \left(\frac{v_w}{v_w \pm v_s} \right)$ approaches zero. At the speed of sound, this result

means that in front of the source, each successive wave is superimposed on the previous one because the source moves forward at the speed of sound. The observer gets them all at the same instant, and so the frequency is infinite. (Before airplanes exceeded the speed of sound, some people argued it would be impossible because such constructive superposition would produce pressures great enough to destroy the airplane.) If the source exceeds the speed of sound, no sound is received by the observer until the source has passed, so that the sounds from the approaching source are mixed with those from it when receding. This mixing appears messy, but something interesting happens—a sonic boom is created. (See [\[link\]](#).)

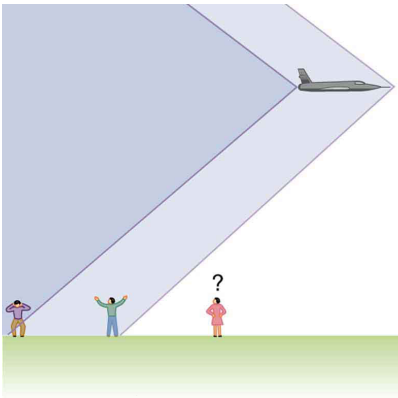


Sound waves from a source that moves faster than the speed of sound spread spherically from the point where they are emitted, but the source moves ahead of each.

Constructive interference along the lines shown (actually a cone in three dimensions) creates a shock wave called a sonic boom. The faster the speed of the source, the smaller the angle θ .

There is constructive interference along the lines shown (a cone in three dimensions) from similar sound waves arriving there simultaneously. This superposition forms a disturbance called a **sonic boom**, a constructive interference of sound created by an object moving faster than sound. Inside the cone, the interference is mostly destructive, and so the sound intensity there is much less than on the shock wave. An aircraft creates two sonic

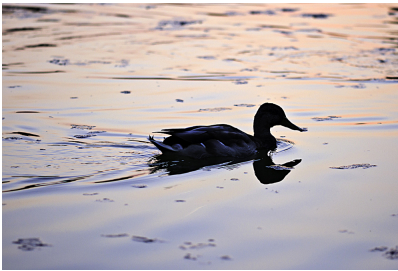
booms, one from its nose and one from its tail. (See [\[link\]](#).) During television coverage of space shuttle landings, two distinct booms could often be heard. These were separated by exactly the time it would take the shuttle to pass by a point. Observers on the ground often do not see the aircraft creating the sonic boom, because it has passed by before the shock wave reaches them, as seen in [\[link\]](#). If the aircraft flies close by at low altitude, pressures in the sonic boom can be destructive and break windows as well as rattle nerves. Because of how destructive sonic booms can be, supersonic flights are banned over populated areas of the United States.



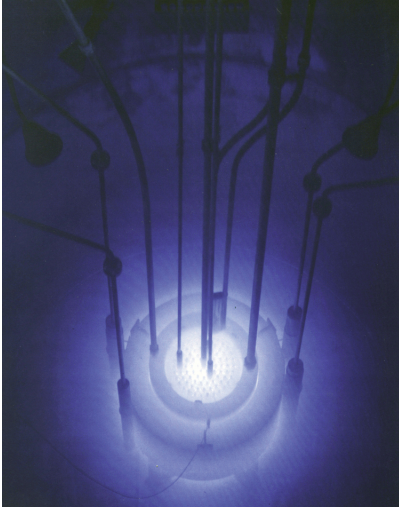
Two sonic booms,
created by the nose
and tail of an
aircraft, are
observed on the
ground after the
plane has passed
by.

Sonic booms are one example of a broader phenomenon called bow wakes. A **bow wake**, such as the one in [\[link\]](#), is created when the wave source moves faster than the wave propagation speed. Water waves spread out in circles from the point where created, and the bow wake is the familiar V-shaped wake trailing the source. A more exotic bow wake is created when a

subatomic particle travels through a medium faster than the speed of light travels in that medium. (In a vacuum, the maximum speed of light will be $c = 3.00 \times 10^8$ m/s; in the medium of water, the speed of light is closer to $0.75c$. If the particle creates light in its passage, that light spreads on a cone with an angle indicative of the speed of the particle, as illustrated in [\[link\]](#). Such a bow wake is called Cerenkov radiation and is commonly observed in particle physics.



Bow wake created
by a duck.
Constructive
interference
produces the rather
structured wake,
while there is
relatively little
wave action inside
the wake, where
interference is
mostly destructive.
(credit: Horia
Varlan, Flickr)



The blue glow in
this research
reactor pool is
Cerenkov radiation
caused by
subatomic particles
traveling faster than
the speed of light in
water. (credit: U.S.
Nuclear Regulatory
Commission)

Doppler shifts and sonic booms are interesting sound phenomena that occur in all types of waves. They can be of considerable use. For example, the Doppler shift in ultrasound can be used to measure blood velocity, while police use the Doppler shift in radar (a microwave) to measure car velocities. In meteorology, the Doppler shift is used to track the motion of storm clouds; such “Doppler Radar” can give velocity and direction and rain or snow potential of imposing weather fronts. In astronomy, we can examine the light emitted from distant galaxies and determine their speed relative to ours. As galaxies move away from us, their light is shifted to a lower frequency, and so to a longer wavelength—the so-called red shift.

Such information from galaxies far, far away has allowed us to estimate the age of the universe (from the Big Bang) as about 14 billion years.

Exercise:

Check Your Understanding

Problem:

Why did scientist Christian Doppler observe musicians both on a moving train and also from a stationary point not on the train?

Solution:

Doppler needed to compare the perception of sound when the observer is stationary and the sound source moves, as well as when the sound source and the observer are both in motion.

Exercise:

Check Your Understanding

Problem:

Describe a situation in your life when you might rely on the Doppler shift to help you either while driving a car or walking near traffic.

Solution:

If I am driving and I hear Doppler shift in an ambulance siren, I would be able to tell when it was getting closer and also if it has passed by. This would help me to know whether I needed to pull over and let the ambulance through.

Test Prep for AP Courses

Exercise:

Problem:

A baggage handler stands on the edge of a runway as a landing plane approaches. Compared to the pitch of the plane as heard by the plane's pilot, which of the following correctly describes the sensation experienced by the handler?

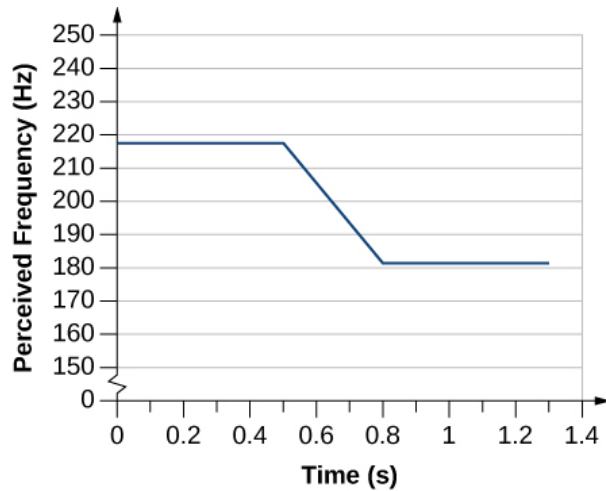
- a. The frequency of the plane will be lower pitched according to the baggage handler and will become even lower pitched as the plane slows to a stop.
- b. The frequency of the plane will be lower pitched according to the baggage handler but will increase in pitch as the plane slows to a stop.
- c. The frequency of the plane will be higher pitched according to the baggage handler but will decrease in pitch as the plane slows to a stop.
- d. The frequency of the plane will be higher pitched according to the baggage handler and will further increase in pitch as the plane slows to a stop.

Solution:

(c)

Exercise:**Problem:**

The following graph represents the perceived frequency of a car as it passes a student.



Plot of time versus perceived frequency to illustrate the Doppler effect.

- If the true frequency of the car's horn is 200 Hz, how fast was the car traveling?
- On the graph above, draw a line demonstrating the perceived frequency for a car traveling twice as fast. Label all intercepts, maximums, and minimums on the graph.

Section Summary

- The Doppler effect is an alteration in the observed frequency of a sound due to motion of either the source or the observer.
- The actual change in frequency is called the Doppler shift.
- A sonic boom is constructive interference of sound created by an object moving faster than sound.
- A sonic boom is a type of bow wake created when any wave source moves faster than the wave propagation speed.
- For a stationary observer and a moving source, the observed frequency

f_{obs} is:

Equation:

$$f_{\text{obs}} = f_s \left(\frac{v_w}{v_w \pm v_s} \right),$$

where f_s is the frequency of the source, v_s is the speed of the source, and v_w is the speed of sound. The minus sign is used for motion toward the observer and the plus sign for motion away.

- For a stationary source and moving observer, the observed frequency is:

Equation:

$$f_{\text{obs}} = f_s \left(\frac{v_w \pm v_{\text{obs}}}{v_w} \right),$$

where v_{obs} is the speed of the observer.

Conceptual Questions

Exercise:

Problem: Is the Doppler shift real or just a sensory illusion?

Exercise:

Problem:

Due to efficiency considerations related to its bow wake, the supersonic transport aircraft must maintain a cruising speed that is a constant ratio to the speed of sound (a constant Mach number). If the aircraft flies from warm air into colder air, should it increase or decrease its speed? Explain your answer.

Exercise:

Problem:

When you hear a sonic boom, you often cannot see the plane that made it. Why is that?

Problems & Exercises

Exercise:

Problem:

(a) What frequency is received by a person watching an oncoming ambulance moving at 110 km/h and emitting a steady 800-Hz sound from its siren? The speed of sound on this day is 345 m/s. (b) What frequency does she receive after the ambulance has passed?

Solution:

(a) 878 Hz

(b) 735 Hz

Exercise:

Problem:

(a) At an air show a jet flies directly toward the stands at a speed of 1200 km/h, emitting a frequency of 3500 Hz, on a day when the speed of sound is 342 m/s. What frequency is received by the observers? (b) What frequency do they receive as the plane flies directly away from them?

Exercise:

Problem:

What frequency is received by a mouse just before being dispatched by a hawk flying at it at 25.0 m/s and emitting a screech of frequency 3500 Hz? Take the speed of sound to be 331 m/s.

Solution:

3.79×10^3 Hz

Exercise:

Problem:

A spectator at a parade receives an 888-Hz tone from an oncoming trumpeter who is playing an 880-Hz note. At what speed is the musician approaching if the speed of sound is 338 m/s?

Exercise:**Problem:**

A commuter train blows its 200-Hz horn as it approaches a crossing. The speed of sound is 335 m/s. (a) An observer waiting at the crossing receives a frequency of 208 Hz. What is the speed of the train? (b) What frequency does the observer receive as the train moves away?

Solution:

(a) 12.9 m/s

(b) 193 Hz

Exercise:**Problem:**

Can you perceive the shift in frequency produced when you pull a tuning fork toward you at 10.0 m/s on a day when the speed of sound is 344 m/s? To answer this question, calculate the factor by which the frequency shifts and see if it is greater than 0.300%.

Exercise:**Problem:**

Two eagles fly directly toward one another, the first at 15.0 m/s and the second at 20.0 m/s. Both screech, the first one emitting a frequency of 3200 Hz and the second one emitting a frequency of 3800 Hz. What frequencies do they receive if the speed of sound is 330 m/s?

Solution:

First eagle hears $4.23 \times 10^3 \text{ Hz}$

Second eagle hears $3.56 \times 10^3 \text{ Hz}$

Exercise:

Problem:

What is the minimum speed at which a source must travel toward you for you to be able to hear that its frequency is Doppler shifted? That is, what speed produces a shift of 0.300% on a day when the speed of sound is 331 m/s?

Glossary

Doppler effect

an alteration in the observed frequency of a sound due to motion of either the source or the observer

Doppler shift

the actual change in frequency due to relative motion of source and observer

sonic boom

a constructive interference of sound created by an object moving faster than sound

bow wake

V-shaped disturbance created when the wave source moves faster than the wave propagation speed

Sound Interference and Resonance: Standing Waves in Air Columns

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define antinode, node, fundamental, overtones, and harmonics.
- Identify instances of sound interference in everyday situations.
- Describe how sound interference occurring inside open and closed tubes changes the characteristics of the sound, and how this applies to sounds produced by musical instruments.
- Calculate the length of a tube using sound wave measurements.

The information presented in this section supports the following AP® learning objectives and science practices:

- **6.D.1.1** The student is able to use representations of individual pulses and construct representations to model the interaction of two wave pulses to analyze the superposition of two pulses. **(S.P. 1.1, 1.4)**
- **6.D.1.2** The student is able to design a suitable experiment and analyze data illustrating the superposition of mechanical waves (only for wave pulses or standing waves). **(S.P. 4.2, 5.1)**
- **6.D.1.3** The student is able to design a plan for collecting data to quantify the amplitude variations when two or more traveling waves or wave pulses interact in a given medium. **(S.P. 4.2)**
- **6.D.3.1** The student is able to refine a scientific question related to standing waves and design a detailed plan for the experiment that can be conducted to examine the phenomenon qualitatively or quantitatively. **(S.P. 2.1, 2.2, 4.2)**

- **6.D.3.2** The student is able to predict properties of standing waves that result from the addition of incident and reflected waves that are confined to a region and have nodes and antinodes. (S.P. 6.4)
- **6.D.3.3** The student is able to plan data collection strategies, predict the outcome based on the relationship under test, perform data analysis, evaluate evidence compared to the prediction, explain any discrepancy and, if necessary, revise the relationship among variables responsible for establishing standing waves on a string or in a column of air. (S.P. 3.2, 4.1, 5.1, 5.2, 5.3)
- **6.D.3.4** The student is able to describe representations and models of situations in which standing waves result from the addition of incident and reflected waves confined to a region. (S.P. 1.2)
- **6.D.4.2** The student is able to calculate wavelengths and frequencies (if given wave speed) of standing waves based on boundary conditions and length of region within which the wave is confined, and calculate numerical values of wavelengths and frequencies. Examples should include musical instruments. (S.P. 2.2)



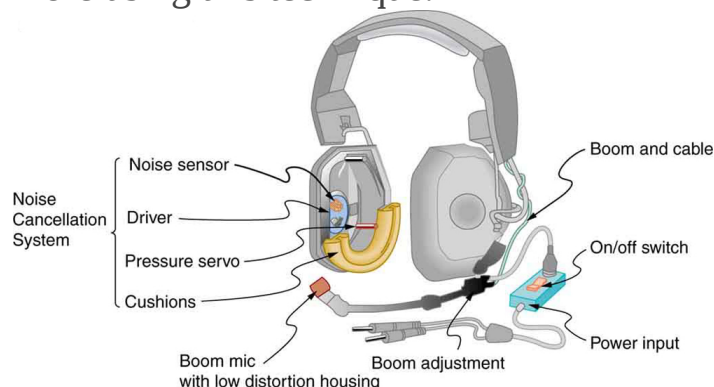
Some types

of
headphones
use the
phenomena
of
constructiv
e and
destructive
interference
to cancel
out outside
noises.
(credit:
JVC
America,
Flickr)

Interference is the hallmark of waves, all of which exhibit constructive and destructive interference exactly analogous to that seen for water waves. In fact, one way to prove something “is a wave” is to observe interference effects. So, sound being a wave, we expect it to exhibit interference; we have already mentioned a few such effects, such as the beats from two similar notes played simultaneously.

[\[link\]](#) shows a clever use of sound interference to cancel noise. Larger-scale applications of active noise reduction by destructive interference are contemplated for entire passenger compartments in commercial aircraft. To obtain destructive interference, a fast electronic analysis is performed, and a second sound is introduced with its maxima and minima exactly reversed from the incoming noise. Sound waves in fluids are pressure waves and consistent with Pascal’s principle; pressures from two different sources add and subtract like simple numbers; that is, positive and negative gauge pressures add to a much smaller pressure, producing a lower-intensity sound. Although completely destructive interference is possible only under

the simplest conditions, it is possible to reduce noise levels by 30 dB or more using this technique.



Headphones designed to cancel noise with destructive interference create a sound wave exactly opposite to the incoming sound. These headphones can be more effective than the simple passive attenuation used in most ear protection. Such headphones were used on the record-setting, around the world nonstop flight of the Voyager aircraft to protect the pilots' hearing from engine noise.

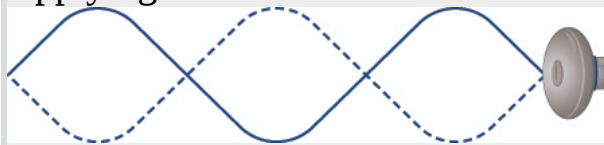
Where else can we observe sound interference? All sound resonances, such as in musical instruments, are due to constructive and destructive interference. Only the resonant frequencies interfere constructively to form standing waves, while others interfere destructively and are absent. From the toot made by blowing over a bottle, to the characteristic flavor of a violin's sounding box, to the recognizability of a great singer's voice, resonance and standing waves play a vital role.

Note:
Interference

Interference is such a fundamental aspect of waves that observing interference is proof that something is a wave. The wave nature of light was established by experiments showing interference. Similarly, when electrons scattered from crystals exhibited interference, their wave nature was confirmed to be exactly as predicted by symmetry with certain wave characteristics of light.

Note:

Applying the Science Practices: Standing Wave



The standing wave pattern of a rubber tube attached to a doorknob.

Tie one end of a strip of long rubber tubing to a stable object (doorknob, fence post, etc.) and shake the other end up and down until a standing wave pattern is achieved. Devise a method to determine the frequency and wavelength generated by your arm shaking. Do your results align with the equation? Do you find that the velocity of the wave generated is consistent for each trial? If not, explain why this is the case.

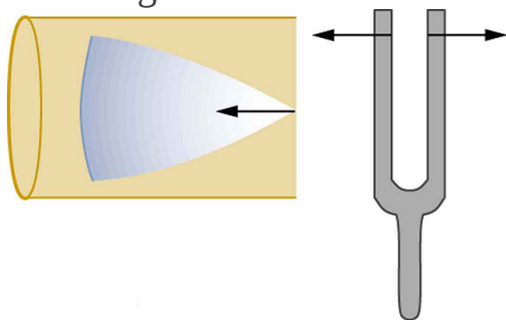
Note:

Answer

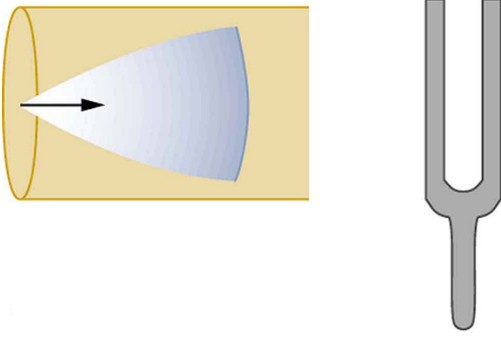
This task will likely require two people. The frequency of the wave pattern can be found by timing how long it takes the student shaking the rubber tubing to move his or her hand up and down one full time. (It may be beneficial to time how long it takes the student to do this ten times, and then divide by ten to reduce error.) The wavelength of the standing wave can be measured with a meter stick by measuring the distance between

two nodes and multiplying by two. This information should be gathered for standing wave patterns of multiple different wavelengths. As students collect their data, they can use the equation to determine if the wave velocity is consistent. There will likely be some error in the experiment yielding velocities of slightly different value. This error is probably due to an inaccuracy in the wavelength and/or frequency measurements.

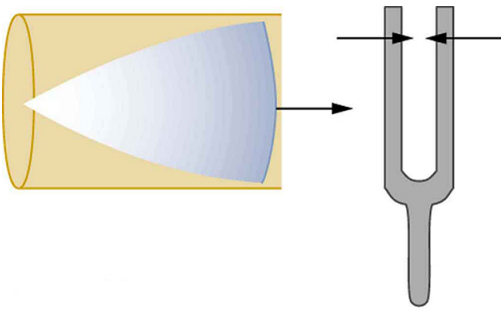
Suppose we hold a tuning fork near the end of a tube that is closed at the other end, as shown in [\[link\]](#), [\[link\]](#), [\[link\]](#), and [\[link\]](#). If the tuning fork has just the right frequency, the air column in the tube resonates loudly, but at most frequencies it vibrates very little. This observation just means that the air column has only certain natural frequencies. The figures show how a resonance at the lowest of these natural frequencies is formed. A disturbance travels down the tube at the speed of sound and bounces off the closed end. If the tube is just the right length, the reflected sound arrives back at the tuning fork exactly half a cycle later, and it interferes constructively with the continuing sound produced by the tuning fork. The incoming and reflected sounds form a standing wave in the tube as shown.



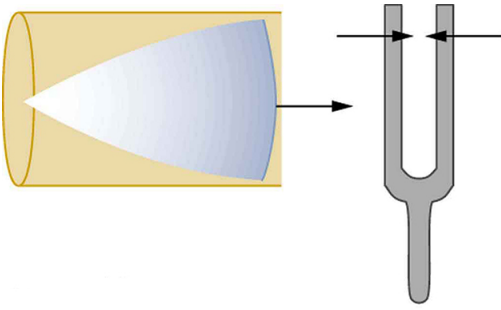
Resonance of air in a tube closed at one end, caused by a tuning fork. A disturbance moves down the tube.



Resonance of air in a tube closed at one end, caused by a tuning fork. The disturbance reflects from the closed end of the tube.

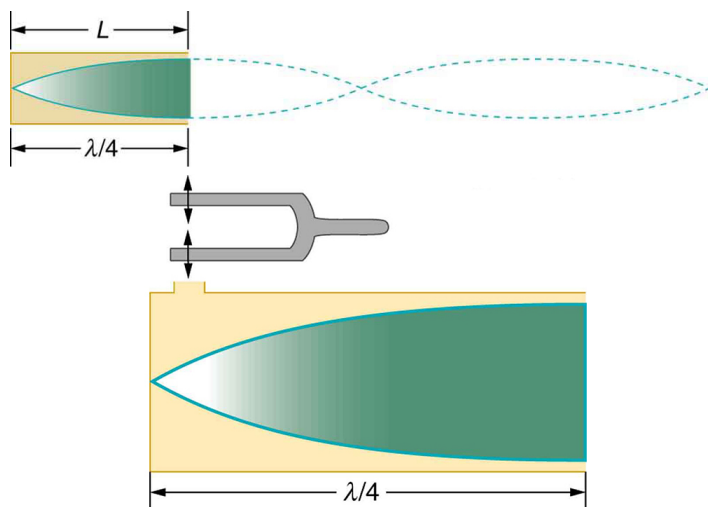


Resonance of air in a tube closed at one end, caused by a tuning fork. If the length of the tube L is just right, the disturbance gets back to the tuning fork half a cycle later and interferes constructively with the continuing sound from the tuning fork. This interference forms a standing wave, and the air column resonates.



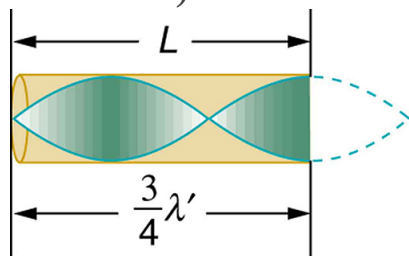
Resonance of air in a tube closed at one end, caused by a tuning fork. A graph of air displacement along the length of the tube shows none at the closed end, where the motion is constrained, and a maximum at the open end. This standing wave has one-fourth of its wavelength in the tube, so that $\lambda = 4L$.

The standing wave formed in the tube has its maximum air displacement (an **antinode**) at the open end, where motion is unconstrained, and no displacement (a **node**) at the closed end, where air movement is halted. The distance from a node to an antinode is one-fourth of a wavelength, and this equals the length of the tube; thus, $\lambda = 4L$. This same resonance can be produced by a vibration introduced at or near the closed end of the tube, as shown in [\[link\]](#). It is best to consider this a natural vibration of the air column independently of how it is induced.

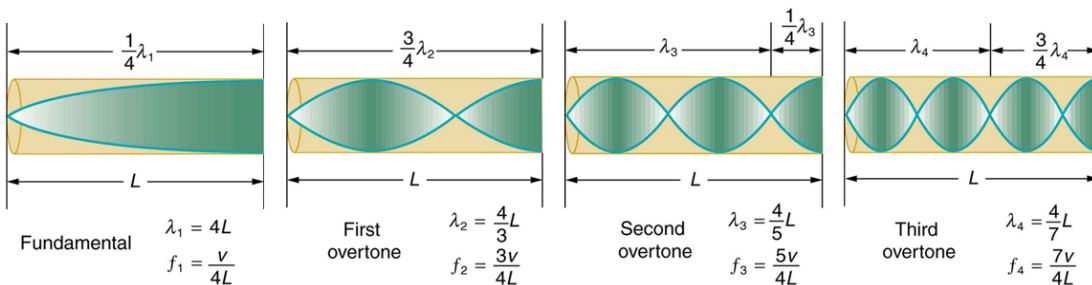


The same standing wave is created in the tube by a vibration introduced near its closed end.

Given that maximum air displacements are possible at the open end and none at the closed end, there are other, shorter wavelengths that can resonate in the tube, such as the one shown in [\[link\]](#). Here the standing wave has three-fourths of its wavelength in the tube, or $L = (3/4)\lambda'$, so that $\lambda' = 4L/3$. Continuing this process reveals a whole series of shorter-wavelength and higher-frequency sounds that resonate in the tube. We use specific terms for the resonances in any system. The lowest resonant frequency is called the **fundamental**, while all higher resonant frequencies are called **overtones**. All resonant frequencies are integral multiples of the fundamental, and they are collectively called **harmonics**. The fundamental is the first harmonic, the first overtone is the second harmonic, and so on. [\[link\]](#) shows the fundamental and the first three overtones (the first four harmonics) in a tube closed at one end.



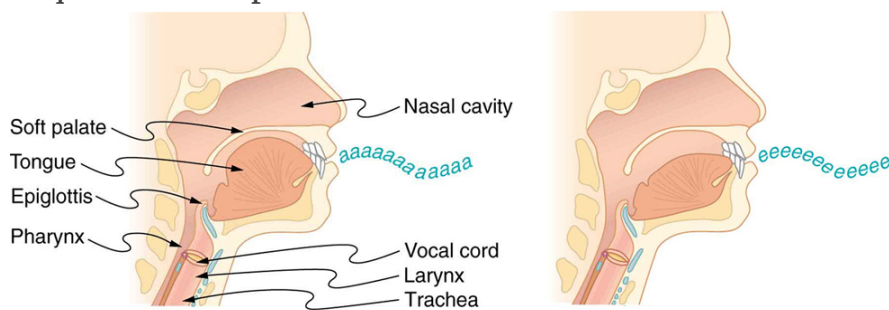
Another resonance for a tube closed at one end. This has maximum air displacements at the open end, and none at the closed end. The wavelength is shorter, with three-fourths λ equaling the length of the tube, so that $\lambda = 4L/3$. This higher-frequency vibration is the first overtone.



The fundamental and three lowest overtones for a tube closed at one end. All have maximum air displacements at the open end and none at the closed end.

The fundamental and overtones can be present simultaneously in a variety of combinations. For example, middle C on a trumpet has a sound distinctively different from middle C on a clarinet, both instruments being modified versions of a tube closed at one end. The fundamental frequency

is the same (and usually the most intense), but the overtones and their mix of intensities are different and subject to shading by the musician. This mix is what gives various musical instruments (and human voices) their distinctive characteristics, whether they have air columns, strings, sounding boxes, or drumheads. In fact, much of our speech is determined by shaping the cavity formed by the throat and mouth and positioning the tongue to adjust the fundamental and combination of overtones. Simple resonant cavities can be made to resonate with the sound of the vowels, for example. (See [\[link\]](#).) In boys, at puberty, the larynx grows and the shape of the resonant cavity changes giving rise to the difference in predominant frequencies in speech between men and women.



The throat and mouth form an air column closed at one end that resonates in response to vibrations in the voice box. The spectrum of overtones and their intensities vary with mouth shaping and tongue position to form different sounds. The voice box can be replaced with a mechanical vibrator, and understandable speech is still possible. Variations in basic shapes make different voices recognizable.

Now let us look for a pattern in the resonant frequencies for a simple tube that is closed at one end. The fundamental has $\lambda = 4L$, and frequency is related to wavelength and the speed of sound as given by:

Equation:

$$v_w = f\lambda.$$

Solving for f in this equation gives

Equation:

$$f = \frac{v_w}{\lambda} = \frac{v_w}{4L},$$

where v_w is the speed of sound in air. Similarly, the first overtone has $\lambda' = 4L/3$ (see [\[link\]](#)), so that

Equation:

$$f' = 3 \frac{v_w}{4L} = 3f.$$

Because $f' = 3f$, we call the first overtone the third harmonic. Continuing this process, we see a pattern that can be generalized in a single expression. The resonant frequencies of a tube closed at one end are

Equation:

$$f_n = n \frac{v_w}{4L}, n = 1, 3, 5,$$

where f_1 is the fundamental, f_3 is the first overtone, and so on. It is interesting that the resonant frequencies depend on the speed of sound and, hence, on temperature. This dependence poses a noticeable problem for organs in old unheated cathedrals, and it is also the reason why musicians commonly bring their wind instruments to room temperature before playing them.

Example:

Find the Length of a Tube with a 128 Hz Fundamental

(a) What length should a tube closed at one end have on a day when the air temperature, is 22.0°C, if its fundamental frequency is to be 128 Hz (C below middle C)?

(b) What is the frequency of its fourth overtone?

Strategy

The length L can be found from the relationship in $f_n = n \frac{v_w}{4L}$, but we will first need to find the speed of sound v_w .

Solution for (a)

(1) Identify knowns:

- the fundamental frequency is 128 Hz
- the air temperature is 22.0°C

(2) Use $f_n = n \frac{v_w}{4L}$ to find the fundamental frequency ($n = 1$).

Equation:

$$f_1 = \frac{v_w}{4L}$$

(3) Solve this equation for length.

Equation:

$$L = \frac{v_w}{4f_1}$$

(4) Find the speed of sound using $v_w = (331 \text{ m/s}) \sqrt{\frac{T}{273 \text{ K}}}$.

Equation:

$$v_w = (331 \text{ m/s}) \sqrt{\frac{295 \text{ K}}{273 \text{ K}}} = 344 \text{ m/s}$$

(5) Enter the values of the speed of sound and frequency into the expression for L .

Equation:

$$L = \frac{v_w}{4f_1} = \frac{344 \text{ m/s}}{4(128 \text{ Hz})} = 0.672 \text{ m}$$

Discussion on (a)

Many wind instruments are modified tubes that have finger holes, valves, and other devices for changing the length of the resonating air column and

hence, the frequency of the note played. Horns producing very low frequencies, such as tubas, require tubes so long that they are coiled into loops.

Solution for (b)

(1) Identify knowns:

- the first overtone has $n = 3$
- the second overtone has $n = 5$
- the third overtone has $n = 7$
- the fourth overtone has $n = 9$

(2) Enter the value for the fourth overtone into $f_n = n \frac{v_w}{4L}$.

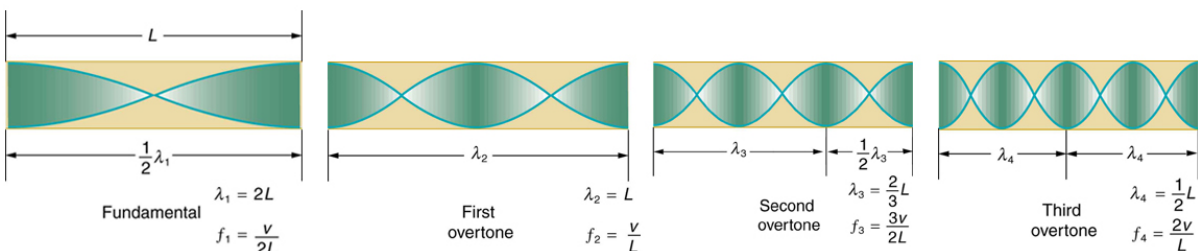
Equation:

$$f_9 = 9 \frac{v_w}{4L} = 9f_1 = 1.15 \text{ kHz}$$

Discussion on (b)

Whether this overtone occurs in a simple tube or a musical instrument depends on how it is stimulated to vibrate and the details of its shape. The trombone, for example, does not produce its fundamental frequency and only makes overtones.

Another type of tube is one that is *open* at both ends. Examples are some organ pipes, flutes, and oboes. The resonances of tubes open at both ends can be analyzed in a very similar fashion to those for tubes closed at one end. The air columns in tubes open at both ends have maximum air displacements at both ends, as illustrated in [\[link\]](#). Standing waves form as shown.



The resonant frequencies of a tube open at both ends are shown, including the fundamental and the first three overtones. In all cases the maximum air displacements occur at both ends of the tube, giving it different natural frequencies than a tube closed at one end.

Based on the fact that a tube open at both ends has maximum air displacements at both ends, and using [\[link\]](#) as a guide, we can see that the resonant frequencies of a tube open at both ends are:

Equation:

$$f_n = n \frac{v_w}{2L}, \quad n = 1, 2, 3, \dots,$$

where f_1 is the fundamental, f_2 is the first overtone, f_3 is the second overtone, and so on. Note that a tube open at both ends has a fundamental frequency twice what it would have if closed at one end. It also has a different spectrum of overtones than a tube closed at one end. So if you had two tubes with the same fundamental frequency but one was open at both ends and the other was closed at one end, they would sound different when played because they have different overtones. Middle C, for example, would sound richer played on an open tube, because it has even multiples of the fundamental as well as odd. A closed tube has only odd multiples.

Note:

Applying the Science Practices: Closed- and Open-Ended Tubes

Strike an open-ended length of plastic pipe while holding it in the air. Now place one end of the pipe on a hard surface, sealing one opening, and strike it again. How does the sound change? Further investigate the sound created by the pipe by striking pipes of different lengths and composition.

Note:

Answer

When the pipe is placed on the ground, the standing wave within the pipe changes from being open on both ends to being closed on one end. As a result, the fundamental frequency will change from $f = \frac{v}{2L}$ to $f = \frac{v}{4L}$. This decrease in frequency results in a decrease in observed pitch.

Note:

Real-World Applications: Resonance in Everyday Systems

Resonance occurs in many different systems, including strings, air columns, and atoms. Resonance is the driven or forced oscillation of a system at its natural frequency. At resonance, energy is transferred rapidly to the oscillating system, and the amplitude of its oscillations grows until the system can no longer be described by Hooke's law. An example of this is the distorted sound intentionally produced in certain types of rock music.

Wind instruments use resonance in air columns to amplify tones made by lips or vibrating reeds. Other instruments also use air resonance in clever ways to amplify sound. [\[link\]](#) shows a violin and a guitar, both of which have sounding boxes but with different shapes, resulting in different overtone structures. The vibrating string creates a sound that resonates in the sounding box, greatly amplifying the sound and creating overtones that give the instrument its characteristic flavor. The more complex the shape of the sounding box, the greater its ability to resonate over a wide range of frequencies. The marimba, like the one shown in [\[link\]](#) uses pots or gourds below the wooden slats to amplify their tones. The resonance of the pot can be adjusted by adding water.



String instruments such as violins and guitars use resonance in their sounding boxes to amplify and enrich the sound created by their vibrating strings. The bridge and supports couple the string vibrations to the sounding boxes and air within.
(credits: guitar, Feliciano Guimares, Fotopedia; violin, Steve Snodgrass, Flickr)



Resonance has been used in musical instruments since prehistoric times. This marimba uses gourds as resonance chambers to amplify its sound.
(credit: APC Events, Flickr)

We have emphasized sound applications in our discussions of resonance and standing waves, but these ideas apply to any system that has wave characteristics. Vibrating strings, for example, are actually resonating and have fundamentals and overtones similar to those for air columns. More subtle are the resonances in atoms due to the wave character of their electrons. Their orbitals can be viewed as standing waves, which have a fundamental (ground state) and overtones (excited states). It is fascinating that wave characteristics apply to such a wide range of physical systems.

Exercise:

Check Your Understanding

Problem:

Describe how noise-canceling headphones differ from standard headphones used to block outside sounds.

Solution:

Regular headphones only block sound waves with a physical barrier. Noise-canceling headphones use destructive interference to reduce the loudness of outside sounds.

Exercise:

Check Your Understanding

Problem:

How is it possible to use a standing wave's node and antinode to determine the length of a closed-end tube?

Solution:

When the tube resonates at its natural frequency, the wave's node is located at the closed end of the tube, and the antinode is located at the open end. The length of the tube is equal to one-fourth of the wavelength of this wave. Thus, if we know the wavelength of the wave, we can determine the length of the tube.

Note:

PhET Explorations: Sound

This simulation lets you see sound waves. Adjust the frequency or volume and you can see and hear how the wave changes. Move the listener around and hear what she hears.

[Sound](#)

Note:

Applying the Science Practices: Variables Affecting Superposition

In the PhET Interactive Simulation above, select the tab titled ‘Two Source Interference.’ Within this tab, manipulate the variables present (frequency, amplitude, and speaker separation) to investigate the relationship the variables have with the superposition pattern constructed on the screen. Record all observations.

Note:

Answer

As frequency is increased, the wavelength within the standing wave pattern will decrease. This results in an increase in nodes and antinodes, as represented in the applet by black and white surfaces. As amplitude is decreased, the contrast between black and white surfaces decreases, in demonstration of the decrease in sound level. There is no impact on sound wavelength, or number of nodes and antinodes shown. Increasing the speaker separation will affect the wavelength constructed. However, by separating the speakers, the number of nodes and antinodes within the applet will increase, as the waves are able to interfere over a greater distance.

Test Prep for AP Courses

Exercise:

Problem:

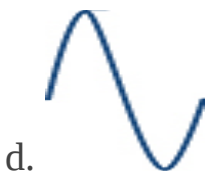
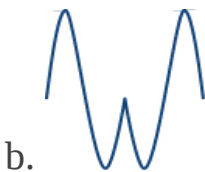
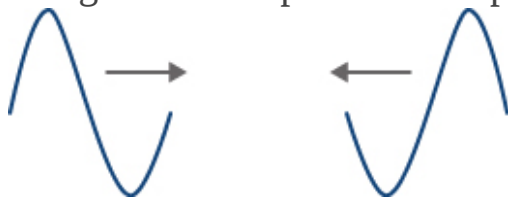
A common misconception is that two wave pulses traveling in opposite directions will reflect off each other. Outline a procedure that you would use to convince someone that the two wave pulses do not reflect off each other, but instead travel through each other. You may use sketches to represent your understanding. Be sure to provide evidence to not only refute the original claim, but to support yours as well.

Solution:

Answers vary. Students could include a sketch showing an increased amplitude when two waves occupy the same location. Students could also cite conceptual evidence such as sound waves passing through each other.

Exercise:**Problem:**

Two wave pulses are traveling toward each other on a string, as shown below. Which of the following representations correctly shows the string as the two pulses overlap?

**Exercise:**

Problem:

A student sends a transverse wave pulse of amplitude A along a rope attached at one end. As the pulse returns to the student, a second pulse of amplitude $3A$ is sent along the opposite side of the rope. What is the resulting amplitude when the two pulses interact?

- a. $4A$
 - b. A
 - c. $2A$, on the side of the original wave pulse
 - d. $2A$, on the side of the second wave pulse
-

Solution:

(d)

Exercise:**Problem:**

A student would like to demonstrate destructive interference using two sound sources. Explain how the student could set up this demonstration and what restrictions they would need to place upon their sources. Be sure to consider both the layout of space and the sounds created in your explanation.

Exercise:**Problem:**

A student is shaking a flexible string attached to a wooden board in a rhythmic manner. Which of the following choices will decrease the wavelength within the rope?

- I. The student could shake her hand back and forth with greater frequency.
- II. The student could shake her hand back in forth with a greater amplitude.

III. The student could increase the tension within the rope by stepping backwards from the board.

- a. I only
- b. I and II
- c. I and III
- d. II and III
- e. I, II, and III

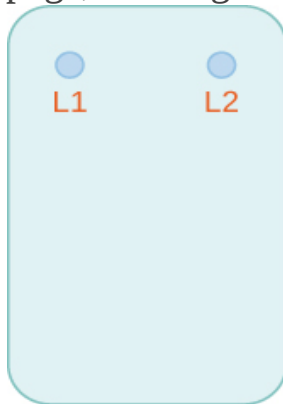
Solution:

(c)

Exercise:

Problem:

A ripple tank has two locations (L1 and L2) that vibrate in tandem as shown below. Both L1 and L2 vibrate in a plane perpendicular to the page, creating a two-dimensional interference pattern.



Describe an experimental procedure to determine the speed of the waves created within the water, including all additional equipment that you would need. You may use the diagram below to help your description, or you may create one of your own. Include enough detail so that another student could carry out your experiment.

Exercise:

Problem:

A string is vibrating between two posts as shown above. Students are to determine the speed of the wave within this string. They have already measured the amount of time necessary for the wave to oscillate up and down. The students must also take what other measurements to determine the speed of the wave?

- a. The distance between the two posts.
- b. The amplitude of the wave
- c. The tension in the string
- d. The amplitude of the wave and the tension in the string
- e. The distance between the two posts, the amplitude of the wave, and the tension in the string

Solution:

(a)

Exercise:**Problem:**

The accepted speed of sound in room temperature air is 346 m/s. Knowing that their school is colder than usual, a group of students is asked to determine the speed of sound in their room. They are permitted to use any materials necessary; however, their lab procedure must utilize standing wave patterns. The students collect the information [\[link\]](#).

Trial Number	Wavelength (m)	Frequency (Hz)	
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Trial Number	Wavelength (m)	Frequency (Hz)	
1	3.45	95	
2	2.32	135	
3	1.70	190	
4	1.45	240	
5	1.08	305	

- Describe an experimental procedure the group of students could have used to obtain this data. Include diagrams of the experimental setup and any equipment used in the process.
- Select a set of data points from the table and plot those points on a graph to determine the speed of sound within the classroom. Fill in the blank column in the table for any quantities you graph other than the given data. Label the axes and indicate the scale for each. Draw a best-fit line or curve through your data points.
- Using information from the graph, determine the speed of sound within the student's classroom, and explain what characteristic of the graph provides this evidence.
- Determine the temperature of the classroom.

Exercise:

Problem:

A tube is open at one end. If the fundamental frequency f is created by a wavelength λ , then which of the following describes the frequency and wavelength associated with the tube's fourth overtone?

	f	λ
(a)	$4f$	$\lambda/4$
(b)	$4f$	λ
(c)	$9f$	$\lambda/9$
(d)	$9f$	λ

Solution:

(c)

Exercise:

Problem:

A group of students were tasked with collecting information about standing waves. [\[link\]](#) a series of their data, showing the length of an air column and a resonant frequency present when the column is struck.

Length (m)	Resonant Frequency (Hz)
1	85.75
2	43
3	29

Length (m)	Resonant Frequency (Hz)
4	21.5

- a. From their data, determine whether the air column was open or closed on each end.
- b. Predict the resonant frequency of the column at a length of 2.5 meters.

Exercise:

Problem:

When a student blows across a glass half-full of water, a resonant frequency is created within the air column remaining in the glass. Which of the following can the student do to increase this resonant frequency?

- I. Add more water to the glass.
 - II. Replace the water with a more dense fluid.
 - III. Increase the temperature of the room.
- a. I only
 - b. I and III
 - c. II and III
 - d. all of the above

Solution:

(b)

Exercise:

Problem:

A student decides to test the speed of sound through wood using a wooden ruler. The student rests the ruler on a desk with half of its length protruding off the desk edge. The student then holds one end in place and strikes the protruding end with his other hand, creating a musical sound, and counts the number of vibrations of the ruler. Explain why the student would not be able to measure the speed of sound through wood using this method.

Solution:

Striking the end of a protruding ruler would create transverse waves, not sound waves. Any measurable, audible sound would come from the repetitive striking of the ruler against the desk. The student is confused about the speed of sound through solids versus the speed of sound in air.

Exercise:**Problem:**

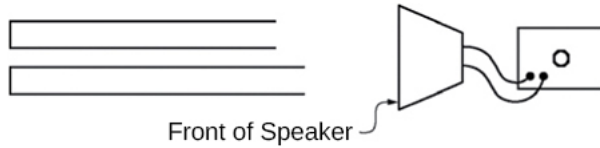
A musician stands outside in a field and plucks a string on an acoustic guitar. Standing waves will most likely occur in which of the following media? Select two answers.

- a. The guitar string
 - b. The air inside the guitar
 - c. The air surrounding the guitar
 - d. The ground beneath the musician
-

Solution:

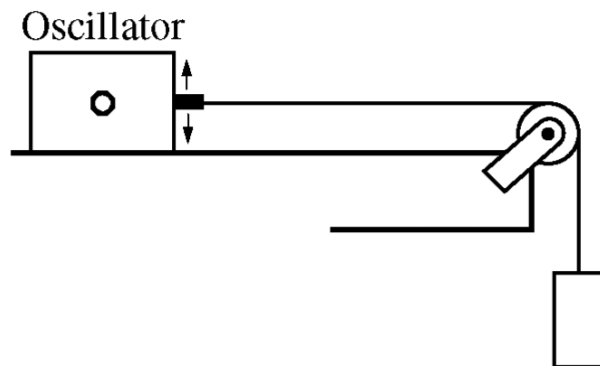
(a), (b)

Exercise:**Problem:**



This figure shows two tubes that are identical except for their slightly different lengths. Both tubes have one open end and one closed end. A speaker connected to a variable frequency generator is placed in front of the tubes, as shown. The speaker is set to produce a note of very low frequency when turned on. The frequency is then slowly increased to produce resonances in the tubes. Students observe that at first only one of the tubes resonates at a time. Later, as the frequency gets very high, there are times when both tubes resonate.

In a clear, coherent, paragraph-length answer, explain why there are some high frequencies, but no low frequencies, at which both tubes resonate. You may include diagrams and/or equations as part of your explanation.



Exercise:

Problem:

A student connects one end of a string with negligible mass to an oscillator. The other end of the string is passed over a pulley and attached to a suspended weight, as shown above. The student finds that a standing wave with one antinode is formed on the string when the frequency of the oscillator is f_0 . The student then moves the oscillator to shorten the horizontal segment of string to half its original length. At what frequency will a standing wave with one antinode now be formed on the string?

- a. $f_0/2$
- b. f_0
- c. $2f_0$
- d. There is no frequency at which a standing wave will be formed.

Solution:

(c)

Exercise:**Problem:**

A guitar string of length L is bound at both ends. [\[link\]](#) shows the string's harmonic frequencies when struck.

Harmonic Number	Frequency
1	$225/L$
2	$450/L$

Harmonic Number	Frequency
3	$675/L$
4	$900/L$

- Based on the information above, what is the speed of the wave within the string?
- The guitarist then slides her finger along the neck of the guitar, changing the string length as a result. Calculate the fundamental frequency of the string and wave speed present if the string length is reduced to $2/3 L$.

Section Summary

- Sound interference and resonance have the same properties as defined for all waves.
- In air columns, the lowest-frequency resonance is called the fundamental, whereas all higher resonant frequencies are called overtones. Collectively, they are called harmonics.
- The resonant frequencies of a tube closed at one end are:

Equation:

$$f_n = n \frac{v_w}{4L}, n = 1, 3, 5 \dots,$$

f_1 is the fundamental and L is the length of the tube.

- The resonant frequencies of a tube open at both ends are:

Equation:

$$f_n = n \frac{v_w}{2L}, n = 1, 2, 3 \dots$$

Conceptual Questions

Exercise:**Problem:**

How does an unamplified guitar produce sounds so much more intense than those of a plucked string held taut by a simple stick?

Exercise:**Problem:**

You are given two wind instruments of identical length. One is open at both ends, whereas the other is closed at one end. Which is able to produce the lowest frequency?

Exercise:**Problem:**

What is the difference between an overtone and a harmonic? Are all harmonics overtones? Are all overtones harmonics?

Problems & Exercises**Exercise:****Problem:**

A “showy” custom-built car has two brass horns that are supposed to produce the same frequency but actually emit 263.8 and 264.5 Hz. What beat frequency is produced?

Solution:

0.7 Hz

Exercise:

Problem:

What beat frequencies will be present: (a) If the musical notes A and C are played together (frequencies of 220 and 264 Hz)? (b) If D and F are played together (frequencies of 297 and 352 Hz)? (c) If all four are played together?

Exercise:**Problem:**

What beat frequencies result if a piano hammer hits three strings that emit frequencies of 127.8, 128.1, and 128.3 Hz?

Solution:

0.3 Hz, 0.2 Hz, 0.5 Hz

Exercise:**Problem:**

A piano tuner hears a beat every 2.00 s when listening to a 264.0-Hz tuning fork and a single piano string. What are the two possible frequencies of the string?

Exercise:**Problem:**

(a) What is the fundamental frequency of a 0.672-m-long tube, open at both ends, on a day when the speed of sound is 344 m/s? (b) What is the frequency of its second harmonic?

Solution:

(a) 256 Hz

(b) 512 Hz

Exercise:

Problem:

If a wind instrument, such as a tuba, has a fundamental frequency of 32.0 Hz, what are its first three overtones? It is closed at one end. (The overtones of a real tuba are more complex than this example, because it is a tapered tube.)

Exercise:**Problem:**

What are the first three overtones of a bassoon that has a fundamental frequency of 90.0 Hz? It is open at both ends. (The overtones of a real bassoon are more complex than this example, because its double reed makes it act more like a tube closed at one end.)

Solution:

180 Hz, 270 Hz, 360 Hz

Exercise:**Problem:**

How long must a flute be in order to have a fundamental frequency of 262 Hz (this frequency corresponds to middle C on the evenly tempered chromatic scale) on a day when air temperature is 20.0°C? It is open at both ends.

Exercise:**Problem:**

What length should an oboe have to produce a fundamental frequency of 110 Hz on a day when the speed of sound is 343 m/s? It is open at both ends.

Solution:

1.56 m

Exercise:**Problem:**

What is the length of a tube that has a fundamental frequency of 176 Hz and a first overtone of 352 Hz if the speed of sound is 343 m/s?

Exercise:**Problem:**

(a) Find the length of an organ pipe closed at one end that produces a fundamental frequency of 256 Hz when air temperature is 18.0°C. (b) What is its fundamental frequency at 25.0°C?

Solution:

(a) 0.334 m

(b) 259 Hz

Exercise:**Problem:**

By what fraction will the frequencies produced by a wind instrument change when air temperature goes from 10.0°C to 30.0°C? That is, find the ratio of the frequencies at those temperatures.

Exercise:**Problem:**

The ear canal resonates like a tube closed at one end. (See [\[link\]](#).) If ear canals range in length from 1.80 to 2.60 cm in an average population, what is the range of fundamental resonant frequencies? Take air temperature to be 37.0°C, which is the same as body temperature. How does this result correlate with the intensity versus frequency graph ([\[link\]](#)) of the human ear?

Solution:

3.39 to 4.90 kHz

Exercise:

Problem:

Calculate the first overtone in an ear canal, which resonates like a 2.40-cm-long tube closed at one end, by taking air temperature to be 37.0°C . Is the ear particularly sensitive to such a frequency? (The resonances of the ear canal are complicated by its nonuniform shape, which we shall ignore.)

Exercise:

Problem:

A crude approximation of voice production is to consider the breathing passages and mouth to be a resonating tube closed at one end. (See [\[link\]](#).) (a) What is the fundamental frequency if the tube is 0.240-m long, by taking air temperature to be 37.0°C ? (b) What would this frequency become if the person replaced the air with helium? Assume the same temperature dependence for helium as for air.

Solution:

(a) 367 Hz

(b) 1.07 kHz

Exercise:

Problem:

(a) Students in a physics lab are asked to find the length of an air column in a tube closed at one end that has a fundamental frequency of 256 Hz. They hold the tube vertically and fill it with water to the top, then lower the water while a 256-Hz tuning fork is rung and listen for the first resonance. What is the air temperature if the resonance occurs for a length of 0.336 m? (b) At what length will they observe the second resonance (first overtone)?

Exercise:

Problem:

What frequencies will a 1.80-m-long tube produce in the audible range at 20.0°C if: (a) The tube is closed at one end? (b) It is open at both ends?

Solution:

(a) $f_n = n(47.6 \text{ Hz}), n = 1, 3, 5, \dots, 419$

(b) $f_n = n(95.3 \text{ Hz}), n = 1, 2, 3, \dots, 210$

Glossary

antinode

point of maximum displacement

node

point of zero displacement

fundamental

the lowest-frequency resonance

overtones

all resonant frequencies higher than the fundamental

harmonics

the term used to refer collectively to the fundamental and its overtones

Hearing

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define hearing, pitch, loudness, timbre, note, tone, phon, ultrasound, and infrasound.
- Compare loudness to frequency and intensity of a sound.
- Identify structures of the inner ear and explain how they relate to sound perception.



Hearing allows this
vocalist, his band, and his
fans to enjoy music.
(credit: West Point Public
Affairs, Flickr)

The human ear has a tremendous range and sensitivity. It can give us a wealth of simple information—such as pitch, loudness, and direction. And from its input we can detect musical quality and nuances of voiced emotion. How is our hearing related to the physical qualities of sound, and how does the hearing mechanism work?

Hearing is the perception of sound. (Perception is commonly defined to be awareness through the senses, a typically circular definition of higher-level processes in living organisms.) Normal human hearing encompasses frequencies from 20 to 20,000 Hz, an impressive range. Sounds below 20 Hz are called **infrasound**, whereas those above 20,000 Hz are **ultrasound**. Neither is perceived by the ear, although infrasound can sometimes be felt as vibrations. When we do hear low-frequency vibrations, such as the sounds of a diving board, we hear the individual vibrations only because there are higher-frequency sounds in each. Other animals have hearing ranges different from that of humans. Dogs can hear sounds as high as 30,000 Hz, whereas bats and dolphins can hear up to 100,000-Hz sounds. You may have noticed that dogs respond to the sound of a dog whistle which produces sound out of the range of human hearing. Elephants are known to respond to frequencies below 20 Hz.

The perception of frequency is called **pitch**. Most of us have excellent relative pitch, which means that we can tell whether one sound has a different frequency from another. Typically, we can discriminate between two sounds if their frequencies differ by 0.3% or more. For example, 500.0 and 501.5 Hz are noticeably different. Pitch perception is directly related to frequency and is not greatly affected by other physical quantities such as intensity. Musical **notes** are particular sounds that can be produced by most instruments and in Western music have particular names. Combinations of notes constitute music. Some people can identify musical notes, such as A-sharp, C, or E-flat, just by listening to them. This uncommon ability is called perfect pitch.

The ear is remarkably sensitive to low-intensity sounds. The lowest audible intensity or threshold is about 10^{-12} W/m^2 or 0 dB. Sounds as much as 10^{12} more intense can be briefly tolerated. Very few measuring devices are capable of observations over a range of a trillion. The perception of

intensity is called **loudness**. At a given frequency, it is possible to discern differences of about 1 dB, and a change of 3 dB is easily noticed. But loudness is not related to intensity alone. Frequency has a major effect on how loud a sound seems. The ear has its maximum sensitivity to frequencies in the range of 2000 to 5000 Hz, so that sounds in this range are perceived as being louder than, say, those at 500 or 10,000 Hz, even when they all have the same intensity. Sounds near the high- and low-frequency extremes of the hearing range seem even less loud, because the ear is even less sensitive at those frequencies. [\[link\]](#) gives the dependence of certain human hearing perceptions on physical quantities.

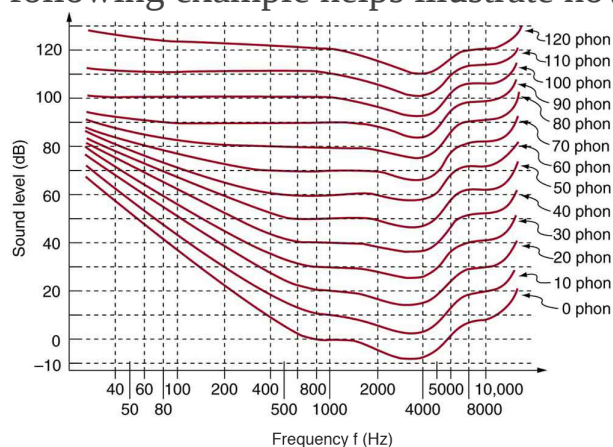
Perception	Physical quantity
Pitch	Frequency
Loudness	Intensity and Frequency
Timbre	Number and relative intensity of multiple frequencies. Subtle craftsmanship leads to non-linear effects and more detail.
Note	Basic unit of music with specific names, combined to generate tunes
Tone	Number and relative intensity of multiple frequencies.

Sound Perceptions

When a violin plays middle C, there is no mistaking it for a piano playing the same note. The reason is that each instrument produces a distinctive set

of frequencies and intensities. We call our perception of these combinations of frequencies and intensities **tone** quality, or more commonly the **timbre** of the sound. It is more difficult to correlate timbre perception to physical quantities than it is for loudness or pitch perception. Timbre is more subjective. Terms such as dull, brilliant, warm, cold, pure, and rich are employed to describe the timbre of a sound. So the consideration of timbre takes us into the realm of perceptual psychology, where higher-level processes in the brain are dominant. This is true for other perceptions of sound, such as music and noise. We shall not delve further into them; rather, we will concentrate on the question of loudness perception.

A unit called a **phon** is used to express loudness numerically. Phons differ from decibels because the phon is a unit of loudness perception, whereas the decibel is a unit of physical intensity. [\[link\]](#) shows the relationship of loudness to intensity (or intensity level) and frequency for persons with normal hearing. The curved lines are equal-loudness curves. Each curve is labeled with its loudness in phons. Any sound along a given curve will be perceived as equally loud by the average person. The curves were determined by having large numbers of people compare the loudness of sounds at different frequencies and sound intensity levels. At a frequency of 1000 Hz, phons are taken to be numerically equal to decibels. The following example helps illustrate how to use the graph:



The relationship of loudness in phons to intensity level (in decibels) and intensity (in watts per meter squared) for persons with normal hearing. The

curved lines are equal-loudness curves—all sounds on a given curve are perceived as equally loud. Phons and decibels are defined to be the same at 1000 Hz.

Example:

Measuring Loudness: Loudness Versus Intensity Level and Frequency

(a) What is the loudness in phons of a 100-Hz sound that has an intensity level of 80 dB? (b) What is the intensity level in decibels of a 4000-Hz sound having a loudness of 70 phons? (c) At what intensity level will an 8000-Hz sound have the same loudness as a 200-Hz sound at 60 dB?

Strategy for (a)

The graph in [\[link\]](#) should be referenced in order to solve this example. To find the loudness of a given sound, you must know its frequency and intensity level and locate that point on the square grid, then interpolate between loudness curves to get the loudness in phons.

Solution for (a)

(1) Identify knowns:

- The square grid of the graph relating phons and decibels is a plot of intensity level versus frequency—both physical quantities.
- 100 Hz at 80 dB lies halfway between the curves marked 70 and 80 phons.

(2) Find the loudness: 75 phons.

Strategy for (b)

The graph in [\[link\]](#) should be referenced in order to solve this example. To find the intensity level of a sound, you must have its frequency and loudness. Once that point is located, the intensity level can be determined from the vertical axis.

Solution for (b)

(1) Identify knowns:

- Values are given to be 4000 Hz at 70 phons.

(2) Follow the 70-phon curve until it reaches 4000 Hz. At that point, it is below the 70 dB line at about 67 dB.

(3) Find the intensity level:

67 dB

Strategy for (c)

The graph in [\[link\]](#) should be referenced in order to solve this example.

Solution for (c)

(1) Locate the point for a 200 Hz and 60 dB sound.

(2) Find the loudness: This point lies just slightly above the 50-phon curve, and so its loudness is 51 phons.

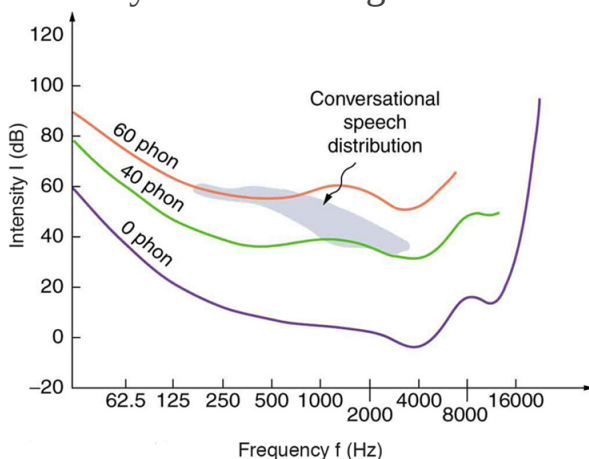
(3) Look for the 51-phon level is at 8000 Hz: 63 dB.

Discussion

These answers, like all information extracted from [\[link\]](#), have uncertainties of several phons or several decibels, partly due to difficulties in interpolation, but mostly related to uncertainties in the equal-loudness curves.

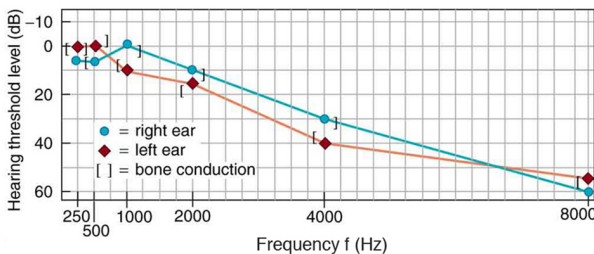
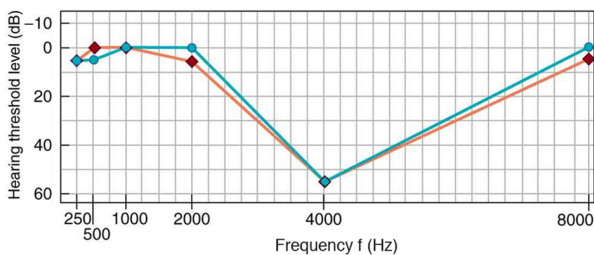
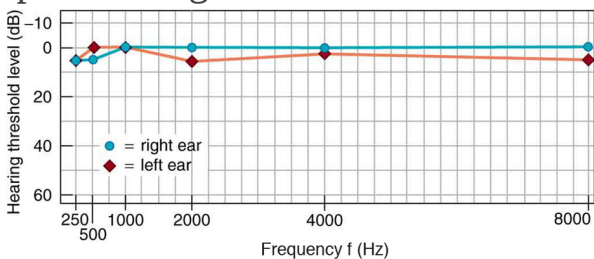
Further examination of the graph in [\[link\]](#) reveals some interesting facts about human hearing. First, sounds below the 0-phon curve are not perceived by most people. So, for example, a 60 Hz sound at 40 dB is inaudible. The 0-phon curve represents the threshold of normal hearing. We can hear some sounds at intensity levels below 0 dB. For example, a 3-dB, 5000-Hz sound is audible, because it lies above the 0-phon curve. The loudness curves all have dips in them between about 2000 and 5000 Hz. These dips mean the ear is most sensitive to frequencies in that range. For example, a 15-dB sound at 4000 Hz has a loudness of 20 phons, the same as a 20-dB sound at 1000 Hz. The curves rise at both extremes of the frequency range, indicating that a greater-intensity level sound is needed at those frequencies to be perceived to be as loud as at middle frequencies. For example, a sound at 10,000 Hz must have an intensity level of 30 dB to seem as loud as a 20 dB sound at 1000 Hz. Sounds above 120 phons are painful as well as damaging.

We do not often utilize our full range of hearing. This is particularly true for frequencies above 8000 Hz, which are rare in the environment and are unnecessary for understanding conversation or appreciating music. In fact, people who have lost the ability to hear such high frequencies are usually unaware of their loss until tested. The shaded region in [\[link\]](#) is the frequency and intensity region where most conversational sounds fall. The curved lines indicate what effect hearing losses of 40 and 60 phons will have. A 40-phon hearing loss at all frequencies still allows a person to understand conversation, although it will seem very quiet. A person with a 60-phon loss at all frequencies will hear only the lowest frequencies and will not be able to understand speech unless it is much louder than normal. Even so, speech may seem indistinct, because higher frequencies are not as well perceived. The conversational speech region also has a gender component, in that female voices are usually characterized by higher frequencies. So the person with a 60-phon hearing impediment might have difficulty understanding the normal conversation of a woman.



The shaded region represents frequencies and intensity levels found in normal conversational speech. The 0-phon line represents the normal hearing threshold, while those at 40 and 60 represent thresholds for people with 40- and 60-phon hearing losses, respectively.

Hearing tests are performed over a range of frequencies, usually from 250 to 8000 Hz, and can be displayed graphically in an audiogram like that in [\[link\]](#). The hearing threshold is measured in dB *relative to the normal threshold*, so that normal hearing registers as 0 dB at all frequencies. Hearing loss caused by noise typically shows a dip near the 4000 Hz frequency, irrespective of the frequency that caused the loss and often affects both ears. The most common form of hearing loss comes with age and is called *presbycusis*—literally elder ear. Such loss is increasingly severe at higher frequencies, and interferes with music appreciation and speech recognition.



Audiograms showing the threshold in intensity level versus frequency for three different individuals. Intensity level is measured relative to the normal threshold. The top left graph is that of a person with

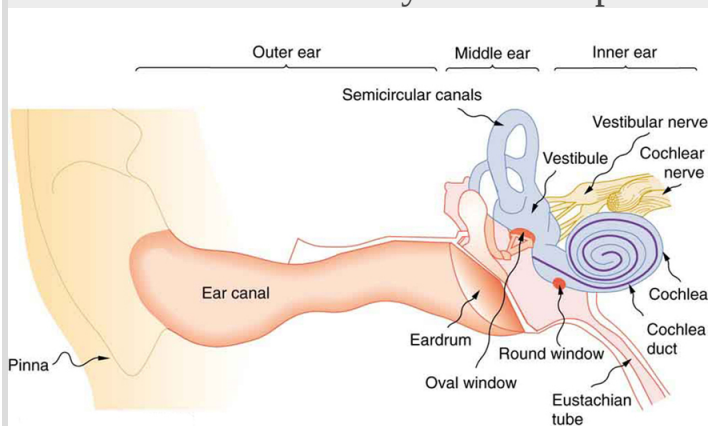
normal hearing. The graph to its right has a dip at 4000 Hz and is that of a child who suffered hearing loss due to a cap gun.

The third graph is typical of presbycusis, the progressive loss of higher frequency hearing with age. Tests performed by bone conduction (brackets) can distinguish nerve damage from middle ear damage.

Note:

The Hearing Mechanism

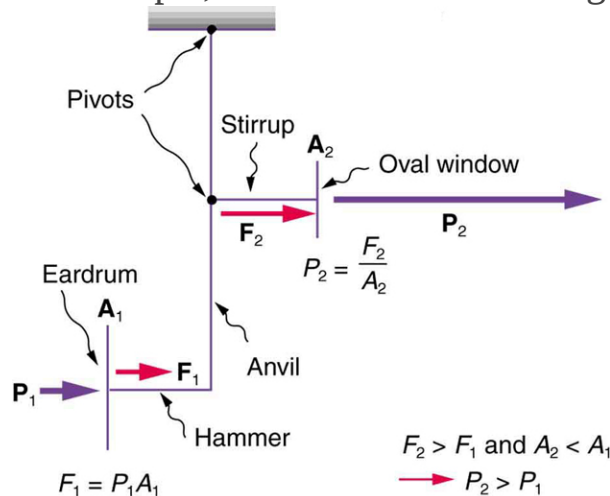
The hearing mechanism involves some interesting physics. The sound wave that impinges upon our ear is a pressure wave. The ear is a transducer that converts sound waves into electrical nerve impulses in a manner much more sophisticated than, but analogous to, a microphone. [\[link\]](#) shows the gross anatomy of the ear with its division into three parts: the outer ear or ear canal; the middle ear, which runs from the eardrum to the cochlea; and the inner ear, which is the cochlea itself. The body part normally referred to as the ear is technically called the pinna.



The illustration shows the gross

anatomy of the human ear.

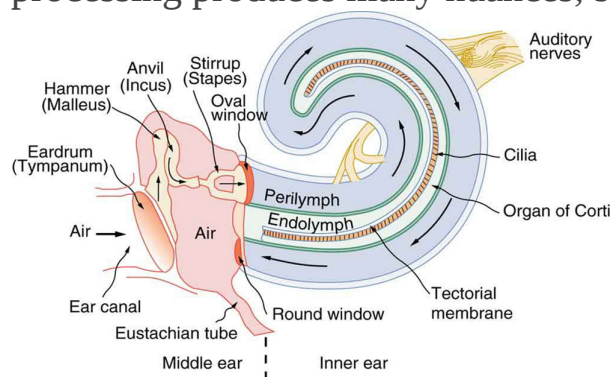
The outer ear, or ear canal, carries sound to the recessed protected eardrum. The air column in the ear canal resonates and is partially responsible for the sensitivity of the ear to sounds in the 2000 to 5000 Hz range. The middle ear converts sound into mechanical vibrations and applies these vibrations to the cochlea. The lever system of the middle ear takes the force exerted on the eardrum by sound pressure variations, amplifies it and transmits it to the inner ear via the oval window, creating pressure waves in the cochlea approximately 40 times greater than those impinging on the eardrum. (See [\[link\]](#).) Two muscles in the middle ear (not shown) protect the inner ear from very intense sounds. They react to intense sound in a few milliseconds and reduce the force transmitted to the cochlea. This protective reaction can also be triggered by your own voice, so that humming while shooting a gun, for example, can reduce noise damage.



This schematic shows the middle ear's system for converting sound pressure into force, increasing that force through a lever system, and applying the increased force to

a small area of the cochlea,
thereby creating a pressure
about 40 times that in the
original sound wave. A
protective muscle reaction to
intense sounds greatly reduces
the mechanical advantage of the
lever system.

[\[link\]](#) shows the middle and inner ear in greater detail. Pressure waves moving through the cochlea cause the tectorial membrane to vibrate, rubbing cilia (called hair cells), which stimulate nerves that send electrical signals to the brain. The membrane resonates at different positions for different frequencies, with high frequencies stimulating nerves at the near end and low frequencies at the far end. The complete operation of the cochlea is still not understood, but several mechanisms for sending information to the brain are known to be involved. For sounds below about 1000 Hz, the nerves send signals at the same frequency as the sound. For frequencies greater than about 1000 Hz, the nerves signal frequency by position. There is a structure to the cilia, and there are connections between nerve cells that perform signal processing before information is sent to the brain. Intensity information is partly indicated by the number of nerve signals and by volleys of signals. The brain processes the cochlear nerve signals to provide additional information such as source direction (based on time and intensity comparisons of sounds from both ears). Higher-level processing produces many nuances, such as music appreciation.



The inner ear, or cochlea, is a coiled tube about 3 mm in diameter and 3 cm in length if uncoiled. When the oval window is forced inward, as shown, a pressure wave travels through the perilymph in the direction of the arrows, stimulating nerves at the base of cilia in the organ of Corti.

Hearing losses can occur because of problems in the middle or inner ear. Conductive losses in the middle ear can be partially overcome by sending sound vibrations to the cochlea through the skull. Hearing aids for this purpose usually press against the bone behind the ear, rather than simply amplifying the sound sent into the ear canal as many hearing aids do. Damage to the nerves in the cochlea is not repairable, but amplification can partially compensate. There is a risk that amplification will produce further damage. Another common failure in the cochlea is damage or loss of the cilia but with nerves remaining functional. Cochlear implants that stimulate the nerves directly are now available and widely accepted. Over 100,000 implants are in use, in about equal numbers of adults and children.

The cochlear implant was pioneered in Melbourne, Australia, by Graeme Clark in the 1970s for his deaf father. The implant consists of three external components and two internal components. The external components are a microphone for picking up sound and converting it into an electrical signal, a speech processor to select certain frequencies and a transmitter to transfer the signal to the internal components through electromagnetic induction. The internal components consist of a receiver/transmitter secured in the bone beneath the skin, which converts the signals into electric impulses and sends them through an internal cable to the cochlea and an array of about 24 electrodes wound through the cochlea. These electrodes in turn send the impulses directly into the brain. The electrodes basically emulate the cilia.

Exercise:

Check Your Understanding

Problem:

Are ultrasound and infrasound imperceptible to all hearing organisms?
Explain your answer.

Solution:

No, the range of perceptible sound is based in the range of human hearing. Many other organisms perceive either infrasound or ultrasound.

Section Summary

- The range of audible frequencies is 20 to 20,000 Hz.
- Those sounds above 20,000 Hz are ultrasound, whereas those below 20 Hz are infrasound.
- The perception of frequency is pitch.
- The perception of intensity is loudness.
- Loudness has units of phons.

Conceptual Questions

Exercise:

Problem:

Why can a hearing test show that your threshold of hearing is 0 dB at 250 Hz, when [\[link\]](#) implies that no one can hear such a frequency at less than 20 dB?

Problems & Exercises

Exercise:

Problem:

The factor of 10^{-12} in the range of intensities to which the ear can respond, from threshold to that causing damage after brief exposure, is truly remarkable. If you could measure distances over the same range with a single instrument and the smallest distance you could measure was 1 mm, what would the largest be?

Solution:

$$1 \times 10^6 \text{ km}$$

Exercise:**Problem:**

The frequencies to which the ear responds vary by a factor of 10^3 . Suppose the speedometer on your car measured speeds differing by the same factor of 10^3 , and the greatest speed it reads is 90.0 mi/h. What would be the slowest nonzero speed it could read?

Exercise:**Problem:**

What are the closest frequencies to 500 Hz that an average person can clearly distinguish as being different in frequency from 500 Hz? The sounds are not present simultaneously.

Solution:

$$498.5 \text{ or } 501.5 \text{ Hz}$$

Exercise:**Problem:**

Can the average person tell that a 2002-Hz sound has a different frequency than a 1999-Hz sound without playing them simultaneously?

Exercise:**Problem:**

If your radio is producing an average sound intensity level of 85 dB, what is the next lowest sound intensity level that is clearly less intense?

Solution:

82 dB

Exercise:**Problem:**

Can you tell that your roommate turned up the sound on the TV if its average sound intensity level goes from 70 to 73 dB?

Exercise:**Problem:**

Based on the graph in [\[link\]](#), what is the threshold of hearing in decibels for frequencies of 60, 400, 1000, 4000, and 15,000 Hz? Note that many AC electrical appliances produce 60 Hz, music is commonly 400 Hz, a reference frequency is 1000 Hz, your maximum sensitivity is near 4000 Hz, and many older TVs produce a 15,750 Hz whine.

Solution:

approximately 48, 9, 0, -7, and 20 dB, respectively

Exercise:**Problem:**

What sound intensity levels must sounds of frequencies 60, 3000, and 8000 Hz have in order to have the same loudness as a 40-dB sound of frequency 1000 Hz (that is, to have a loudness of 40 phons)?

Exercise:

Problem:

What is the approximate sound intensity level in decibels of a 600-Hz tone if it has a loudness of 20 phons? If it has a loudness of 70 phons?

Solution:

(a) 23 dB

(b) 70 dB

Exercise:**Problem:**

(a) What are the loudnesses in phons of sounds having frequencies of 200, 1000, 5000, and 10,000 Hz, if they are all at the same 60.0-dB sound intensity level? (b) If they are all at 110 dB? (c) If they are all at 20.0 dB?

Exercise:**Problem:**

Suppose a person has a 50-dB hearing loss at all frequencies. By how many factors of 10 will low-intensity sounds need to be amplified to seem normal to this person? Note that smaller amplification is appropriate for more intense sounds to avoid further hearing damage.

Solution:

Five factors of 10

Exercise:

Problem:

If a woman needs an amplification of 5.0×10^{12} times the threshold intensity to enable her to hear at all frequencies, what is her overall hearing loss in dB? Note that smaller amplification is appropriate for more intense sounds to avoid further damage to her hearing from levels above 90 dB.

Exercise:**Problem:**

(a) What is the intensity in watts per meter squared of a just barely audible 200-Hz sound? (b) What is the intensity in watts per meter squared of a barely audible 4000-Hz sound?

Solution:

(a) $2 \times 10^{-10} \text{ W/m}^2$

(b) $2 \times 10^{-13} \text{ W/m}^2$

Exercise:**Problem:**

(a) Find the intensity in watts per meter squared of a 60.0-Hz sound having a loudness of 60 phons. (b) Find the intensity in watts per meter squared of a 10,000-Hz sound having a loudness of 60 phons.

Exercise:**Problem:**

A person has a hearing threshold 10 dB above normal at 100 Hz and 50 dB above normal at 4000 Hz. How much more intense must a 100-Hz tone be than a 4000-Hz tone if they are both barely audible to this person?

Solution:

2.5

Exercise:

Problem:

A child has a hearing loss of 60 dB near 5000 Hz, due to noise exposure, and normal hearing elsewhere. How much more intense is a 5000-Hz tone than a 400-Hz tone if they are both barely audible to the child?

Exercise:

Problem:

What is the ratio of intensities of two sounds of identical frequency if the first is just barely discernible as louder to a person than the second?

Solution:

1.26

Glossary

loudness

the perception of sound intensity

timbre

number and relative intensity of multiple sound frequencies

note

basic unit of music with specific names, combined to generate tunes

tone

number and relative intensity of multiple sound frequencies

phon

the numerical unit of loudness

ultrasound

sounds above 20,000 Hz

infrasound

sounds below 20 Hz

Ultrasound

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define acoustic impedance and intensity reflection coefficient.
- Describe medical and other uses of ultrasound technology.
- Calculate acoustic impedance using density values and the speed of ultrasound.
- Calculate the velocity of a moving object using Doppler-shifted ultrasound.



Ultrasound is used in medicine to painlessly and noninvasively monitor patient health and diagnose a wide range of disorders. (credit: abbybatchelder, Flickr)

Any sound with a frequency above 20,000 Hz (or 20 kHz)—that is, above the highest audible frequency—is defined to be ultrasound. In practice, it is possible to create

ultrasound frequencies up to more than a gigahertz. (Higher frequencies are difficult to create; furthermore, they propagate poorly because they are very strongly absorbed.) Ultrasound has a tremendous number of applications, which range from burglar alarms to use in cleaning delicate objects to the guidance systems of bats. We begin our discussion of ultrasound with some of its applications in medicine, in which it is used extensively both for diagnosis and for therapy.

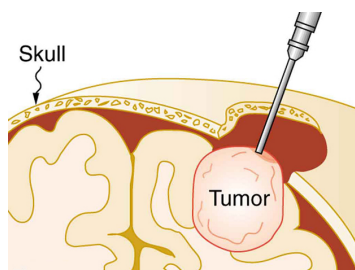
Note:

Characteristics of Ultrasound

The characteristics of ultrasound, such as frequency and intensity, are wave properties common to all types of waves. Ultrasound also has a wavelength that limits the fineness of detail it can detect. This characteristic is true of all waves. We can never observe details significantly smaller than the wavelength of our probe; for example, we will never see individual atoms with visible light, because the atoms are so small compared with the wavelength of light.

Ultrasound in Medical Therapy

Ultrasound, like any wave, carries energy that can be absorbed by the medium carrying it, producing effects that vary with intensity. When focused to intensities of 10^3 to 10^5 W/m², ultrasound can be used to shatter gallstones or pulverize cancerous tissue in surgical procedures. (See [\[link\]](#).) Intensities this great can damage individual cells, variously causing their protoplasm to stream inside them, altering their permeability, or rupturing their walls through *cavitation*. Cavitation is the creation of vapor cavities in a fluid—the longitudinal vibrations in ultrasound alternatively compress and expand the medium, and at sufficient amplitudes the expansion separates molecules. Most cavitation damage is done when the cavities collapse, producing even greater shock pressures.



The tip of this
small probe
oscillates at 23 kHz
with such a large
amplitude that it
pulverizes tissue on
contact. The debris
is then aspirated.
The speed of the tip
may exceed the
speed of sound in
tissue, thus creating
shock waves and
cavitation, rather
than a smooth
simple harmonic
oscillator-type
wave.

Most of the energy carried by high-intensity ultrasound in tissue is converted to thermal energy. In fact, intensities of 10^3 to 10^4 W/m² are commonly used for deep-heat treatments called ultrasound diathermy. Frequencies of 0.8 to 1 MHz are typical. In both athletics and physical therapy, ultrasound diathermy is most often applied to injured or overworked muscles to relieve pain and improve flexibility. Skill is needed by the therapist to avoid “bone burns” and other tissue damage caused by overheating and cavitation, sometimes made worse by reflection and focusing of the ultrasound by joint and bone tissue.

In some instances, you may encounter a different decibel scale, called the sound *pressure* level, when ultrasound travels in water or in human and other biological tissues. We shall not use the scale here, but it is notable that numbers for sound pressure levels range 60 to 70 dB higher than you would quote for β , the sound intensity level used in this text. Should you encounter a sound pressure level of 220 decibels, then, it is not an astronomically high intensity, but equivalent to about 155 dB—high enough to destroy tissue, but not as unreasonably high as it might seem at first.

Ultrasound in Medical Diagnostics

When used for imaging, ultrasonic waves are emitted from a transducer, a crystal exhibiting the piezoelectric effect (the expansion and contraction of a substance when a voltage is applied across it, causing a vibration of the crystal). These high-frequency vibrations are transmitted into any tissue in contact with the transducer. Similarly, if a pressure is applied to the crystal (in the form of a wave reflected off tissue layers), a voltage is produced which can be recorded. The crystal therefore acts as both a transmitter and a receiver of sound. Ultrasound is also partially absorbed by tissue on its path, both on its journey away from the transducer and on its return journey. From the time between when the original signal is sent and when the reflections from various boundaries between media are received, (as well as a measure of the intensity loss of the signal), the nature and position of each boundary between tissues and organs may be deduced.

Reflections at boundaries between two different media occur because of differences in a characteristic known as the **acoustic impedance** Z of each substance. Impedance is defined as

Equation:

$$Z = \rho v,$$

where ρ is the density of the medium (in kg/m^3) and v is the speed of sound through the medium (in m/s). The units for Z are therefore $\text{kg}/(\text{m}^2 \cdot \text{s})$.

[\[link\]](#) shows the density and speed of sound through various media (including various soft tissues) and the associated acoustic impedances. Note that the acoustic impedances for soft tissue do not vary much but that there is a big difference between the acoustic impedance of soft tissue and air and also between soft tissue and bone.

Medium	Density (kg/m^3)	Speed of Ultrasound (m/s)	Acoustic Impedance ($\text{kg}/(\text{m}^2 \cdot \text{s})$)
Air	1.3	330	429
Water	1000	1500	1.5×10^6

Medium	Density (kg/m ³)	Speed of Ultrasound (m/s)	Acoustic Impedance (kg/(m ² · s))
Blood	1060	1570	1.66×10^6
Fat	925	1450	1.34×10^6
Muscle (average)	1075	1590	1.70×10^6
Bone (varies)	1400– 1900	4080	5.7×10^6 to 7.8×10^6
Barium titanate (transducer material)	5600	5500	30.8×10^6

The Ultrasound Properties of Various Media, Including Soft Tissue Found in the Body

At the boundary between media of different acoustic impedances, some of the wave energy is reflected and some is transmitted. The greater the *difference* in acoustic impedance between the two media, the greater the reflection and the smaller the transmission.

The **intensity reflection coefficient** a is defined as the ratio of the intensity of the reflected wave relative to the incident (transmitted) wave. This statement can be written mathematically as

Equation:

$$a = \frac{(Z_2 - Z_1)^2}{(Z_1 + Z_2)^2},$$

where Z_1 and Z_2 are the acoustic impedances of the two media making up the boundary. A reflection coefficient of zero (corresponding to total transmission and no reflection) occurs when the acoustic impedances of the two media are the same. An impedance “match” (no reflection) provides an efficient coupling of sound energy from one medium to another. The image formed in an ultrasound is made by tracking reflections (as shown in [\[link\]](#)) and mapping the intensity of the reflected sound waves in a two-dimensional plane.

Example:**Calculate Acoustic Impedance and Intensity Reflection Coefficient: Ultrasound and Fat Tissue**

(a) Using the values for density and the speed of ultrasound given in [\[link\]](#), show that the acoustic impedance of fat tissue is indeed $1.34 \times 10^6 \text{ kg}/(\text{m}^2 \cdot \text{s})$.

(b) Calculate the intensity reflection coefficient of ultrasound when going from fat to muscle tissue.

Strategy for (a)

The acoustic impedance can be calculated using $Z = \rho v$ and the values for ρ and v found in [\[link\]](#).

Solution for (a)

(1) Substitute known values from [\[link\]](#) into $Z = \rho v$.

Equation:

$$Z = \rho v = (925 \text{ kg}/\text{m}^3)(1450 \text{ m}/\text{s})$$

(2) Calculate to find the acoustic impedance of fat tissue.

Equation:

$$1.34 \times 10^6 \text{ kg}/(\text{m}^2 \cdot \text{s})$$

This value is the same as the value given for the acoustic impedance of fat tissue.

Strategy for (b)

The intensity reflection coefficient for any boundary between two media is given by $a = \frac{(Z_2 - Z_1)^2}{(Z_1 + Z_2)^2}$, and the acoustic impedance of muscle is given in [\[link\]](#).

Solution for (b)

Substitute known values into $a = \frac{(Z_2 - Z_1)^2}{(Z_1 + Z_2)^2}$ to find the intensity reflection coefficient:

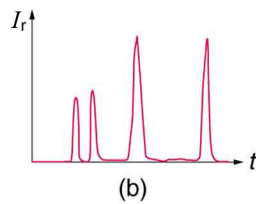
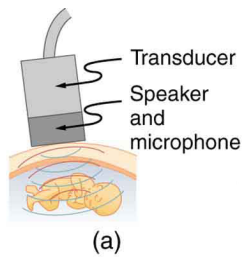
Equation:

$$a = \frac{(Z_2 - Z_1)^2}{(Z_1 + Z_2)^2} = \frac{(1.34 \times 10^6 \text{ kg}/(\text{m}^2 \cdot \text{s}) - 1.70 \times 10^6 \text{ kg}/(\text{m}^2 \cdot \text{s}))^2}{(1.70 \times 10^6 \text{ kg}/(\text{m}^2 \cdot \text{s}) + 1.34 \times 10^6 \text{ kg}/(\text{m}^2 \cdot \text{s}))^2} = 0.014$$

Discussion

This result means that only 1.4% of the incident intensity is reflected, with the remaining being transmitted.

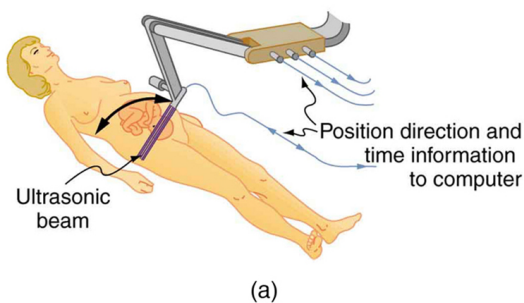
The applications of ultrasound in medical diagnostics have produced untold benefits with no known risks. Diagnostic intensities are too low (about 10^{-2} W/m^2) to cause thermal damage. More significantly, ultrasound has been in use for several decades and detailed follow-up studies do not show evidence of ill effects, quite unlike the case for x-rays.



(a) An ultrasound speaker doubles as a microphone. Brief bleeps are broadcast, and echoes are recorded from various depths. (b) Graph of echo intensity versus time. The time for echoes to return is directly proportional to the distance of

the reflector,
yielding this
information
noninvasively

The most common ultrasound applications produce an image like that shown in [\[link\]](#). The speaker-microphone broadcasts a directional beam, sweeping the beam across the area of interest. This is accomplished by having multiple ultrasound sources in the probe's head, which are phased to interfere constructively in a given, adjustable direction. Echoes are measured as a function of position as well as depth. A computer constructs an image that reveals the shape and density of internal structures.



(a) An ultrasonic image is produced by sweeping the ultrasonic beam across the area of interest, in this case the woman's abdomen. Data are recorded and analyzed in a computer, providing a two-

dimensional image. (b)
Ultrasound image of 12-week-old fetus. (credit: Margaret W. Carruthers, Flickr)

How much detail can ultrasound reveal? The image in [\[link\]](#) is typical of low-cost systems, but that in [\[link\]](#) shows the remarkable detail possible with more advanced systems, including 3D imaging. Ultrasound today is commonly used in prenatal care. Such imaging can be used to see if the fetus is developing at a normal rate, and help in the determination of serious problems early in the pregnancy. Ultrasound is also in wide use to image the chambers of the heart and the flow of blood within the beating heart, using the Doppler effect (echocardiology).

Whenever a wave is used as a probe, it is very difficult to detect details smaller than its wavelength λ . Indeed, current technology cannot do quite this well. Abdominal scans may use a 7-MHz frequency, and the speed of sound in tissue is about 1540 m/s—so the wavelength limit to detail would be $\lambda = \frac{v_w}{f} = \frac{1540 \text{ m/s}}{7 \times 10^6 \text{ Hz}} = 0.22 \text{ mm}$. In practice, 1-mm detail is attainable, which is sufficient for many purposes. Higher-frequency ultrasound would allow greater detail, but it does not penetrate as well as lower frequencies do. The accepted rule of thumb is that you can effectively scan to a depth of about 500λ into tissue. For 7 MHz, this penetration limit is $500 \times 0.22 \text{ mm}$, which is 0.11 m. Higher frequencies may be employed in smaller organs, such as the eye, but are not practical for looking deep into the body.

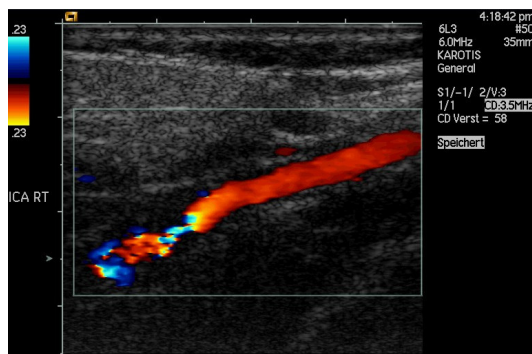


A 3D ultrasound image of a fetus. As well as for the detection of any abnormalities,

such scans have also been shown to be useful for strengthening the emotional bonding between parents and their unborn child. (credit: Jennie Cu, Wikimedia Commons)

In addition to shape information, ultrasonic scans can produce density information superior to that found in X-rays, because the intensity of a reflected sound is related to changes in density. Sound is most strongly reflected at places where density changes are greatest.

Another major use of ultrasound in medical diagnostics is to detect motion and determine velocity through the Doppler shift of an echo, known as **Doppler-shifted ultrasound**. This technique is used to monitor fetal heartbeat, measure blood velocity, and detect occlusions in blood vessels, for example. (See [\[link\]](#).) The magnitude of the Doppler shift in an echo is directly proportional to the velocity of whatever reflects the sound. Because an echo is involved, there is actually a double shift. The first occurs because the reflector (say a fetal heart) is a moving observer and receives a Doppler-shifted frequency. The reflector then acts as a moving source, producing a second Doppler shift.



This Doppler-shifted ultrasonic image of a partially occluded artery uses color to indicate velocity. The highest velocities are in red, while the lowest are blue. The blood must move

faster through the constriction
to carry the same flow. (credit:
Arning C, Grzyska U,
Wikimedia Commons)

A clever technique is used to measure the Doppler shift in an echo. The frequency of the echoed sound is superimposed on the broadcast frequency, producing beats. The beat frequency is $F_B = |f_1 - f_2|$, and so it is directly proportional to the Doppler shift ($f_1 - f_2$) and hence, the reflector's velocity. The advantage in this technique is that the Doppler shift is small (because the reflector's velocity is small), so that great accuracy would be needed to measure the shift directly. But measuring the beat frequency is easy, and it is not affected if the broadcast frequency varies somewhat. Furthermore, the beat frequency is in the audible range and can be amplified for audio feedback to the medical observer.

Note:

Uses for Doppler-Shifted Radar

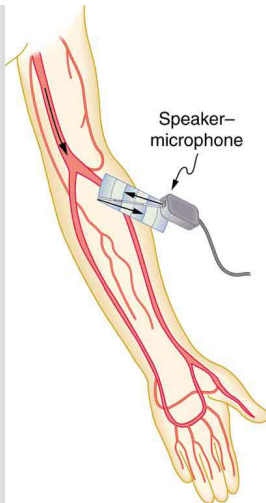
Doppler-shifted radar echoes are used to measure wind velocities in storms as well as aircraft and automobile speeds. The principle is the same as for Doppler-shifted ultrasound. There is evidence that bats and dolphins may also sense the velocity of an object (such as prey) reflecting their ultrasound signals by observing its Doppler shift.

Example:

Calculate Velocity of Blood: Doppler-Shifted Ultrasound

Ultrasound that has a frequency of 2.50 MHz is sent toward blood in an artery that is moving toward the source at 20.0 cm/s, as illustrated in [\[link\]](#). Use the speed of sound in human tissue as 1540 m/s. (Assume that the frequency of 2.50 MHz is accurate to seven significant figures.)

- What frequency does the blood receive?
- What frequency returns to the source?
- What beat frequency is produced if the source and returning frequencies are mixed?



Ultrasound is partly reflected by blood cells and plasma back toward the speaker-microphone. Because the cells are moving, two Doppler shifts are produced—one for blood as a moving observer, and the other for the reflected sound coming from a moving source. The magnitude of the shift is directly proportional to blood velocity.

Strategy

The first two questions can be answered using $f_{\text{obs}} = f_s \left(\frac{v_w}{v_w \pm v_s} \right)$ and

$f_{\text{obs}} = f_s \left(\frac{v_w \pm v_{\text{obs}}}{v_w} \right)$ for the Doppler shift. The last question asks for beat frequency, which is the difference between the original and returning frequencies.

Solution for (a)

(1) Identify knowns:

- The blood is a moving observer, and so the frequency it receives is given by

Equation:

$$f_{\text{obs}} = f_s \left(\frac{v_w \pm v_{\text{obs}}}{v_w} \right).$$

- v_b is the blood velocity (v_{obs} here) and the plus sign is chosen because the motion is toward the source.

(2) Enter the given values into the equation.

Equation:

$$f_{\text{obs}} = (2,500,000 \text{ Hz}) \left(\frac{1540 \text{ m/s} + 0.2 \text{ m/s}}{1540 \text{ m/s}} \right)$$

(3) Calculate to find the frequency: 2,500,325 Hz.

Solution for (b)

(1) Identify knowns:

- The blood acts as a moving source.
- The microphone acts as a stationary observer.
- The frequency leaving the blood is 2,500,325 Hz, but it is shifted upward as given by

Equation:

$$f_{\text{obs}} = f_s \left(\frac{v_w}{v_w - v_b} \right).$$

f_{obs} is the frequency received by the speaker-microphone.

- The source velocity is v_b .
- The minus sign is used because the motion is toward the observer.

The minus sign is used because the motion is toward the observer.

(2) Enter the given values into the equation:

Equation:

$$f_{\text{obs}} = (2,500,325 \text{ Hz}) \left(\frac{1540 \text{ m/s}}{1540 \text{ m/s} - 0.200 \text{ m/s}} \right)$$

(3) Calculate to find the frequency returning to the source: 2,500,649 Hz.

Solution for (c)

(1) Identify knowns:

- The beat frequency is simply the absolute value of the difference between f_s and f_{obs} , as stated in:

Equation:

$$f_B = | f_{\text{obs}} - f_s |.$$

(2) Substitute known values:

Equation:

$$| 2,500,649 \text{ Hz} - 2,500,000 \text{ Hz} |$$

(3) Calculate to find the beat frequency: 649 Hz.

Discussion

The Doppler shifts are quite small compared with the original frequency of 2.50 MHz. It is far easier to measure the beat frequency than it is to measure the echo frequency with an accuracy great enough to see shifts of a few hundred hertz out of a couple of megahertz. Furthermore, variations in the source frequency do not greatly affect the beat frequency, because both f_s and f_{obs} would increase or decrease. Those changes subtract out in $f_B = | f_{\text{obs}} - f_s |$.

Note:

Industrial and Other Applications of Ultrasound

Industrial, retail, and research applications of ultrasound are common. A few are discussed here. Ultrasonic cleaners have many uses. Jewelry, machined parts, and other objects that have odd shapes and crevices are immersed in a cleaning fluid that is agitated with ultrasound typically about 40 kHz in frequency. The intensity is great enough to cause cavitation, which is responsible for most of the cleansing action. Because cavitation-produced shock pressures are large and well transmitted in a fluid,

they reach into small crevices where even a low-surface-tension cleaning fluid might not penetrate.

Sonar is a familiar application of ultrasound. Sonar typically employs ultrasonic frequencies in the range from 30.0 to 100 kHz. Bats, dolphins, submarines, and even some birds use ultrasonic sonar. Echoes are analyzed to give distance and size information both for guidance and finding prey. In most sonar applications, the sound reflects quite well because the objects of interest have significantly different density than the medium in which they travel. When the Doppler shift is observed, velocity information can also be obtained. Submarine sonar can be used to obtain such information, and there is evidence that some bats also sense velocity from their echoes.

Similarly, there are a range of relatively inexpensive devices that measure distance by timing ultrasonic echoes. Many cameras, for example, use such information to focus automatically. Some doors open when their ultrasonic ranging devices detect a nearby object, and certain home security lights turn on when their ultrasonic rangefinders observe motion. Ultrasonic “measuring tapes” also exist to measure such things as room dimensions. Sinks in public restrooms are sometimes automated with ultrasound devices to turn faucets on and off when people wash their hands. These devices reduce the spread of germs and can conserve water.

Ultrasound is used for nondestructive testing in industry and by the military. Because ultrasound reflects well from any large change in density, it can reveal cracks and voids in solids, such as aircraft wings, that are too small to be seen with x-rays. For similar reasons, ultrasound is also good for measuring the thickness of coatings, particularly where there are several layers involved.

Basic research in solid state physics employs ultrasound. Its attenuation is related to a number of physical characteristics, making it a useful probe. Among these characteristics are structural changes such as those found in liquid crystals, the transition of a material to a superconducting phase, as well as density and other properties.

These examples of the uses of ultrasound are meant to whet the appetites of the curious, as well as to illustrate the underlying physics of ultrasound. There are many more applications, as you can easily discover for yourself.

Exercise:

Check Your Understanding

Problem:

Why is it possible to use ultrasound both to observe a fetus in the womb and also to destroy cancerous tumors in the body?

Solution:

Ultrasound can be used medically at different intensities. Lower intensities do not cause damage and are used for medical imaging. Higher intensities can pulverize and destroy targeted substances in the body, such as tumors.

Section Summary

- The acoustic impedance is defined as:

Equation:

$$Z = \rho v,$$

ρ is the density of a medium through which the sound travels and v is the speed of sound through that medium.

- The intensity reflection coefficient a , a measure of the ratio of the intensity of the wave reflected off a boundary between two media relative to the intensity of the incident wave, is given by

Equation:

$$a = \frac{(Z_2 - Z_1)^2}{(Z_1 + Z_2)^2}.$$

- The intensity reflection coefficient is a unitless quantity.

Conceptual Questions**Exercise:****Problem:**

If audible sound follows a rule of thumb similar to that for ultrasound, in terms of its absorption, would you expect the high or low frequencies from your neighbor's stereo to penetrate into your house? How does this expectation compare with your experience?

Exercise:**Problem:**

Elephants and whales are known to use infrasound to communicate over very large distances. What are the advantages of infrasound for long distance communication?

Exercise:**Problem:**

It is more difficult to obtain a high-resolution ultrasound image in the abdominal region of someone who is overweight than for someone who has a slight build. Explain why this statement is accurate.

Exercise:**Problem:**

Suppose you read that 210-dB ultrasound is being used to pulverize cancerous tumors. You calculate the intensity in watts per centimeter squared and find it is unreasonably high (10^5 W/cm^2). What is a possible explanation?

Problems & Exercises

Unless otherwise indicated, for problems in this section, assume that the speed of sound through human tissues is 1540 m/s.

Exercise:**Problem:**

What is the sound intensity level in decibels of ultrasound of intensity 10^5 W/m^2 , used to pulverize tissue during surgery?

Solution:

170 dB

Exercise:**Problem:**

Is 155-dB ultrasound in the range of intensities used for deep heating? Calculate the intensity of this ultrasound and compare this intensity with values quoted in the text.

Exercise:**Problem:**

Find the sound intensity level in decibels of $2.00 \times 10^{-2} \text{ W/m}^2$ ultrasound used in medical diagnostics.

Solution:

103 dB

Exercise:**Problem:**

The time delay between transmission and the arrival of the reflected wave of a signal using ultrasound traveling through a piece of fat tissue was 0.13 ms. At what depth did this reflection occur?

Exercise:**Problem:**

In the clinical use of ultrasound, transducers are always coupled to the skin by a thin layer of gel or oil, replacing the air that would otherwise exist between the transducer and the skin. (a) Using the values of acoustic impedance given in [\[link\]](#) calculate the intensity reflection coefficient between transducer material and air. (b) Calculate the intensity reflection coefficient between transducer material and gel (assuming for this problem that its acoustic impedance is identical to that of water). (c) Based on the results of your calculations, explain why the gel is used.

Solution:

(a) 1.00

(b) 0.823

(c) Gel is used to facilitate the transmission of the ultrasound between the transducer and the patient's body.

Exercise:**Problem:**

(a) Calculate the minimum frequency of ultrasound that will allow you to see details as small as 0.250 mm in human tissue. (b) What is the effective depth to which this sound is effective as a diagnostic probe?

Exercise:

Problem:

(a) Find the size of the smallest detail observable in human tissue with 20.0-MHz ultrasound. (b) Is its effective penetration depth great enough to examine the entire eye (about 3.00 cm is needed)? (c) What is the wavelength of such ultrasound in 0°C air?

Solution:

(a) 77.0 μm

(b) Effective penetration depth = 3.85 cm, which is enough to examine the eye.

(c) 16.6 μm

Exercise:**Problem:**

(a) Echo times are measured by diagnostic ultrasound scanners to determine distances to reflecting surfaces in a patient. What is the difference in echo times for tissues that are 3.50 and 3.60 cm beneath the surface? (This difference is the minimum resolving time for the scanner to see details as small as 0.100 cm, or 1.00 mm. Discrimination of smaller time differences is needed to see smaller details.) (b) Discuss whether the period T of this ultrasound must be smaller than the minimum time resolution. If so, what is the minimum frequency of the ultrasound and is that out of the normal range for diagnostic ultrasound?

Exercise:**Problem:**

(a) How far apart are two layers of tissue that produce echoes having round-trip times (used to measure distances) that differ by 0.750 μs ? (b) What minimum frequency must the ultrasound have to see detail this small?

Solution:

(a) $5.78 \times 10^{-4} \text{ m}$

(b) $2.67 \times 10^6 \text{ Hz}$

Exercise:

Problem:

(a) A bat uses ultrasound to find its way among trees. If this bat can detect echoes 1.00 ms apart, what minimum distance between objects can it detect? (b) Could this distance explain the difficulty that bats have finding an open door when they accidentally get into a house?

Exercise:**Problem:**

A dolphin is able to tell in the dark that the ultrasound echoes received from two sharks come from two different objects only if the sharks are separated by 3.50 m, one being that much farther away than the other. (a) If the ultrasound has a frequency of 100 kHz, show this ability is not limited by its wavelength. (b) If this ability is due to the dolphin's ability to detect the arrival times of echoes, what is the minimum time difference the dolphin can perceive?

Solution:

(a) $v_w = 1540 \text{ m/s} = f\lambda \Rightarrow \lambda = \frac{1540 \text{ m/s}}{100 \times 10^3 \text{ Hz}} = 0.0154 \text{ m} < 3.50 \text{ m}$. Because the wavelength is much shorter than the distance in question, the wavelength is not the limiting factor.

(b) 4.55 ms

Exercise:**Problem:**

A diagnostic ultrasound echo is reflected from moving blood and returns with a frequency 500 Hz higher than its original 2.00 MHz. What is the velocity of the blood? (Assume that the frequency of 2.00 MHz is accurate to seven significant figures and 500 Hz is accurate to three significant figures.)

Exercise:**Problem:**

Ultrasound reflected from an oncoming bloodstream that is moving at 30.0 cm/s is mixed with the original frequency of 2.50 MHz to produce beats. What is the beat frequency? (Assume that the frequency of 2.50 MHz is accurate to seven significant figures.)

Solution:

974 Hz

(Note: extra digits were retained in order to show the difference.)

Glossary

acoustic impedance

property of medium that makes the propagation of sound waves more difficult

intensity reflection coefficient

a measure of the ratio of the intensity of the wave reflected off a boundary between two media relative to the intensity of the incident wave

Doppler-shifted ultrasound

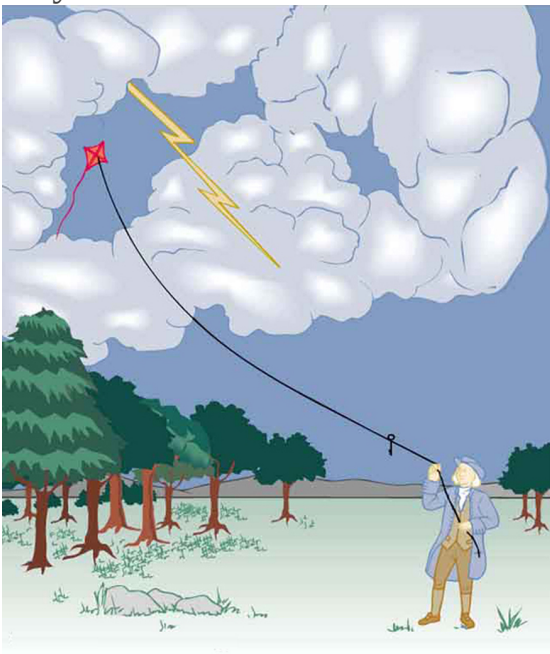
a medical technique to detect motion and determine velocity through the Doppler shift of an echo

Connection for AP® Courses
class="introduction"

Static electricity from this plastic slide causes the child's hair to stand on end. The sliding motion stripped electrons away from the child's body, leaving an excess of positive charges, which repel each other along each strand of hair. (credit: Ken Bosma/Wikimedia Commons)



The image of American politician and scientist Benjamin Franklin (1706–1790) flying a kite in a thunderstorm (shown in Figure 18.2) is familiar to every schoolchild. In this experiment, Franklin demonstrated a connection between lightning and static electricity. Sparks were drawn from a key hung on a kite string during an electrical storm. These sparks were like those produced by static electricity, such as the spark that jumps from your finger to a metal doorknob after you walk across a wool carpet. Much has been written about Franklin. His experiments were only part of the life of a man who was a scientist, inventor, revolutionary, statesman, and writer. Franklin's experiments were not performed in isolation, nor were they the only ones to reveal connections.



Benjamin Franklin, his kite,
and electricity.

When Benjamin Franklin demonstrated that lightning was related to static electricity, he made a connection that is now part of the evidence that all directly experienced forces (except gravitational force) are manifestations of the electromagnetic force. For example, the Italian scientist Luigi Galvani (1737-1798) performed a series of experiments in which static electricity was used to stimulate contractions of leg muscles of dead frogs,

an effect already known in humans subjected to static discharges. But Galvani also found that if he joined one end of two metal wires (say copper and zinc) and touched the other ends of the wires to muscles; he produced the same effect in frogs as static discharge. Alessandro Volta (1745-1827), partly inspired by Galvani's work, experimented with various combinations of metals and developed the battery.

During the same era, other scientists made progress in discovering fundamental connections. The periodic table was developed as systematic properties of the elements were discovered. This influenced the development and refinement of the concept of atoms as the basis of matter. Such submicroscopic descriptions of matter also help explain a great deal more. Atomic and molecular interactions, such as the forces of friction, cohesion, and adhesion, are now known to be manifestations of the electromagnetic force.

Static electricity is just one aspect of the electromagnetic force, which also includes moving electricity and magnetism. All the macroscopic forces that we experience directly, such as the sensations of touch and the tension in a rope, are due to the electromagnetic force, one of the four fundamental forces in nature. The gravitational force, another fundamental force, is actually sensed through the electromagnetic interaction of molecules, such as between those in our feet and those on the top of a bathroom scale. (The other two fundamental forces, the strong nuclear force and the weak nuclear force, cannot be sensed on the human scale.)

This chapter begins the study of electromagnetic phenomena at a fundamental level. The next several chapters will cover static electricity, moving electricity, and magnetism – collectively known as electromagnetism. In this chapter, we begin with the study of electric phenomena due to charges that are at least temporarily stationary, called electrostatics, or static electricity.

The chapter introduces several very important concepts of charge, electric force, and electric field, as well as defining the relationships between these concepts. The charge is defined as a property of a system (Big Idea 1) that can affect its interaction with other charged systems (Enduring Understanding 1.B). The law of conservation of electric charge is also

discussed (Essential Knowledge 1.B.1). The two kinds of electric charge are defined as positive and negative, providing an explanation for having positively charged, negatively charged, or neutral objects (containing equal quantities of positive and negative charges) (Essential Knowledge 1.B.2). The discrete nature of the electric charge is introduced in this chapter by defining the elementary charge as the smallest observed unit of charge that can be isolated, which is the electron charge (Essential Knowledge 1.B.3). The concepts of a system (having internal structure) and of an object (having no internal structure) are implicitly introduced to explain charges carried by the electron and proton (Enduring Understanding 1.A, Essential Knowledge 1.A.1).

An electric field is caused by the presence of charged objects (Enduring Understanding 2.C) and can be used to explain interactions between electrically charged objects (Big Idea 2). The electric force represents the effect of an electric field on a charge placed in the field. The magnitude and direction of the electric force are defined by the magnitude and direction of the electric field and magnitude and sign of the charge (Essential Knowledge 2.C.1). The magnitude of the electric field is proportional to the net charge of the objects that created that field (Essential Knowledge 2.C.2). For the special case of a spherically symmetric charged object, the electric field outside the object is radial, and its magnitude varies as the inverse square of the radial distance from the center of that object (Essential Knowledge 2.C.3). The chapter provides examples of vector field maps for various charged systems, including point charges, spherically symmetric charge distributions, and uniformly charged parallel plates (Essential Knowledge 2.C.1, Essential Knowledge 2.C.2). For multiple point charges, the chapter explains how to find the vector field map by adding the electric field vectors of each individual object, including the special case of two equal charges having opposite signs, known as an electric dipole (Essential Knowledge 2.C.4). The special case of two oppositely charged parallel plates with uniformly distributed electric charge when the electric field is perpendicular to the plates and is constant in both magnitude and direction is described in detail, providing many opportunities for problem solving and applications (Essential Knowledge 2.C.5).

The idea that interactions can be described by forces is also reinforced in this chapter (Big Idea 3). Like all other forces that you have learned about so far, electric force is a vector that affects the motion according to Newton's laws (Enduring Understanding 3.A). It is clearly stated in the chapter that electric force appears as a result of interactions between two charged objects (Essential Knowledge 3.A.3, Essential Knowledge 3.C.2). At the macroscopic level the electric force is a long-range force (Enduring Understanding 3.C); however, at the microscopic level many contact forces, such as friction, can be explained by interatomic electric forces (Essential Knowledge 3.C.4). This understanding of friction is helpful when considering properties of conductors and insulators and the transfer of charge by conduction.

Interactions between systems can result in changes in those systems (Big Idea 4). In the case of charged systems, such interactions can lead to changes of electric properties (Enduring Understanding 4.E), such as charge distribution (Essential Knowledge 4.E.3). Any changes are governed by conservation laws (Big Idea 5). Depending on whether the system is closed or open, certain quantities of the system remain the same or changes in those quantities are equal to the amount of transfer of this quantity from or to the system (Enduring Understanding 5.A). The electric charge is one of these quantities (Essential Knowledge 5.A.2). Therefore, the electric charge of a system is conserved (Enduring Understanding 5.C) and the exchange of electric charge between objects in a system does not change the total electric charge of the system (Essential Knowledge 5.C.2).

Big Idea 1 Objects and systems have properties such as mass and charge. Systems may have internal structure.

Enduring Understanding 1.A The internal structure of a system determines many properties of the system.

Essential Knowledge 1.A.1 A system is an object or a collection of objects. Objects are treated as having no internal structure.

Enduring Understanding 1.B Electric charge is a property of an object or system that affects its interactions with other objects or systems containing charge.

Essential Knowledge 1.B.1 Electric charge is conserved. The net charge of a system is equal to the sum of the charges of all the objects in the system.

Essential Knowledge 1.B.2 There are only two kinds of electric charge. Neutral objects or systems contain equal quantities of positive and negative charge, with the exception of some fundamental particles that have no electric charge.

Essential Knowledge 1.B.3 The smallest observed unit of charge that can be isolated is the electron charge, also known as the elementary charge.

Big Idea 2 Fields existing in space can be used to explain interactions.

Enduring Understanding 2.C An electric field is caused by an object with electric charge.

Essential Knowledge 2.C.1 The magnitude of the electric force F exerted on an object with electric charge q by an electric field (\vec{E}) is $\vec{F} = q\vec{E}$. The direction of the force is determined by the direction of the field and the sign of the charge, with positively charged objects accelerating in the direction of the field and negatively charged objects accelerating in the direction opposite the field. This should include a vector field map for positive point charges, negative point charges, spherically symmetric charge distribution, and uniformly charged parallel plates.

Essential Knowledge 2.C.2 The magnitude of the electric field vector is proportional to the net electric charge of the object(s) creating that field. This includes positive point charges, negative point charges, spherically symmetric charge distributions, and uniformly charged parallel plates.

Essential Knowledge 2.C.3 The electric field outside a spherically symmetric charged object is radial, and its magnitude varies as the inverse square of the radial distance from the center of that object. Electric field lines are not in the curriculum. Students will be expected to rely only on the rough intuitive sense underlying field lines, wherein the field is viewed as analogous to something emanating uniformly from a source.

Essential Knowledge 2.C.4 The electric field around dipoles and other systems of electrically charged objects (that can be modeled as point objects) is found by vector addition of the field of each individual object. Electric dipoles are treated qualitatively in this course as a teaching analogy to facilitate student understanding of magnetic dipoles.

Essential Knowledge 2.C.5 Between two oppositely charged parallel plates with uniformly distributed electric charge, at points far from the edges of the plates, the electric field is perpendicular to the plates and is constant in both magnitude and direction.

Big Idea 3 The interactions of an object with other objects can be described by forces.

Enduring Understanding 3.A All forces share certain common characteristics when considered by observers in inertial reference frames.

Essential Knowledge 3.A.3 A force exerted on an object is always due to the interaction of that object with another object.

Enduring Understanding 3.C At the macroscopic level, forces can be categorized as either long-range (action-at-a-distance) forces or contact forces.

Essential Knowledge 3.C.2 Electric force results from the interaction of one object that has an electric charge with another object that has an electric charge.

Essential Knowledge 3.C.4 Contact forces result from the interaction of one object touching another object, and they arise from interatomic electric forces. These forces include tension, friction, normal, spring (Physics 1), and buoyant (Physics 2).

Big Idea 4 Interactions between systems can result in changes in those systems.

Enduring Understanding 4.E The electric and magnetic properties of a system can change in response to the presence of, or changes in, other

objects or systems.

Essential Knowledge 4.E.3 The charge distribution in a system can be altered by the effects of electric forces produced by a charged object.

Big Idea 5 Changes that occur as a result of interactions are constrained by conservation laws.

Enduring Understanding 5.A Certain quantities are conserved, in the sense that the changes of those quantities in a given system are always equal to the transfer of that quantity to or from the system by all possible interactions with other systems.

Essential Knowledge 5.A.2 For all systems under all circumstances, energy, charge, linear momentum, and angular momentum are conserved.

Enduring Understanding 5.C The electric charge of a system is conserved.

Essential Knowledge 5.C.2 The exchange of electric charges among a set of objects in a system conserves electric charge.

Glossary

static electricity

a buildup of electric charge on the surface of an object

electromagnetic force

one of the four fundamental forces of nature; the electromagnetic force consists of static electricity, moving electricity and magnetism

Static Electricity and Charge: Conservation of Charge

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define electric charge, and describe how the two types of charge interact.
- Describe three common situations that generate static electricity.
- State the law of conservation of charge.

The information presented in this section supports the following AP[®] learning objectives and science practices:

- **1.B.1.1** The student is able to make claims about natural phenomena based on conservation of electric charge. **(S.P. 6.4)**
- **1.B.1.2** The student is able to make predictions, using the conservation of electric charge, about the sign and relative quantity of net charge of objects or systems after various charging processes, including conservation of charge in simple circuits. **(S.P. 6.4, 7.2)**
- **1.B.2.1** The student is able to construct an explanation of the two-charge model of electric charge based on evidence produced through scientific practices. **(S.P. 6.4)**
- **1.B.3.1** The student is able to challenge the claim that an electric charge smaller than the elementary charge has been isolated. **(S.P. 1.5, 6.1, 7.2)**
- **5.A.2.1** The student is able to define open and closed systems for everyday situations and apply conservation concepts for energy, charge, and linear momentum to those situations. **(S.P. 6.4, 7.2)**

- **5.C.2.1** The student is able to predict electric charges on objects within a system by application of the principle of charge conservation within a system. **(S.P. 6.4)**
- **5.C.2.2** The student is able to design a plan to collect data on the electrical charging of objects and electric charge induction on neutral objects and qualitatively analyze that data. **(S.P. 4.2, 5.1)**
- **5.C.2.3** The student is able to justify the selection of data relevant to an investigation of the electrical charging of objects and electric charge induction on neutral objects. **(S.P. 4.1)**



Borneo amber was mined in Sabah, Malaysia, from shale-sandstone-mudstone veins.

When a piece of amber is rubbed with a piece of silk, the amber gains more electrons, giving it a net negative charge.

At the same time, the silk, having lost electrons, becomes positively charged. (credit:

What makes plastic wrap cling? Static electricity. Not only are applications of static electricity common these days, its existence has been known since ancient times. The first record of its effects dates to ancient Greeks who noted more than 500 years B.C. that polishing amber temporarily enabled it to attract bits of straw (see [\[link\]](#)). The very word *electric* derives from the Greek word for amber (*electron*).

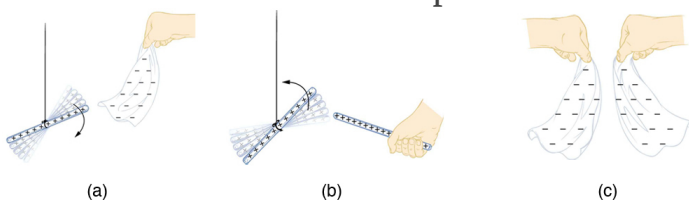
Many of the characteristics of static electricity can be explored by rubbing things together. Rubbing creates the spark you get from walking across a wool carpet, for example. Static cling generated in a clothes dryer and the attraction of straw to recently polished amber also result from rubbing. Similarly, lightning results from air movements under certain weather conditions. You can also rub a balloon on your hair, and the static electricity created can then make the balloon cling to a wall. We also have to be cautious of static electricity, especially in dry climates. When we pump gasoline, we are warned to discharge ourselves (after sliding across the seat) on a metal surface before grabbing the gas nozzle. Attendants in hospital operating rooms must wear booties with aluminum foil on the bottoms to avoid creating sparks which may ignite the oxygen being used.

Some of the most basic characteristics of static electricity include:

- The effects of static electricity are explained by a physical quantity not previously introduced, called electric charge.
- There are only two types of charge, one called positive and the other called negative.
- Like charges repel, whereas unlike charges attract.
- The force between charges decreases with distance.

How do we know there are two types of **electric charge**? When various materials are rubbed together in controlled ways, certain combinations of materials always produce one type of charge on one material and the opposite type on the other. By convention, we call one type of charge “positive”, and the other type “negative.” For example, when glass is

rubbed with silk, the glass becomes positively charged and the silk negatively charged. Since the glass and silk have opposite charges, they attract one another like clothes that have rubbed together in a dryer. Two glass rods rubbed with silk in this manner will repel one another, since each rod has positive charge on it. Similarly, two silk cloths so rubbed will repel, since both cloths have negative charge. [\[link\]](#) shows how these simple materials can be used to explore the nature of the force between charges.



A glass rod becomes positively charged when rubbed with silk, while the silk becomes negatively charged.

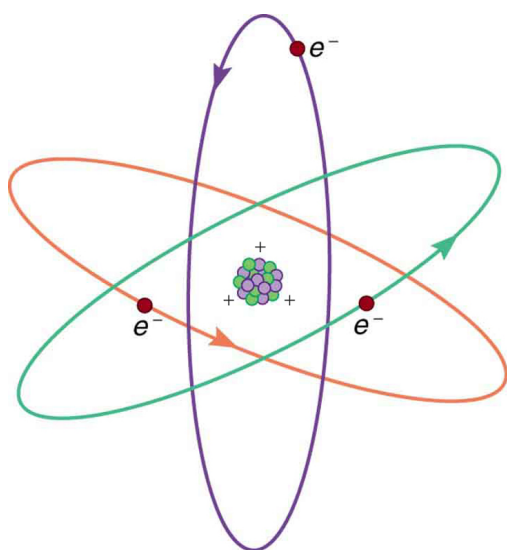
(a) The glass rod is attracted to the silk because their charges are opposite. (b) Two similarly charged glass rods repel. (c) Two similarly charged silk cloths repel.

More sophisticated questions arise. Where do these charges come from? Can you create or destroy charge? Is there a smallest unit of charge? Exactly how does the force depend on the amount of charge and the distance between charges? Such questions obviously occurred to Benjamin Franklin and other early researchers, and they interest us even today.

Charge Carried by Electrons and Protons

Franklin wrote in his letters and books that he could see the effects of electric charge but did not understand what caused the phenomenon. Today we have the advantage of knowing that normal matter is made of atoms, and that atoms contain positive and negative charges, usually in equal amounts.

[\[link\]](#) shows a simple model of an atom with negative **electrons** orbiting its positive nucleus. The nucleus is positive due to the presence of positively charged **protons**. Nearly all charge in nature is due to electrons and protons, which are two of the three building blocks of most matter. (The third is the neutron, which is neutral, carrying no charge.) Other charge-carrying particles are observed in cosmic rays and nuclear decay, and are created in particle accelerators. All but the electron and proton survive only a short time and are quite rare by comparison.



This simplified (and not to scale) view of an atom is called the planetary model of the atom.

Negative electrons orbit a much heavier positive nucleus, as the planets orbit the much heavier sun. There the similarity ends, because forces in the atom are electromagnetic, whereas those in the planetary system are gravitational.

Normal macroscopic amounts of matter contain immense numbers of atoms and molecules and, hence, even greater numbers of individual negative and positive charges.

The charges of electrons and protons are identical in magnitude but opposite in sign. Furthermore, all charged objects in nature are integral multiples of this basic quantity of charge, meaning that all charges are made of combinations of a basic unit of charge. Usually, charges are formed by combinations of electrons and protons. The magnitude of this basic charge is

Equation:

$$|q_e| = 1.60 \times 10^{-19} \text{ C}.$$

The symbol q is commonly used for charge and the subscript e indicates the charge of a single electron (or proton).

The SI unit of charge is the coulomb (C). The number of protons needed to make a charge of 1.00 C is

Equation:

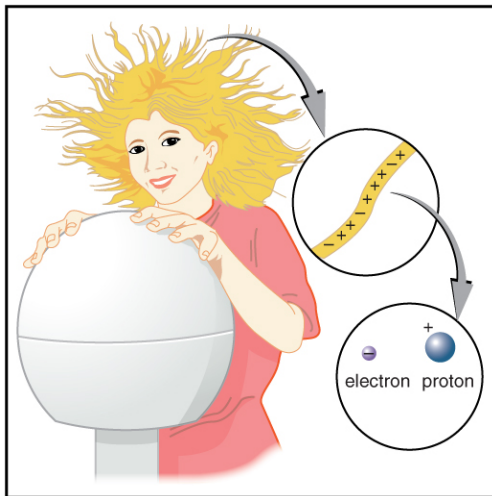
$$1.00 \text{ C} \times \frac{1 \text{ proton}}{1.60 \times 10^{-19} \text{ C}} = 6.25 \times 10^{18} \text{ protons}.$$

Similarly, 6.25×10^{18} electrons have a combined charge of -1.00 coulomb. Just as there is a smallest bit of an element (an atom), there is a smallest bit of charge. There is no directly observed charge smaller than $|q_e|$ (see [Things Great and Small: The Submicroscopic Origin of Charge](#)), and all observed charges are integral multiples of $|q_e|$.

Note:**Things Great and Small: The Submicroscopic Origin of Charge**

With the exception of exotic, short-lived particles, all charge in nature is carried by electrons and protons. Electrons carry the charge we have named negative. Protons carry an equal-magnitude charge that we call positive. (See [\[link\]](#).) Electron and proton charges are considered fundamental building blocks, since all other charges are integral multiples of those carried by electrons and protons. Electrons and protons are also two of the three fundamental building blocks of ordinary matter. The neutron is the third and has zero total charge.

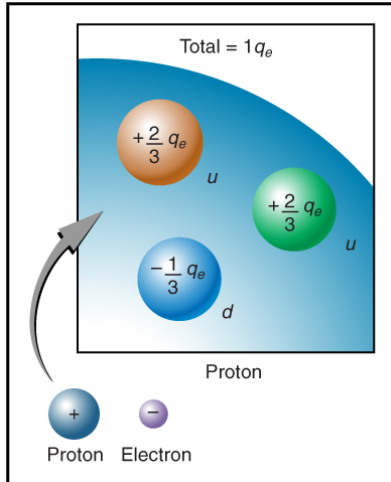
[\[link\]](#) shows a person touching a Van de Graaff generator and receiving excess positive charge. The expanded view of a hair shows the existence of both types of charges but an excess of positive. The repulsion of these positive like charges causes the strands of hair to repel other strands of hair and to stand up. The further blowup shows an artist's conception of an electron and a proton perhaps found in an atom in a strand of hair.



When this person touches
a Van de Graaff
generator, she receives an
excess of positive charge,

causing her hair to stand on end. The charges in one hair are shown. An artist's conception of an electron and a proton illustrate the particles carrying the negative and positive charges. We cannot really see these particles with visible light because they are so small (the electron seems to be an infinitesimal point), but we know a great deal about their measurable properties, such as the charges they carry.

The electron seems to have no substructure; in contrast, when the substructure of protons is explored by scattering extremely energetic electrons from them, it appears that there are point-like particles inside the proton. These sub-particles, named quarks, have never been directly observed, but they are believed to carry fractional charges as seen in [\[link\]](#). Charges on electrons and protons and all other directly observable particles are unitary, but these quark substructures carry charges of either $-\frac{1}{3}$ or $+\frac{2}{3}$. There are continuing attempts to observe fractional charge directly and to learn of the properties of quarks, which are perhaps the ultimate substructure of matter.



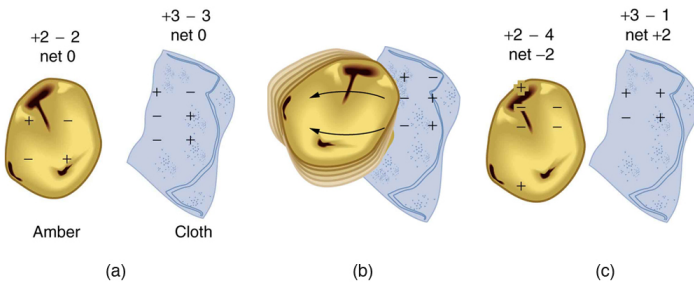
Artist's conception of fractional quark charges inside a proton. A group of three quark charges add up to the single positive charge on the proton:

$$-\frac{1}{3}q_e + \frac{2}{3}q_e + \frac{2}{3}q_e = +1q_e$$

.

Separation of Charge in Atoms

Charges in atoms and molecules can be separated—for example, by rubbing materials together. Some atoms and molecules have a greater affinity for electrons than others and will become negatively charged by close contact in rubbing, leaving the other material positively charged. (See [link](#).) Positive charge can similarly be induced by rubbing. Methods other than rubbing can also separate charges. Batteries, for example, use combinations of substances that interact in such a way as to separate charges. Chemical interactions may transfer negative charge from one substance to the other, making one battery terminal negative and leaving the first one positive.



When materials are rubbed together, charges can be separated, particularly if one material has a greater affinity for electrons than another. (a) Both the amber and cloth are originally neutral, with equal positive and negative charges. Only a tiny fraction of the charges are involved, and only a few of them are shown here. (b) When rubbed together, some negative charge is transferred to the amber, leaving the cloth with a net positive charge. (c) When separated, the amber and cloth now have net charges, but the absolute value of the net positive and negative charges will be equal.

No charge is actually created or destroyed when charges are separated as we have been discussing. Rather, existing charges are moved about. In fact, in all situations the total amount of charge is always constant. This universally obeyed law of nature is called the **law of conservation of charge**.

Note:

Law of Conservation of Charge

Total charge is constant in any process.

Note:**Making Connections: Net Charge**

Hence if a closed system is neutral, it will remain neutral. Similarly, if a closed system has a charge, say, $-10e$, it will always have that charge. The only way to change the charge of a system is to transfer charge outside, either by bringing in charge or removing charge. If it is possible to transfer charge outside, the system is no longer closed/isolated and is known as an open system. However, charge is always conserved, for both open and closed systems. Consequently, the charge transferred to/from an open system is equal to the change in the system's charge.

For example, each of the two materials (amber and cloth) discussed in [\[link\]](#) have no net charge initially. The only way to change their charge is to transfer charge from outside each object. When they are rubbed together, negative charge is transferred to the amber and the final charge of the amber is the sum of the initial charge and the charge transferred to it. On the other hand, the final charge on the cloth is equal to its initial charge minus the charge transferred out.

Similarly when glass is rubbed with silk, the net charge on the silk is its initial charge plus the incoming charge and the charge on the glass is the initial charge minus the outgoing charge. Also the charge gained by the silk will be equal to the charge lost by the glass, which means that if the silk gains $-5e$ charge, the glass would have lost $-5e$ charge.

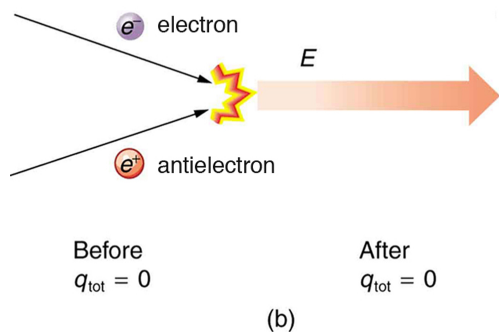
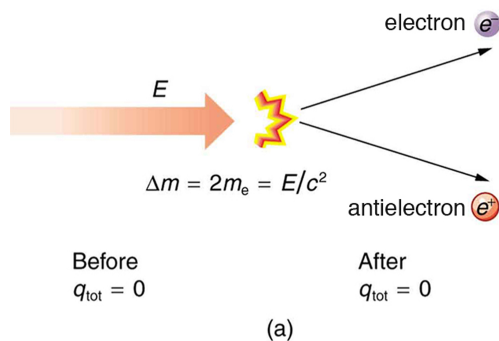
In more exotic situations, such as in particle accelerators, mass, Δm , can be created from energy in the amount $\Delta m = \frac{E}{c^2}$. Sometimes, the created mass is charged, such as when an electron is created. Whenever a charged particle is created, another having an opposite charge is always created along with it, so that the total charge created is zero. Usually, the two particles are “matter-antimatter” counterparts. For example, an antielectron would usually be created at the same time as an electron. The antielectron has a positive charge (it is called a positron), and so the total charge created is zero. (See [\[link\]](#).) All particles have antimatter counterparts with opposite signs. When matter and antimatter counterparts are brought together, they completely annihilate one another. By annihilate, we mean that the mass of the two particles is converted to energy E , again obeying the relationship

$\Delta m = \frac{E}{c^2}$. Since the two particles have equal and opposite charge, the total charge is zero before and after the annihilation; thus, total charge is conserved.

Note:

Making Connections: Conservation Laws

Only a limited number of physical quantities are universally conserved. Charge is one—energy, momentum, and angular momentum are others. Because they are conserved, these physical quantities are used to explain more phenomena and form more connections than other, less basic quantities. We find that conserved quantities give us great insight into the rules followed by nature and hints to the organization of nature. Discoveries of conservation laws have led to further discoveries, such as the weak nuclear force and the quark substructure of protons and other particles.



(a) When enough energy is present, it can be converted into matter.

Here the matter created is an electron–antielectron pair. (m_e is the electron's mass.) The total charge before and after this event is zero. (b) When matter and antimatter collide, they annihilate each other; the total charge is conserved at zero before and after the annihilation.

The law of conservation of charge is absolute—it has never been observed to be violated. Charge, then, is a special physical quantity, joining a very short list of other quantities in nature that are always conserved. Other conserved quantities include energy, momentum, and angular momentum.

Note:

PhET Explorations: Balloons and Static Electricity

Why does a balloon stick to your sweater? Rub a balloon on a sweater, then let go of the balloon and it flies over and sticks to the sweater. View the charges in the sweater, balloons, and the wall.

[Balloons
and Static
Electricity](#)

Note:**Applying the Science Practices: Electrical Charging**

Design an experiment to demonstrate the electrical charging of objects, by using a glass rod, a balloon, small bits of paper, and different pieces of cloth (like silk, wool, or nylon). Also show that like charges repel each other whereas unlike charges attract each other.

Test Prep for AP Courses**Exercise:****Problem:**

When a glass rod is rubbed against silk, which of the following statements is true?

- a. Electrons are removed from the silk.
- b. Electrons are removed from the rod.
- c. Protons are removed from the silk.
- d. Protons are removed from the rod.

Solution:

(b)

Exercise:**Problem:**

In an experiment, three microscopic latex spheres are sprayed into a chamber and become charged with $+3e$, $+5e$, and $-3e$, respectively. Later, all three spheres collide simultaneously and then separate. Which of the following are possible values for the final charges on the spheres? Select two answers.

	X	Y	Z
(a)	$+4e$	$-4e$	$+5e$
(b)	$-4e$	$+4.5e$	$+5.5e$
(c)	$+5e$	$-8e$	$+7e$
(d)	$+6e$	$+6e$	$-7e$

Exercise:

Problem:

If objects X and Y attract each other, which of the following will be false?

- a. X has positive charge and Y has negative charge.
- b. X has negative charge and Y has positive charge.
- c. X and Y both have positive charge.
- d. X is neutral and Y has a charge.

Solution:

(c)

Exercise:

Problem:

Suppose a positively charged object A is brought in contact with an uncharged object B in a closed system. What type of charge will be left on object B?

- a. negative
- b. positive
- c. neutral

d. cannot be determined

Exercise:

Problem:

What will be the net charge on an object which attracts neutral pieces of paper but repels a negatively charged balloon?

- a. negative
- b. positive
- c. neutral
- d. cannot be determined

Solution:

(a)

Exercise:

Problem:

When two neutral objects are rubbed against each other, the first one gains a net charge of $3e$. Which of the following statements is true?

- a. The second object gains $3e$ and is negatively charged.
- b. The second object loses $3e$ and is negatively charged.
- c. The second object gains $3e$ and is positively charged.
- d. The second object loses $3e$ and is positively charged.

Exercise:

Problem:

In an experiment, a student runs a comb through his hair several times and brings it close to small pieces of paper. Which of the following will he observe?

- a. Pieces of paper repel the comb.

- b. Pieces of paper are attracted to the comb.
 - c. Some pieces of paper are attracted and some repel the comb.
 - d. There is no attraction or repulsion between the pieces of paper and the comb.
-

Solution:

(b)

Exercise:

Problem:

In an experiment a negatively charged balloon (balloon X) is repelled by another charged balloon Y. However, an object Z is attracted to balloon Y. Which of the following can be the charge on Z? Select *two* answers.

- a. negative
- b. positive
- c. neutral
- d. cannot be determined

Exercise:

Problem:

Suppose an object has a charge of 1 C and gains 6.88×10^{18} electrons.

- a. What will be the net charge of the object?
 - b. If the object has gained electrons from a neutral object, what will be the charge on the neutral object?
 - c. Find and explain the relationship between the total charges of the two objects before and after the transfer.
 - d. When a third object is brought in contact with the first object (after it gains the electrons), the resulting charge on the third object is 0.4 C. What was its initial charge?
-

Solution:

(a) -0.1 C , (b) 1.1 C , (c) Both charges will be equal to 1 C , law of conservation of charge, (d) 0.9 C

Exercise:**Problem:**

The charges on two identical metal spheres (placed in a closed system) are $-2.4 \times 10^{-17}\text{ C}$ and $-4.8 \times 10^{-17}\text{ C}$.

- How many electrons will be equivalent to the charge on each sphere?
- If the two spheres are brought in contact and then separated, find the charge on each sphere.
- Calculate the number of electrons that would be equivalent to the resulting charge on each sphere.

Exercise:**Problem:**

In an experiment the following observations are made by a student for four charged objects W, X, Y, and Z:

- A glass rod rubbed with silk attracts W.
- W attracts Z but repels X.
- X attracts Z but repels Y.
- Y attracts W and Z.

Estimate whether the charges on each of the four objects are positive, negative, or neutral.

Solution:

W is negative, X is positive, Y is negative, Z is neutral.

Section Summary

- There are only two types of charge, which we call positive and negative.
- Like charges repel, unlike charges attract, and the force between charges decreases with the square of the distance.
- The vast majority of positive charge in nature is carried by protons, while the vast majority of negative charge is carried by electrons.
- The electric charge of one electron is equal in magnitude and opposite in sign to the charge of one proton.
- An ion is an atom or molecule that has nonzero total charge due to having unequal numbers of electrons and protons.
- The SI unit for charge is the coulomb (C), with protons and electrons having charges of opposite sign but equal magnitude; the magnitude of this basic charge $|q_e|$ is

Equation:

$$|q_e| = 1.60 \times 10^{-19} \text{ C}.$$

- Whenever charge is created or destroyed, equal amounts of positive and negative are involved.
- Most often, existing charges are separated from neutral objects to obtain some net charge.
- Both positive and negative charges exist in neutral objects and can be separated by rubbing one object with another. For macroscopic objects, negatively charged means an excess of electrons and positively charged means a depletion of electrons.
- The law of conservation of charge ensures that whenever a charge is created, an equal charge of the opposite sign is created at the same time.

Conceptual Questions

Exercise:

Problem:

There are very large numbers of charged particles in most objects. Why, then, don't most objects exhibit static electricity?

Exercise:**Problem:**

Why do most objects tend to contain nearly equal numbers of positive and negative charges?

Problems & Exercises**Exercise:****Problem:**

Common static electricity involves charges ranging from nanocoulombs to microcoulombs. (a) How many electrons are needed to form a charge of -2.00 nC (b) How many electrons must be removed from a neutral object to leave a net charge of $0.500 \mu\text{C}$?

Solution:

(a) 1.25×10^{10}

(b) 3.13×10^{12}

Exercise:**Problem:**

If 1.80×10^{20} electrons move through a pocket calculator during a full day's operation, how many coulombs of charge moved through it?

Exercise:

Problem:

To start a car engine, the car battery moves 3.75×10^{21} electrons through the starter motor. How many coulombs of charge were moved?

Solution:

-600 C

Exercise:**Problem:**

A certain lightning bolt moves 40.0 C of charge. How many fundamental units of charge $|q_e|$ is this?

Glossary

electric charge

a physical property of an object that causes it to be attracted toward or repelled from another charged object; each charged object generates and is influenced by a force called an electromagnetic force

law of conservation of charge

states that whenever a charge is created, an equal amount of charge with the opposite sign is created simultaneously

electron

a particle orbiting the nucleus of an atom and carrying the smallest unit of negative charge

proton

a particle in the nucleus of an atom and carrying a positive charge equal in magnitude and opposite in sign to the amount of negative charge carried by an electron

Conductors and Insulators

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define conductor and insulator, explain the difference, and give examples of each.
- Describe three methods for charging an object.
- Explain what happens to an electric force as you move farther from the source.
- Define polarization.

The information presented in this section supports the following AP® learning objectives and science practices:

- **1.B.2.2** The student is able to make a qualitative prediction about the distribution of positive and negative electric charges within neutral systems as they undergo various processes. **(S.P. 6.4, 7.2)**
- **1.B.2.3** The student is able to challenge claims that polarization of electric charge or separation of charge must result in a net charge on the object. **(S.P. 6.1)**
- **4.E.3.1** The student is able to make predictions about the redistribution of charge during charging by friction, conduction, and induction. **(S.P. 6.4)**
- **4.E.3.2** The student is able to make predictions about the redistribution of charge caused by the electric field due to other systems, resulting in charged or polarized objects. **(S.P. 6.4, 7.2)**
- **4.E.3.3** The student is able to construct a representation of the distribution of fixed and mobile charge in insulators and conductors. **(S.P. 1.1, 1.4, 6.4)**

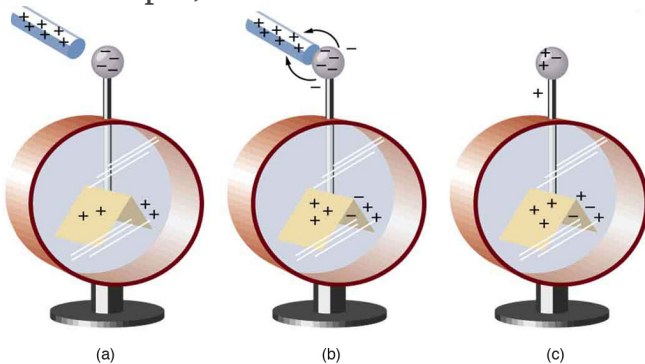
- **4.E.3.4** The student is able to construct a representation of the distribution of fixed and mobile charge in insulators and conductors that predicts charge distribution in processes involving induction or conduction. (S.P. 1.1, 1.4, 6.4)
- **4.E.3.5** The student is able to plan and/or analyze the results of experiments in which electric charge rearrangement occurs by electrostatic induction, or is able to refine a scientific question relating to such an experiment by identifying anomalies in a data set or procedure. (S.P. 3.2, 4.1, 4.2, 5.1, 5.3)



This power adapter uses metal wires and connectors to conduct electricity from the wall socket to a laptop computer. The conducting wires allow electrons to move freely through the cables, which are shielded by rubber and plastic. These materials act as insulators that don't allow electric charge to escape outward. (credit: Evan-Amos, Wikimedia Commons)

Some substances, such as metals and salty water, allow charges to move through them with relative ease. Some of the electrons in metals and similar conductors are not bound to individual atoms or sites in the material. These **free electrons** can move through the material much as air moves through loose sand. Any substance that has free electrons and allows charge to move relatively freely through it is called a **conductor**. The moving electrons may collide with fixed atoms and molecules, losing some energy, but they can move in a conductor. Superconductors allow the movement of charge without any loss of energy. Salty water and other similar conducting materials contain free ions that can move through them. An ion is an atom or molecule having a positive or negative (nonzero) total charge. In other words, the total number of electrons is not equal to the total number of protons.

Other substances, such as glass, do not allow charges to move through them. These are called **insulators**. Electrons and ions in insulators are bound in the structure and cannot move easily—as much as 10^{23} times more slowly than in conductors. Pure water and dry table salt are insulators, for example, whereas molten salt and salty water are conductors.



An electroscope is a favorite instrument in physics demonstrations and student laboratories. It is typically made with gold foil leaves hung from a (conducting) metal stem and is insulated from the room air in a glass-walled container. (a) A positively charged glass rod is

brought near the tip of the electroscope, attracting electrons to the top and leaving a net positive charge on the leaves. Like charges in the light flexible gold leaves repel, separating them. (b) When the rod is touched against the ball, electrons are attracted and transferred, reducing the net charge on the glass rod but leaving the electroscope positively charged. (c) The excess charges are evenly distributed in the stem and leaves of the electroscope once the glass rod is removed.

Charging by Contact

[\[link\]](#) shows an electroscope being charged by touching it with a positively charged glass rod. Because the glass rod is an insulator, it must actually touch the electroscope to transfer charge to or from it. (Note that the extra positive charges reside on the surface of the glass rod as a result of rubbing it with silk before starting the experiment.) Since only electrons move in metals, we see that they are attracted to the top of the electroscope. There, some are transferred to the positive rod by touch, leaving the electroscope with a net positive charge.

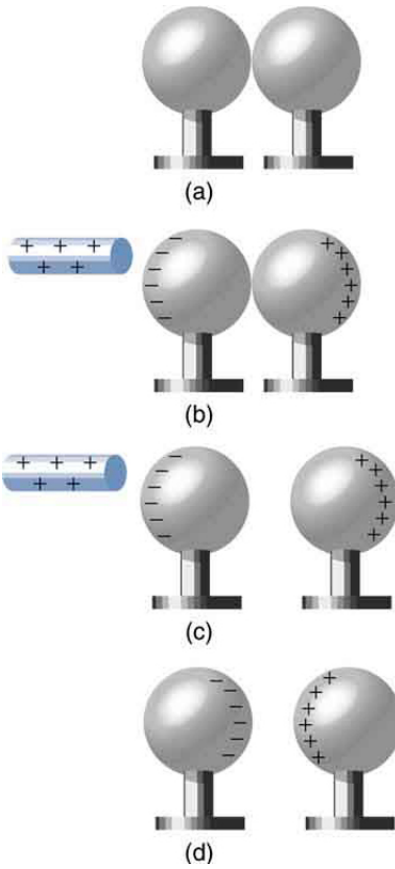
Electrostatic repulsion in the leaves of the charged electroscope separates them. The electrostatic force has a horizontal component that results in the leaves moving apart as well as a vertical component that is balanced by the gravitational force. Similarly, the electroscope can be negatively charged by contact with a negatively charged object.

Charging by Induction

It is not necessary to transfer excess charge directly to an object in order to charge it. [\[link\]](#) shows a method of **induction** wherein a charge is created in a nearby object, without direct contact. Here we see two neutral metal spheres in contact with one another but insulated from the rest of the world. A positively charged rod is brought near one of them, attracting negative charge to that side, leaving the other sphere positively charged.

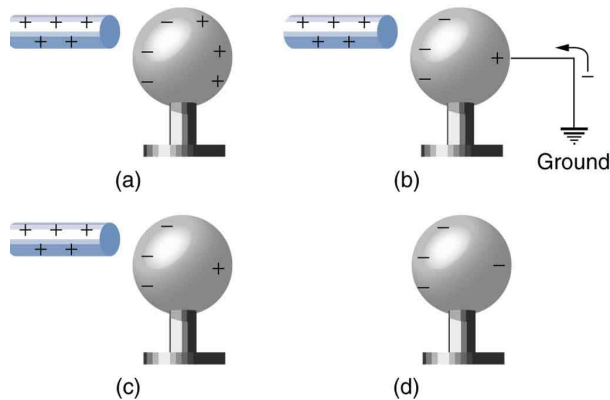
This is an example of induced **polarization** of neutral objects. Polarization is the separation of charges in an object that remains neutral. If the spheres are now separated (before the rod is pulled away), each sphere will have a net charge. Note that the object closest to the charged rod receives an opposite charge when charged by induction. Note also that no charge is removed from the charged rod, so that this process can be repeated without depleting the supply of excess charge.

Another method of charging by induction is shown in [\[link\]](#). The neutral metal sphere is polarized when a charged rod is brought near it. The sphere is then grounded, meaning that a conducting wire is run from the sphere to the ground. Since the earth is large and most ground is a good conductor, it can supply or accept excess charge easily. In this case, electrons are attracted to the sphere through a wire called the ground wire, because it supplies a conducting path to the ground. The ground connection is broken before the charged rod is removed, leaving the sphere with an excess charge opposite to that of the rod. Again, an opposite charge is achieved when charging by induction and the charged rod loses none of its excess charge.



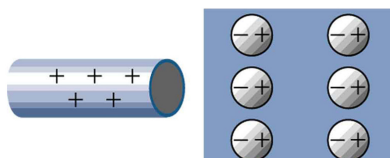
Charging by induction. (a) Two uncharged or neutral metal spheres are in contact with each other but insulated from the rest of the world. (b) A positively charged glass rod is brought near the sphere on the left, attracting negative charge and leaving the other sphere positively charged. (c) The

spheres are separated before the rod is removed, thus separating negative and positive charge. (d) The spheres retain net charges after the inducing rod is removed—without ever having been touched by a charged object.

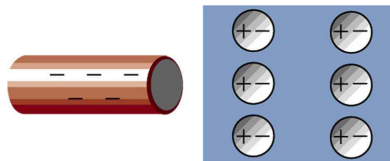


Charging by induction, using a ground connection. (a) A positively charged rod is brought near a neutral metal sphere, polarizing it. (b) The sphere is grounded, allowing electrons to be attracted from the earth's ample supply. (c) The ground connection is broken. (d) The positive rod is removed,

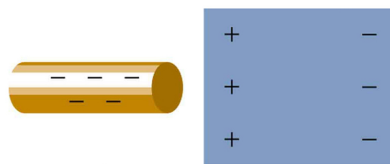
leaving the sphere with an induced negative charge.



(a)



(b)



(c)

Both positive and negative objects attract a neutral object by polarizing its molecules. (a) A positive object brought near a neutral insulator polarizes its molecules. There is a slight shift in the distribution of the electrons orbiting the molecule, with unlike charges

being brought nearer and like charges moved away. Since the electrostatic force decreases with distance, there is a net attraction. (b) A negative object produces the opposite polarization, but again attracts the neutral object. (c) The same effect occurs for a conductor; since the unlike charges are closer, there is a net attraction.

Neutral objects can be attracted to any charged object. The pieces of straw attracted to polished amber are neutral, for example. If you run a plastic comb through your hair, the charged comb can pick up neutral pieces of paper. [\[link\]](#) shows how the polarization of atoms and molecules in neutral objects results in their attraction to a charged object.

When a charged rod is brought near a neutral substance, an insulator in this case, the distribution of charge in atoms and molecules is shifted slightly. Opposite charge is attracted nearer the external charged rod, while like charge is repelled. Since the electrostatic force decreases with distance, the repulsion of like charges is weaker than the attraction of unlike charges, and so there is a net attraction. Thus a positively charged glass rod attracts neutral pieces of paper, as will a negatively charged rubber rod. Some molecules, like water, are polar molecules. Polar molecules have a natural

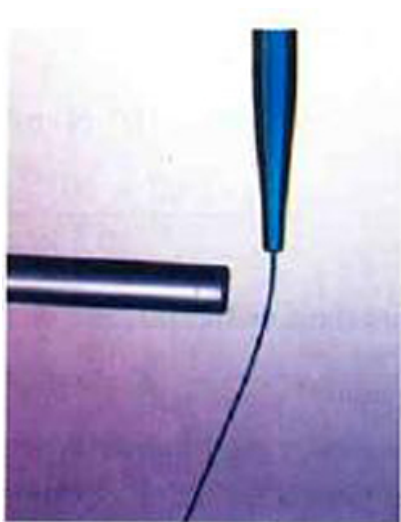
or inherent separation of charge, although they are neutral overall. Polar molecules are particularly affected by other charged objects and show greater polarization effects than molecules with naturally uniform charge distributions.

Exercise:

Check Your Understanding

Problem:

Can you explain the attraction of water to the charged rod in the figure below?



Solution:

Answer

Water molecules are polarized, giving them slightly positive and slightly negative sides. This makes water even more susceptible to a charged rod's attraction. As the water flows downward, due to the force of gravity, the charged conductor exerts a net attraction to the opposite charges in the stream of water, pulling it closer.

Note:

Applying the Science Practices: Electrostatic Induction

Plan an experiment to demonstrate electrostatic induction using household items, like balloons, woolen cloth, aluminum drink cans, or foam cups. Explain the process of induction in your experiment by discussing details of (and making diagrams relating to) the movement and alignment of charges.

Note:

PhET Explorations: John Travoltage

Make sparks fly with John Travoltage. Wiggle Johnnie's foot and he picks up charges from the carpet. Bring his hand close to the door knob and get rid of the excess charge.

[John
Travoltage](#)
[e](#)

Test Prep for AP Courses

Exercise:

Problem:

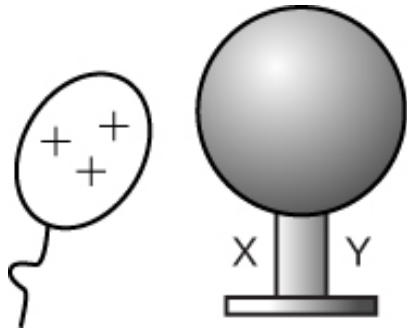
Some students experimenting with an uncharged metal sphere want to give the sphere a net charge using a charged aluminum pie plate. Which of the following steps would give the sphere a net charge of the same sign as the pie plate?

- a. bringing the pie plate close to, but not touching, the metal sphere, then moving the pie plate away.
- b. bringing the pie plate close to, but not touching, the metal sphere, then momentarily touching a grounding wire to the metal sphere.

- c. bringing the pie plate close to, but not touching, the metal sphere, then momentarily touching a grounding wire to the pie plate.
- d. touching the pie plate to the metal sphere.

Exercise:

Problem:



Balloon and sphere.

When the balloon is brought closer to the sphere, there will be a redistribution of charges. What is this phenomenon called?

- a. electrostatic repulsion
- b. conduction
- c. polarization
- d. none of the above

Solution:

(c)

Exercise:

Problem:

What will be the charge at Y (i.e., the part of the sphere furthest from the balloon)?

- a. positive
- b. negative
- c. zero
- d. It can be positive or negative depending on the material.

Exercise:

Problem: What will be the net charge on the sphere?

- a. positive
- b. negative
- c. zero
- d. It can be positive or negative depending on the material.

Solution:

(c)

Exercise:

Problem:

If Y is grounded while the balloon is still close to X, which of the following will be true?

- a. Electrons will flow from the sphere to the ground.
- b. Electrons will flow from the ground to the sphere.
- c. Protons will flow from the sphere to the ground.
- d. Protons will flow from the ground to the sphere.

Exercise:

Problem:

If the balloon is moved away after grounding, what will be the net charge on the sphere?

- a. positive

- b. negative
- c. zero
- d. It can be positive or negative depending on the material.

Solution:

(b)

Exercise:

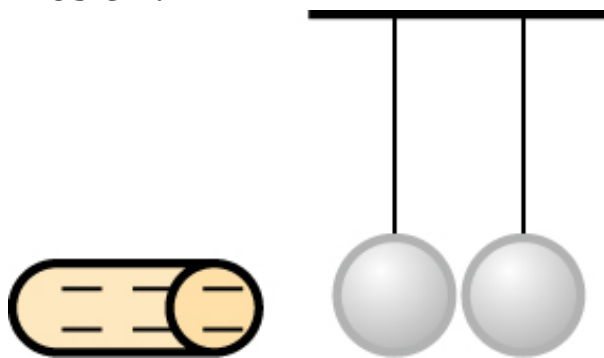
Problem:

A positively charged rod is used to charge a sphere by induction. Which of the following is true?

- a. The sphere must be a conductor.
- b. The sphere must be an insulator.
- c. The sphere can be a conductor or insulator but must be connected to ground.
- d. The sphere can be a conductor or insulator but must be already charged.

Exercise:

Problem:



Rod and metal balls.

As shown in the figure above, two metal balls are suspended and a negatively charged rod is brought close to them.

- a. If the two balls are in contact with each other what will be the charges on each ball?
- b. Explain how the balls get these charges.
- c. What will happen to the charge on the second ball (i.e., the ball further away from the rod) if it is momentarily grounded while the rod is still there?
- d. If (instead of grounding) the second ball is moved away and then the rod is removed from the first ball, will the two balls have induced charges? If yes, what will be the charges? If no, why not?

Solution:

a) Ball 1 will have positive charge and Ball 2 will have negative charge. b) The negatively charged rod attracts positive charge of Ball 1. The electrons of Ball 1 are transferred to Ball 2, making it negatively charged. c) If Ball 2 is grounded while the rod is still there, it will lose its negative charge to the ground. d) Yes, Ball 1 will be positively charged and Ball 2 will be negatively charge.

Exercise:

Problem:

Two experiments are performed using positively charged glass rods and neutral electroscopes. In the first experiment the rod is brought in contact with the electroscope. In the second experiment the rod is only brought close to the electroscope but not in contact. However, while the rod is close, the electroscope is momentarily grounded and then the rod is removed. In both experiments the needles of the electroscopes deflect, which indicates the presence of charges.

- a. What is the charging method in each of the two experiments?
- b. What is the net charge on the electroscope in the first experiment? Explain how the electroscope obtains that charge.

- c. Is the net charge on the electroscope in the second experiment different from that of the first experiment? Explain why.

Section Summary

- Polarization is the separation of positive and negative charges in a neutral object.
- A conductor is a substance that allows charge to flow freely through its atomic structure.
- An insulator holds charge within its atomic structure.
- Objects with like charges repel each other, while those with unlike charges attract each other.
- A conducting object is said to be grounded if it is connected to the Earth through a conductor. Grounding allows transfer of charge to and from the earth's large reservoir.
- Objects can be charged by contact with another charged object and obtain the same sign charge.
- If an object is temporarily grounded, it can be charged by induction, and obtains the opposite sign charge.
- Polarized objects have their positive and negative charges concentrated in different areas, giving them a non-symmetrical charge.
- Polar molecules have an inherent separation of charge.

Conceptual Questions

Exercise:

Problem:

An eccentric inventor attempts to levitate by first placing a large negative charge on himself and then putting a large positive charge on the ceiling of his workshop. Instead, while attempting to place a large negative charge on himself, his clothes fly off. Explain.

Exercise:

Problem:

If you have charged an electroscope by contact with a positively charged object, describe how you could use it to determine the charge of other objects. Specifically, what would the leaves of the electroscope do if other charged objects were brought near its knob?

Exercise:**Problem:**

When a glass rod is rubbed with silk, it becomes positive and the silk becomes negative—yet both attract dust. Does the dust have a third type of charge that is attracted to both positive and negative? Explain.

Exercise:**Problem:**

Why does a car always attract dust right after it is polished? (Note that car wax and car tires are insulators.)

Exercise:**Problem:**

Describe how a positively charged object can be used to give another object a negative charge. What is the name of this process?

Exercise:**Problem:**

What is grounding? What effect does it have on a charged conductor? On a charged insulator?

Problems & Exercises**Exercise:**

Problem:

Suppose a speck of dust in an electrostatic precipitator has 1.0000×10^{12} protons in it and has a net charge of -5.00 nC (a very large charge for a small speck). How many electrons does it have?

Solution:

$$1.03 \times 10^{12}$$

Exercise:**Problem:**

An amoeba has 1.00×10^{16} protons and a net charge of 0.300 pC . (a) How many fewer electrons are there than protons? (b) If you paired them up, what fraction of the protons would have no electrons?

Exercise:**Problem:**

A 50.0 g ball of copper has a net charge of $2.00 \mu\text{C}$. What fraction of the copper's electrons has been removed? (Each copper atom has 29 protons, and copper has an atomic mass of 63.5.)

Solution:

$$9.09 \times 10^{-13}$$

Exercise:**Problem:**

What net charge would you place on a 100 g piece of sulfur if you put an extra electron on $1 \text{ in } 10^{12}$ of its atoms? (Sulfur has an atomic mass of 32.1.)

Exercise:

Problem:

How many coulombs of positive charge are there in 4.00 kg of plutonium, given its atomic mass is 244 and that each plutonium atom has 94 protons?

Solution:

$$1.48 \times 10^8 \text{ C}$$

Glossary

free electron

an electron that is free to move away from its atomic orbit

conductor

a material that allows electrons to move separately from their atomic orbits

insulator

a material that holds electrons securely within their atomic orbits

grounded

when a conductor is connected to the Earth, allowing charge to freely flow to and from Earth's unlimited reservoir

induction

the process by which an electrically charged object brought near a neutral object creates a charge in that object

polarization

slight shifting of positive and negative charges to opposite sides of an atom or molecule

electrostatic repulsion

the phenomenon of two objects with like charges repelling each other

Conductors and Electric Fields in Static Equilibrium

LEARNING OBJECTIVES

By the end of this section, you will be able to:

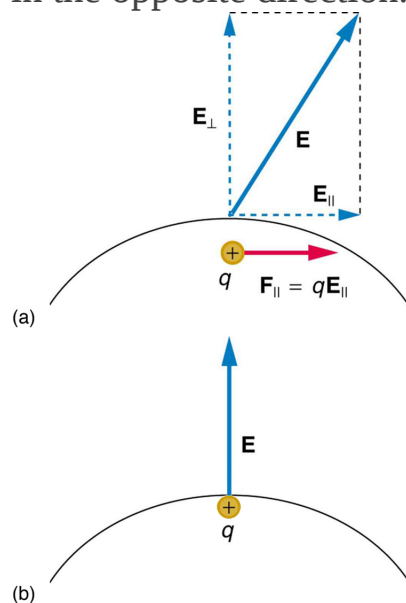
- List the three properties of a conductor in electrostatic equilibrium.
- Explain the effect of an electric field on free charges in a conductor.
- Explain why no electric field may exist inside a conductor.
- Describe the electric field surrounding Earth.
- Explain what happens to an electric field applied to an irregular conductor.
- Describe how a lightning rod works.
- Explain how a metal car may protect passengers inside from the dangerous electric fields caused by a downed line touching the car.

The information presented in this section supports the following AP learning objectives:

- **2.C.3.1** The student is able to explain the inverse square dependence of the electric field surrounding a spherically symmetric electrically charged object.
- **2.C.5.1** The student is able to create representations of the magnitude and direction of the electric field at various distances (small compared to plate size) from two electrically charged plates of equal magnitude and opposite signs and is able to recognize that the assumption of uniform field is not appropriate near edges of plates.

Conductors contain **free charges** that move easily. When excess charge is placed on a conductor or the conductor is put into a static electric field, charges in the conductor quickly respond to reach a steady state called **electrostatic equilibrium**.

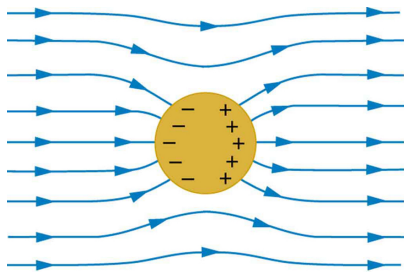
[\[link\]](#) shows the effect of an electric field on free charges in a conductor. The free charges move until the field is perpendicular to the conductor's surface. There can be no component of the field parallel to the surface in electrostatic equilibrium, since, if there were, it would produce further movement of charge. A positive free charge is shown, but free charges can be either positive or negative and are, in fact, negative in metals. The motion of a positive charge is equivalent to the motion of a negative charge in the opposite direction.



When an electric field \mathbf{E} is applied to a conductor, free charges inside the conductor move until the field is perpendicular to the surface. (a) The electric field is a vector quantity,

with both parallel and perpendicular components. The parallel component (\mathbf{E}_{\parallel}) exerts a force (\mathbf{F}_{\parallel}) on the free charge q , which moves the charge until $\mathbf{F}_{\parallel} = 0$. (b) The resulting field is perpendicular to the surface. The free charge has been brought to the conductor's surface, leaving electrostatic forces in equilibrium.

A conductor placed in an **electric field** will be **polarized**. [\[link\]](#) shows the result of placing a neutral conductor in an originally uniform electric field. The field becomes stronger near the conductor but entirely disappears inside it.



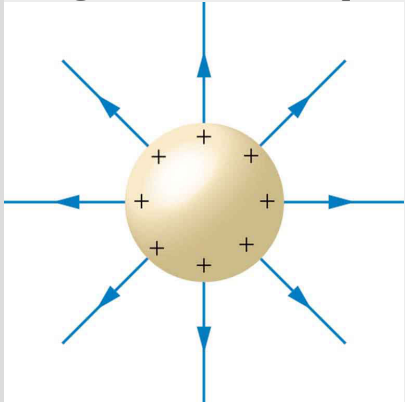
This illustration shows a spherical conductor in static equilibrium with an

originally uniform electric field. Free charges move within the conductor, polarizing it, until the electric field lines are perpendicular to the surface. The field lines end on excess negative charge on one section of the surface and begin again on excess positive charge on the opposite side. No electric field exists inside the conductor, since free charges in the conductor would continue moving in response to any field until it was neutralized.

Note:**Misconception Alert: Electric Field inside a Conductor**

Excess charges placed on a spherical conductor repel and move until they are evenly distributed, as shown in [\[link\]](#). Excess charge is forced to the surface until the field inside the conductor is zero. Outside the conductor,

the field is exactly the same as if the conductor were replaced by a point charge at its center equal to the excess charge.



The mutual repulsion of excess positive charges on a spherical conductor distributes them uniformly on its surface. The resulting electric field is perpendicular to the surface and zero inside. Outside the conductor, the field is identical to that of a point charge at the center equal to the excess charge.

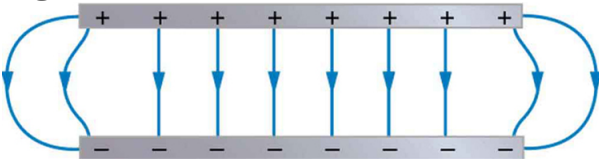
Note:

Properties of a Conductor in Electrostatic Equilibrium

1. The electric field is zero inside a conductor.
2. Just outside a conductor, the electric field lines are perpendicular to its surface, ending or beginning on charges on the surface.
3. Any excess charge resides entirely on the surface or surfaces of a conductor.

The properties of a conductor are consistent with the situations already discussed and can be used to analyze any conductor in electrostatic equilibrium. This can lead to some interesting new insights, such as described below.

How can a very uniform electric field be created? Consider a system of two metal plates with opposite charges on them, as shown in [\[link\]](#). The properties of conductors in electrostatic equilibrium indicate that the electric field between the plates will be uniform in strength and direction. Except near the edges, the excess charges distribute themselves uniformly, producing field lines that are uniformly spaced (hence uniform in strength) and perpendicular to the surfaces (hence uniform in direction, since the plates are flat). The edge effects are less important when the plates are close together.



Two metal plates with equal,
but opposite, excess charges.

The field between them is
uniform in strength and
direction except near the edges.

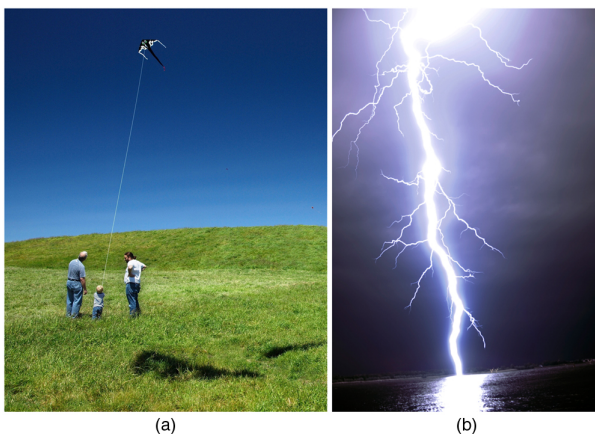
One use of such a field is to
produce uniform acceleration of
charges between the plates,
such as in the electron gun of a
TV tube.

Earth's Electric Field

A near uniform electric field of approximately 150 N/C, directed downward, surrounds Earth, with the magnitude increasing slightly as we get closer to the surface. What causes the electric field? At around 100 km above the surface of Earth we have a layer of charged particles, called the **ionosphere**. The ionosphere is responsible for a range of phenomena including the electric field surrounding Earth. In fair weather the ionosphere is positive and the Earth largely negative, maintaining the electric field ([link](#))(a)).

In storm conditions clouds form and localized electric fields can be larger and reversed in direction ([link](#))(b)). The exact charge distributions depend on the local conditions, and variations of [link](#)(b) are possible.

If the electric field is sufficiently large, the insulating properties of the surrounding material break down and it becomes conducting. For air this occurs at around 3×10^6 N/C. Air ionizes ions and electrons recombine, and we get discharge in the form of lightning sparks and corona discharge.



Earth's electric field. (a) Fair weather field. Earth and the

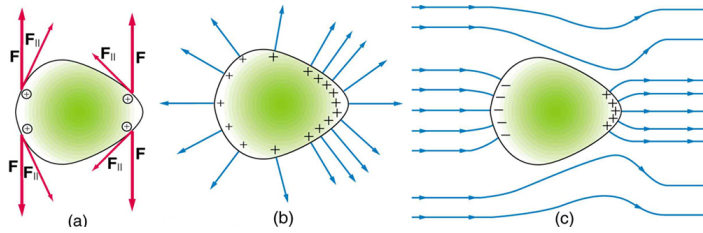
ionosphere (a layer of charged particles) are both conductors. They produce a uniform electric field of about 150 N/C . (credit: D. H. Parks) (b) Storm fields. In the presence of storm clouds, the local electric fields can be larger. At very high fields, the insulating properties of the air break down and lightning can occur. (credit: Jan-Joost Verhoef)

Electric Fields on Uneven Surfaces

So far we have considered excess charges on a smooth, symmetrical conductor surface. What happens if a conductor has sharp corners or is pointed? Excess charges on a nonuniform conductor become concentrated at the sharpest points. Additionally, excess charge may move on or off the conductor at the sharpest points.

To see how and why this happens, consider the charged conductor in [\[link\]](#). The electrostatic repulsion of like charges is most effective in moving them apart on the flattest surface, and so they become least concentrated there. This is because the forces between identical pairs of charges at either end of the conductor are identical, but the components of the forces parallel to the surfaces are different. The component parallel to the surface is greatest on the flattest surface and, hence, more effective in moving the charge.

The same effect is produced on a conductor by an externally applied electric field, as seen in [\[link\]](#) (c). Since the field lines must be perpendicular to the surface, more of them are concentrated on the most curved parts.



Excess charge on a nonuniform conductor becomes most concentrated at the location of greatest curvature.

(a) The forces between identical pairs of charges at either end of the conductor are identical, but the components of the forces parallel to the surface are different. It is \mathbf{F}_{\parallel} that moves the charges apart once they have reached the surface. (b) \mathbf{F}_{\parallel} is smallest at the more pointed end, the charges are left closer together, producing the electric field shown. (c)

An uncharged conductor in an originally uniform electric field is polarized, with the most concentrated charge at its most pointed end.

Applications of Conductors

On a very sharply curved surface, such as shown in [\[link\]](#), the charges are so concentrated at the point that the resulting electric field can be great enough to remove them from the surface. This can be useful.

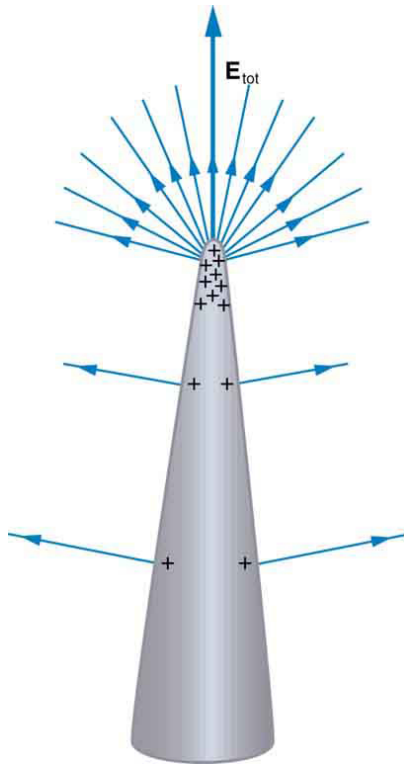
Lightning rods work best when they are most pointed. The large charges created in storm clouds induce an opposite charge on a building that can result in a lightning bolt hitting the building. The induced charge is bled

away continually by a lightning rod, preventing the more dramatic lightning strike.

Of course, we sometimes wish to prevent the transfer of charge rather than to facilitate it. In that case, the conductor should be very smooth and have as large a radius of curvature as possible. (See [\[link\]](#).) Smooth surfaces are used on high-voltage transmission lines, for example, to avoid leakage of charge into the air.

Another device that makes use of some of these principles is a **Faraday cage**. This is a metal shield that encloses a volume. All electrical charges will reside on the outside surface of this shield, and there will be no electrical field inside. A Faraday cage is used to prohibit stray electrical fields in the environment from interfering with sensitive measurements, such as the electrical signals inside a nerve cell.

During electrical storms if you are driving a car, it is best to stay inside the car as its metal body acts as a Faraday cage with zero electrical field inside. If in the vicinity of a lightning strike, its effect is felt on the outside of the car and the inside is unaffected, provided you remain totally inside. This is also true if an active (“hot”) electrical wire was broken (in a storm or an accident) and fell on your car.



A very pointed conductor has a large charge concentration at the point. The electric field is very strong at the point and can exert a force large enough to transfer charge on or off the conductor.

Lightning rods are used to prevent the buildup of large excess charges on structures and, thus, are pointed.



(a)

(b)

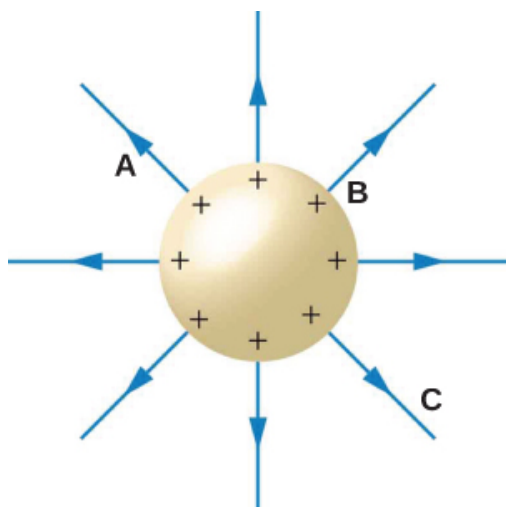
(a) A lightning rod is pointed to facilitate the transfer of charge.

(credit: Romaine, Wikimedia Commons) (b) This Van de Graaff generator has a smooth surface with a large radius of curvature to prevent the transfer of charge and allow a large voltage to be generated. The mutual repulsion of like charges is evident in the person's hair while touching the metal sphere. (credit: Jon 'ShakataGaNai' Davis/Wikimedia Commons).

Test Prep for AP Courses

Exercise:

Problem:



A sphere conductor.

An electric field due to a positively charged spherical conductor is shown above. Where will the electric field be weakest?

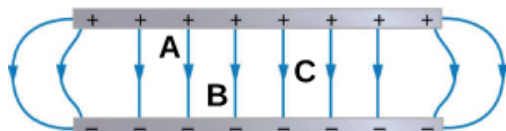
- a. Point A
- b. Point B
- c. Point C
- d. Same at all points

Solution:

(c)

Exercise:

Problem:



Electric field between
two parallel metal plates.

The electric field created by two parallel metal plates is shown above. Where will the electric field be strongest?

- a. Point A
- b. Point B
- c. Point C
- d. Same at all points

Exercise:

Problem:

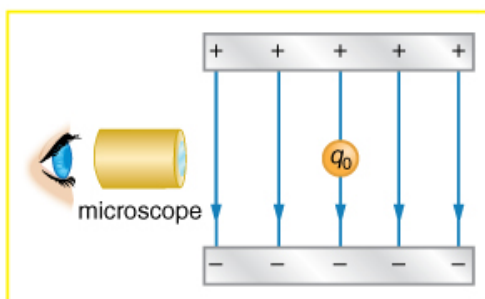
Suppose that the electric field experienced due to a positively charged small spherical conductor at a certain distance is E . What will be the percentage change in electric field experienced at thrice the distance if the charge on the conductor is doubled?

Solution:

decrease by 77.78%.

Exercise:

Problem:



Millikan oil drop
experiment.

The classic Millikan oil drop experiment setup is shown above. In this experiment oil drops are suspended in a vertical electric field against the gravitational force to measure their charge. If the mass of a negatively charged drop suspended in an electric field of 1.18×10^{-4} N/C strength is 3.85×10^{-21} g, find the number of excess electrons in the drop.

Section Summary

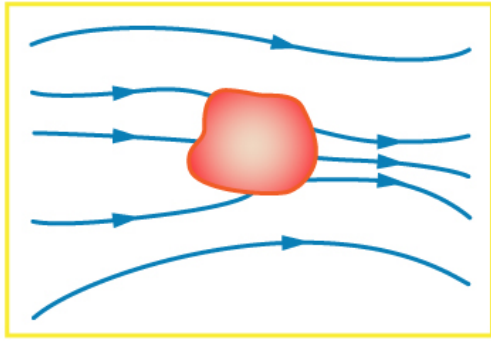
- A conductor allows free charges to move about within it.
- The electrical forces around a conductor will cause free charges to move around inside the conductor until static equilibrium is reached.
- Any excess charge will collect along the surface of a conductor.
- Conductors with sharp corners or points will collect more charge at those points.
- A lightning rod is a conductor with sharply pointed ends that collect excess charge on the building caused by an electrical storm and allow it to dissipate back into the air.
- Electrical storms result when the electrical field of Earth's surface in certain locations becomes more strongly charged, due to changes in the insulating effect of the air.
- A Faraday cage acts like a shield around an object, preventing electric charge from penetrating inside.

Conceptual Questions

Exercise:

Problem:

Is the object in a conductor or an insulator? Justify your answer.



Exercise:

Problem:

If the electric field lines in the figure above were perpendicular to the object, would it necessarily be a conductor? Explain.

Exercise:

Problem:

The discussion of the electric field between two parallel conducting plates, in this module states that edge effects are less important if the plates are close together. What does close mean? That is, is the actual plate separation crucial, or is the ratio of plate separation to plate area crucial?

Exercise:

Problem:

Would the self-created electric field at the end of a pointed conductor, such as a lightning rod, remove positive or negative charge from the conductor? Would the same sign charge be removed from a neutral pointed conductor by the application of a similar externally created electric field? (The answers to both questions have implications for charge transfer utilizing points.)

Exercise:

Problem:

Why is a golfer with a metal club over her shoulder vulnerable to lightning in an open fairway? Would she be any safer under a tree?

Exercise:**Problem:**

Can the belt of a Van de Graaff accelerator be a conductor? Explain.

Exercise:**Problem:**

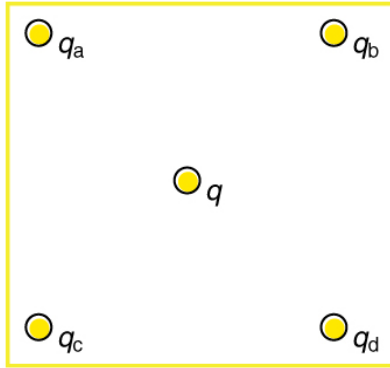
Are you relatively safe from lightning inside an automobile? Give two reasons.

Exercise:**Problem:**

Discuss pros and cons of a lightning rod being grounded versus simply being attached to a building.

Exercise:**Problem:**

Using the symmetry of the arrangement, show that the net Coulomb force on the charge q at the center of the square below ([\[link\]](#)) is zero if the charges on the four corners are exactly equal.



Four point charges q_a , q_b , q_c , and q_d lie on the corners of a square and q is located at its center.

Exercise:

Problem:

(a) Using the symmetry of the arrangement, show that the electric field at the center of the square in [\[link\]](#) is zero if the charges on the four corners are exactly equal. (b) Show that this is also true for any combination of charges in which $q_a = q_b$ and $q_b = q_c$

Exercise:

Problem:

(a) What is the direction of the total Coulomb force on q in [\[link\]](#) if q is negative, $q_a = q_c$ and both are negative, and $q_b = q_d$ and both are positive? (b) What is the direction of the electric field at the center of the square in this situation?

Exercise:

Problem:

Considering [\[link\]](#), suppose that $q_a = q_d$ and $q_b = q_c$. First show that q is in static equilibrium. (You may neglect the gravitational force.) Then discuss whether the equilibrium is stable or unstable, noting that this may depend on the signs of the charges and the direction of displacement of q from the center of the square.

Exercise:**Problem:**

If $q_a = 0$ in [\[link\]](#), under what conditions will there be no net Coulomb force on q ?

Exercise:**Problem:**

In regions of low humidity, one develops a special “grip” when opening car doors, or touching metal door knobs. This involves placing as much of the hand on the device as possible, not just the ends of one's fingers. Discuss the induced charge and explain why this is done.

Exercise:**Problem:**

Tollbooth stations on roadways and bridges usually have a piece of wire stuck in the pavement before them that will touch a car as it approaches. Why is this done?

Exercise:**Problem:**

Suppose a woman carries an excess charge. To maintain her charged status can she be standing on ground wearing just any pair of shoes? How would you discharge her? What are the consequences if she simply walks away?

Problems & Exercises

Exercise:

Problem:

Sketch the electric field lines in the vicinity of the conductor in [\[link\]](#) given the field was originally uniform and parallel to the object's long axis. Is the resulting field small near the long side of the object?



Exercise:

Problem:

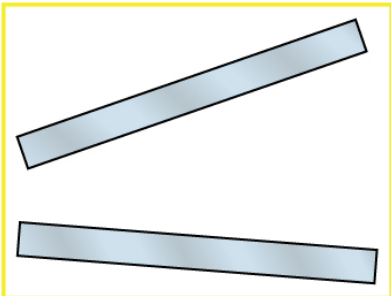
Sketch the electric field lines in the vicinity of the conductor in [\[link\]](#) given the field was originally uniform and parallel to the object's long axis. Is the resulting field small near the long side of the object?



Exercise:

Problem:

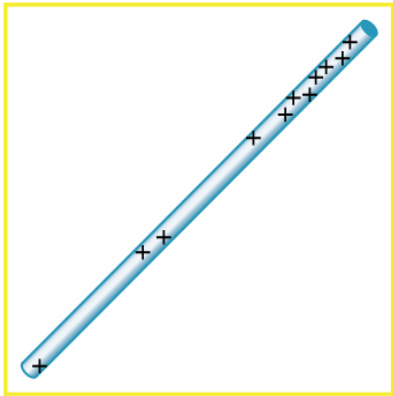
Sketch the electric field between the two conducting plates shown in [\[link\]](#), given the top plate is positive and an equal amount of negative charge is on the bottom plate. Be certain to indicate the distribution of charge on the plates.



Exercise:

Problem:

Sketch the electric field lines in the vicinity of the charged insulator in [\[link\]](#) noting its nonuniform charge distribution.

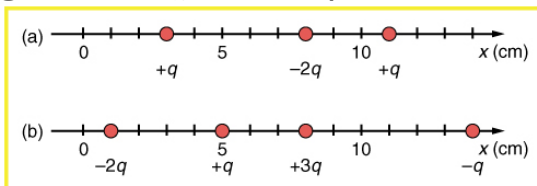


A charged insulating rod such as might be used in a classroom demonstration.

Exercise:

Problem:

What is the force on the charge located at $x = 8.00$ cm in [\[link\]](#)(a) given that $q = 1.00 \mu\text{C}$?



(a) Point charges located at 3.00, 8.00, and 11.0 cm along the x-axis. (b) Point charges located at 1.00, 5.00,

8.00, and 14.0 cm along the x -axis.

Exercise:

Problem:

(a) Find the total electric field at $x = 1.00$ cm in [\[link\]](#)(b) given that $q = 5.00$ nC. (b) Find the total electric field at $x = 11.00$ cm in [\[link\]](#)(b). (c) If the charges are allowed to move and eventually be brought to rest by friction, what will the final charge configuration be? (That is, will there be a single charge, double charge, etc., and what will its value(s) be?)

Solution:

(a) $E_{x=1.00 \text{ cm}} = -\infty$

(b) $2.12 \times 10^5 \text{ N/C}$

(c) one charge of $+q$

Exercise:

Problem:

(a) Find the electric field at $x = 5.00$ cm in [\[link\]](#)(a), given that $q = 1.00$ μC . (b) At what position between 3.00 and 8.00 cm is the total electric field the same as that for $-2q$ alone? (c) Can the electric field be zero anywhere between 0.00 and 8.00 cm? (d) At very large positive or negative values of x , the electric field approaches zero in both (a) and (b). In which does it most rapidly approach zero and why? (e) At what position to the right of 11.0 cm is the total electric field zero, other than at infinity? (Hint: A graphing calculator can yield considerable insight in this problem.)

Exercise:

Problem:

(a) Find the total Coulomb force on a charge of 2.00 nC located at $x = 4.00\text{ cm}$ in [\[link\]](#) (b), given that $q = 1.00\text{ }\mu\text{C}$. (b) Find the x -position at which the electric field is zero in [\[link\]](#) (b).

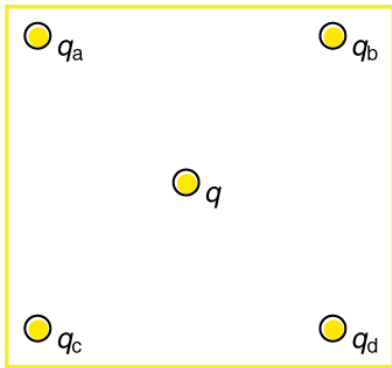
Solution:

(a) 0.252 N to the left

(b) $x = 6.07\text{ cm}$

Exercise:**Problem:**

Using the symmetry of the arrangement, determine the direction of the force on q in the figure below, given that $q_a = q_b = +7.50\text{ }\mu\text{C}$ and $q_c = q_d = -7.50\text{ }\mu\text{C}$. (b) Calculate the magnitude of the force on the charge q , given that the square is 10.0 cm on a side and $q = 2.00\text{ }\mu\text{C}$.

**Exercise:****Problem:**

(a) Using the symmetry of the arrangement, determine the direction of the electric field at the center of the square in [\[link\]](#), given that $q_a = q_b = -1.00\text{ }\mu\text{C}$ and $q_c = q_d = +1.00\text{ }\mu\text{C}$. (b) Calculate the magnitude of the electric field at the location of q , given that the square is 5.00 cm on a side.

Solution:

(a) The electric field at the center of the square will be straight up, since q_a and q_b are positive and q_c and q_d are negative and all have the same magnitude.

(b) $2.04 \times 10^7 \text{ N/C}$ (upward)

Exercise:**Problem:**

Find the electric field at the location of q_a in [\[link\]](#) given that $q_b = q_c = q_d = +2.00 \text{ nC}$, $q = -1.00 \text{ nC}$, and the square is 20.0 cm on a side.

Exercise:**Problem:**

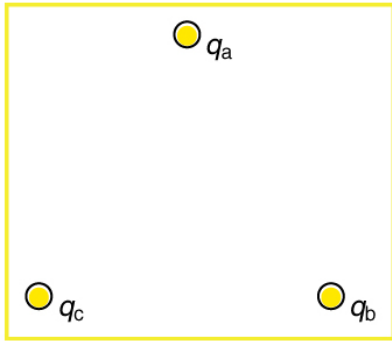
Find the total Coulomb force on the charge q in [\[link\]](#), given that $q = 1.00 \text{ } \mu\text{C}$, $q_a = 2.00 \text{ } \mu\text{C}$, $q_b = -3.00 \text{ } \mu\text{C}$, $q_c = -4.00 \text{ } \mu\text{C}$, and $q_d = +1.00 \text{ } \mu\text{C}$. The square is 50.0 cm on a side.

Solution:

0.102 N, in the $-y$ direction

Exercise:**Problem:**

(a) Find the electric field at the location of q_a in [\[link\]](#), given that $q_b = +10.00 \text{ } \mu\text{C}$ and $q_c = -5.00 \text{ } \mu\text{C}$. (b) What is the force on q_a , given that $q_a = +1.50 \text{ nC}$?



Point charges
located at the
corners of an
equilateral triangle
25.0 cm on a side.

Exercise:

Problem:

(a) Find the electric field at the center of the triangular configuration of charges in [\[link\]](#), given that $q_a = +2.50 \text{ nC}$, $q_b = -8.00 \text{ nC}$, and $q_c = +1.50 \text{ nC}$. (b) Is there any combination of charges, other than $q_a = q_b = q_c$, that will produce a zero strength electric field at the center of the triangular configuration?

Solution:

(a) $\vec{E} = 4.36 \times 10^3 \text{ N/C}$, 35.0° , below the horizontal.

(b) No

Glossary

conductor

an object with properties that allow charges to move about freely within it

free charge

an electrical charge (either positive or negative) which can move about separately from its base molecule

electrostatic equilibrium

an electrostatically balanced state in which all free electrical charges have stopped moving about

polarized

a state in which the positive and negative charges within an object have collected in separate locations

ionosphere

a layer of charged particles located around 100 km above the surface of Earth, which is responsible for a range of phenomena including the electric field surrounding Earth

Faraday cage

a metal shield which prevents electric charge from penetrating its surface

Coulomb's Law

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- State Coulomb's law in terms of how the electrostatic force changes with the distance between two objects.
- Calculate the electrostatic force between two point charges, such as electrons or protons.
- Compare the electrostatic force to the gravitational attraction for a proton and an electron; for a human and the Earth.

The information presented in this section supports the following AP® learning objectives and science practices:

- **3.A.3.3** The student is able to describe a force as an interaction between two objects and identify both objects for any force. **(S.P. 1.4)**
- **3.A.3.4** The student is able to make claims about the force on an object due to the presence of other objects with the same property: mass, electric charge. **(S.P. 6.1, 6.4)**
- **3.C.2.1** The student is able to use Coulomb's law qualitatively and quantitatively to make predictions about the interaction between two electric point charges (interactions between collections of electric point charges are not covered in Physics 1 and instead are restricted to Physics 2). **(S.P. 2.2, 6.4)**
- **3.C.2.2** The student is able to connect the concepts of gravitational force and electric force to compare similarities and differences between the forces. **(S.P. 7.2)**



This NASA image of Arp 87 shows the result of a strong gravitational attraction between two galaxies. In contrast, at the subatomic level, the electrostatic attraction between two objects, such as an electron and a proton, is far greater than their mutual attraction due to gravity. (credit: NASA/HST)

Through the work of scientists in the late 18th century, the main features of the **electrostatic force**—the existence of two types of charge, the observation that like charges repel, unlike charges attract, and the decrease of force with distance—were eventually refined, and expressed as a mathematical formula. The mathematical formula for the electrostatic force is called **Coulomb's law** after the French physicist Charles Coulomb (1736–1806), who performed experiments and first proposed a formula to calculate it.

Note:

Coulomb's Law

Equation:

$$F = k \frac{|q_1 q_2|}{r^2}.$$

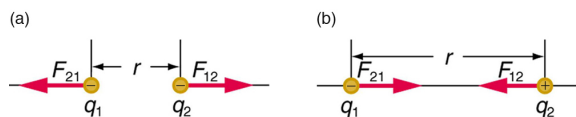
Coulomb's law calculates the magnitude of the force F between two point charges, q_1 and q_2 , separated by a distance r . In SI units, the constant k is equal to

Equation:

$$k = 8.988 \times 10^9 \frac{\text{N} \cdot \text{m}^2}{\text{C}^2} \approx 8.99 \times 10^9 \frac{\text{N} \cdot \text{m}^2}{\text{C}^2}.$$

The electrostatic force is a vector quantity and is expressed in units of newtons. The force is understood to be along the line joining the two charges. (See [\[link\]](#).)

Although the formula for Coulomb's law is simple, it was no mean task to prove it. The experiments Coulomb did, with the primitive equipment then available, were difficult. Modern experiments have verified Coulomb's law to great precision. For example, it has been shown that the force is inversely proportional to distance between two objects squared ($F \propto 1/r^2$) to an accuracy of 1 part in 10^{16} . No exceptions have ever been found, even at the small distances within the atom.



The magnitude of the electrostatic force F between point charges q_1 and q_2 separated by a distance r is given by Coulomb's law. Note that Newton's third law (every force exerted creates an equal and opposite force) applies as usual—the force on q_1 is equal in magnitude and opposite in direction to the force it exerts on q_2 . (a) Like charges. (b) Unlike charges.

Note:

Making Connections: Comparing Gravitational and Electrostatic Forces

Recall that the gravitational force (Newton's law of gravitation) quantifies force as

$$F_g = G \frac{mM}{r^2}.$$

The comparison between the two forces—gravitational and electrostatic—shows some similarities and differences. Gravitational force is proportional to the masses of interacting objects, and the electrostatic force is proportional to the magnitudes of the charges of interacting objects. Hence both forces are proportional to a property that represents the strength of interaction for a given field. In addition, both forces are inversely proportional to the square of the distances between them. It may seem that the two forces are related but that is not the case. In fact, there are huge variations in the magnitudes of the two forces as they depend on different parameters and different mechanisms. For electrons (or protons), electrostatic force is dominant and is much greater than the gravitational force. On the other hand, gravitational force is generally dominant for objects with large masses. Another major difference between the two forces is that gravitational force can only be attractive, whereas electrostatic could be attractive or repulsive (depending on the sign of charges; unlike charges attract and like charges repel).

Example:

How Strong is the Coulomb Force Relative to the Gravitational Force?

Compare the electrostatic force between an electron and proton separated by 0.530×10^{-10} m with the gravitational force between them. This distance is their average separation in a hydrogen atom.

Strategy

To compare the two forces, we first compute the electrostatic force using Coulomb's law, $F = k \frac{|q_1 q_2|}{r^2}$. We then calculate the gravitational force using Newton's universal law of gravitation. Finally, we take a ratio to see how the forces compare in magnitude.

Solution

Entering the given and known information about the charges and separation of the electron and proton into the expression of Coulomb's law yields

Equation:

$$F = k \frac{|q_1 q_2|}{r^2}$$

Equation:

$$= (8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2) \times \frac{(1.60 \times 10^{-19} \text{ C})(1.60 \times 10^{-19} \text{ C})}{(0.530 \times 10^{-10} \text{ m})^2}$$

Thus the Coulomb force is

Equation:

$$F = 8.19 \times 10^{-8} \text{ N}.$$

The charges are opposite in sign, so this is an attractive force. This is a very large force for an electron—it would cause an acceleration of $8.99 \times 10^{22} \text{ m/s}^2$ (verification is left as an end-of-section problem). The gravitational force is given by Newton's law of gravitation as:

Equation:

$$F_G = G \frac{mM}{r^2},$$

where $G = 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2$. Here m and M represent the electron and proton masses, which can be found in the appendices. Entering values for the knowns yields

Equation:

$$F_G = (6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2) \times \frac{(9.11 \times 10^{-31} \text{ kg})(1.67 \times 10^{-27} \text{ kg})}{(0.530 \times 10^{-10} \text{ m})^2} = 3.61 \times 10^{-47} \text{ N}$$

This is also an attractive force, although it is traditionally shown as positive since gravitational force is always attractive. The ratio of the magnitude of the electrostatic force to gravitational force in this case is, thus,

Equation:

$$\frac{F}{F_G} = 2.27 \times 10^{39}.$$

Discussion

This is a remarkably large ratio! Note that this will be the ratio of electrostatic force to gravitational force for an electron and a proton at any distance (taking the ratio before entering numerical values shows that the distance cancels). This ratio gives some indication of just how much larger the Coulomb force is than the gravitational force between two of the most common particles in nature.

As the example implies, gravitational force is completely negligible on a small scale, where the interactions of individual charged particles are important. On a large scale, such as between the Earth and a person, the reverse is true. Most objects are nearly electrically neutral, and so attractive and repulsive **Coulomb forces** nearly cancel. Gravitational force on a large scale dominates interactions between large objects because it is always attractive, while Coulomb forces tend to cancel.

Test Prep for AP Courses

Exercise:

Problem:

For questions 25–27, suppose that the electrostatics force between two charges is F .

What will be the force if the distance between them is halved?

- a. $4F$
- b. $2F$
- c. $F/4$
- d. $F/2$

Solution:

(a)

Exercise:

Problem: Which of the following is false?

- a. If the charge of one of the particles is doubled and that of the second is unchanged, the force will become $2F$.
- b. If the charge of one of the particles is doubled and that of the second is halved, the force will remain F .
- c. If the charge of both the particles is doubled, the force will become $4F$.
- d. None of the above.

Exercise:

Problem:

Which of the following is true about the gravitational force between the particles?

- a. It will be $3.25 \times 10^{-38} F$.
- b. It will be $3.25 \times 10^{38} F$.
- c. It will be equal to F .
- d. It is not possible to determine the gravitational force as the masses of the particles are not given.

Solution:

(d)

Exercise:**Problem:**

Two massive, positively charged particles are initially held a fixed distance apart. When they are moved farther apart, the magnitude of their mutual gravitational force changes by a factor of n . Which of the following indicates the factor by which the magnitude of their mutual electrostatic force changes?

- a. $1/n^2$
- b. $1/n$
- c. n
- d. n^2

Exercise:**Problem:**

- a. What is the electrostatic force between two charges of 1 C each, separated by a distance of 0.5 m?
- b. How will this force change if the distance is increased to 1 m?

Solution:

(a) 3.60×10^{10} N, (b) It will become 1/4 of the original value; hence it will be equal to 8.99×10^9 N

Exercise:**Problem:**

- a. Find the ratio of the electrostatic force to the gravitational force between two electrons.

- b. Will this ratio change if the two electrons are replaced by protons? If yes, find the new ratio.

Section Summary

- Frenchman Charles Coulomb was the first to publish the mathematical equation that describes the electrostatic force between two objects.
- Coulomb's law gives the magnitude of the force between point charges. It is

Equation:

$$F = k \frac{|q_1 q_2|}{r^2},$$

where q_1 and q_2 are two point charges separated by a distance r , and $k \approx 8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2$

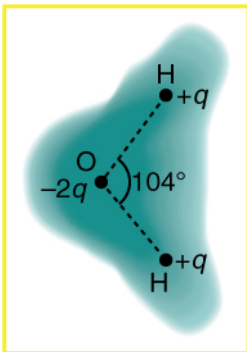
- This Coulomb force is extremely basic, since most charges are due to point-like particles. It is responsible for all electrostatic effects and underlies most macroscopic forces.
- The Coulomb force is extraordinarily strong compared with the gravitational force, another basic force—but unlike gravitational force it can cancel, since it can be either attractive or repulsive.
- The electrostatic force between two subatomic particles is far greater than the gravitational force between the same two particles.

Conceptual Questions

Exercise:

Problem:

[\[link\]](#) shows the charge distribution in a water molecule, which is called a polar molecule because it has an inherent separation of charge. Given water's polar character, explain what effect humidity has on removing excess charge from objects.



Schematic representation of the outer electron cloud of a neutral water molecule. The electrons spend more time near the oxygen than the hydrogens, giving a permanent charge separation as shown. Water is thus a *polar molecule*. It is more easily affected by electrostatic forces than molecules with uniform charge distributions.

Exercise:

Problem:

Using [\[link\]](#), explain, in terms of Coulomb's law, why a polar molecule (such as in [\[link\]](#)) is attracted by both positive and negative charges.

Exercise:

Problem:

Given the polar character of water molecules, explain how ions in the air form nucleation centers for rain droplets.

Problems & Exercises

Exercise:

Problem:

What is the repulsive force between two pith balls that are 8.00 cm apart and have equal charges of -30.0 nC ?

Exercise:**Problem:**

(a) How strong is the attractive force between a glass rod with a $0.700\text{ }\mu\text{C}$ charge and a silk cloth with a $-0.600\text{ }\mu\text{C}$ charge, which are 12.0 cm apart, using the approximation that they act like point charges? (b) Discuss how the answer to this problem might be affected if the charges are distributed over some area and do not act like point charges.

Solution:

(a) 0.263 N

(b) If the charges are distributed over some area, there will be a concentration of charge along the side closest to the oppositely charged object. This effect will increase the net force.

Exercise:**Problem:**

Two point charges exert a 5.00 N force on each other. What will the force become if the distance between them is increased by a factor of three?

Exercise:**Problem:**

Two point charges are brought closer together, increasing the force between them by a factor of 25. By what factor was their separation decreased?

Solution:

The separation decreased by a factor of 5.

Exercise:**Problem:**

How far apart must two point charges of 75.0 nC (typical of static electricity) be to have a force of 1.00 N between them?

Exercise:

Problem:

If two equal charges each of 1 C each are separated in air by a distance of 1 km, what is the magnitude of the force acting between them? You will see that even at a distance as large as 1 km, the repulsive force is substantial because 1 C is a very significant amount of charge.

Exercise:**Problem:**

A test charge of $+2 \mu\text{C}$ is placed halfway between a charge of $+6 \mu\text{C}$ and another of $+4 \mu\text{C}$ separated by 10 cm. (a) What is the magnitude of the force on the test charge? (b) What is the direction of this force (away from or toward the $+6 \mu\text{C}$ charge)?

Exercise:**Problem:**

Bare free charges do not remain stationary when close together. To illustrate this, calculate the acceleration of two isolated protons separated by 2.00 nm (a typical distance between gas atoms). Explicitly show how you follow the steps in the Problem-Solving Strategy for electrostatics.

Solution:

$$\begin{aligned} F &= k \frac{|q_1 q_2|}{r^2} = ma \Rightarrow a = \frac{kq^2}{mr^2} \\ &= \frac{(9.00 \times 10^9 \text{ N}\cdot\text{m}^2/\text{C}^2)(1.60 \times 10^{-19} \text{ m})^2}{(1.67 \times 10^{-27} \text{ kg})(2.00 \times 10^{-9} \text{ m})^2} \\ &= 3.45 \times 10^{16} \text{ m/s}^2 \end{aligned}$$

Exercise:**Problem:**

(a) By what factor must you change the distance between two point charges to change the force between them by a factor of 10? (b) Explain how the distance can either increase or decrease by this factor and still cause a factor of 10 change in the force.

Solution:

(a) 3.2

(b) If the distance increases by 3.2, then the force will decrease by a factor of 10 ; if the distance decreases by 3.2, then the force will increase by a factor of 10. Either way, the force changes by a factor of 10.

Exercise:

Problem:

Suppose you have a total charge q_{tot} that you can split in any manner. Once split, the separation distance is fixed. How do you split the charge to achieve the greatest force?

Exercise:**Problem:**

(a) Common transparent tape becomes charged when pulled from a dispenser. If one piece is placed above another, the repulsive force can be great enough to support the top piece's weight. Assuming equal point charges (only an approximation), calculate the magnitude of the charge if electrostatic force is great enough to support the weight of a 10.0 mg piece of tape held 1.00 cm above another. (b) Discuss whether the magnitude of this charge is consistent with what is typical of static electricity.

Solution:

(a) $1.04 \times 10^{-9} \text{ C}$

(b) This charge is approximately 1 nC, which is consistent with the magnitude of charge typical for static electricity

Exercise:**Problem:**

(a) Find the ratio of the electrostatic to gravitational force between two electrons. (b) What is this ratio for two protons? (c) Why is the ratio different for electrons and protons?

Exercise:**Problem:**

At what distance is the electrostatic force between two protons equal to the weight of one proton?

Exercise:**Problem:**

A certain five cent coin contains 5.00 g of nickel. What fraction of the nickel atoms' electrons, removed and placed 1.00 m above it, would support the weight of this coin? The atomic mass of nickel is 58.7, and each nickel atom contains 28 electrons and 28 protons.

Solution:

1.02×10^{-11}

Exercise:**Problem:**

(a) Two point charges totaling $8.00\ \mu\text{C}$ exert a repulsive force of $0.150\ \text{N}$ on one another when separated by $0.500\ \text{m}$. What is the charge on each? (b) What is the charge on each if the force is attractive?

Exercise:**Problem:**

Point charges of $5.00\ \mu\text{C}$ and $-3.00\ \mu\text{C}$ are placed $0.250\ \text{m}$ apart. (a) Where can a third charge be placed so that the net force on it is zero? (b) What if both charges are positive?

Solution:

- a. $0.859\ \text{m}$ beyond negative charge on line connecting two charges
- b. $0.109\ \text{m}$ from lesser charge on line connecting two charges

Exercise:**Problem:**

Two point charges q_1 and q_2 are $3.00\ \text{m}$ apart, and their total charge is $20\ \mu\text{C}$. (a) If the force of repulsion between them is $0.075\ \text{N}$, what are magnitudes of the two charges? (b) If one charge attracts the other with a force of $0.525\ \text{N}$, what are the magnitudes of the two charges? Note that you may need to solve a quadratic equation to reach your answer.

Glossary

Coulomb's law

the mathematical equation calculating the electrostatic force vector between two charged particles

Coulomb force

another term for the electrostatic force

electrostatic force

the amount and direction of attraction or repulsion between two charged bodies

Electric Field: Concept of a Field Revisited

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Describe a force field and calculate the strength of an electric field due to a point charge.
- Calculate the force exerted on a test charge by an electric field.
- Explain the relationship between electrical force (F) on a test charge and electrical field strength (E).

The information presented in this section supports the following AP[®] learning objectives and science practices:

- **2.C.1.1** The student is able to predict the direction and the magnitude of the force exerted on an object with an electric charge q placed in an electric field E using the mathematical model of the relation between an electric force and an electric field: $\vec{F} = q\vec{E}$, a vector relation. **(S.P. 2.2)**
- **2.C.1.2** The student is able to calculate any one of the variables – electric force, electric charge, and electric field – at a point given the values and sign or direction of the other two quantities. **(S.P. 2.2)**
- **2.C.2.1** The student is able to qualitatively and semiquantitatively apply the vector relationship between the electric field and the net electric charge creating that field. **(S.P. 2.2, 6.4)**
- **3.C.4.1** The student is able to make claims about various contact forces between objects based on the microscopic cause of those forces. **(S.P. 6.1)**
- **3.C.4.2** The student is able to explain contact forces (tension, friction, normal, buoyant, spring) as arising from interatomic

electric forces and that they therefore have certain directions.
(S.P. 6.2)

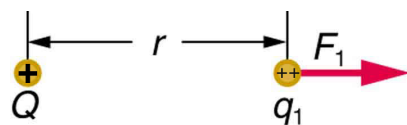
Contact forces, such as between a baseball and a bat, are explained on the small scale by the interaction of the charges in atoms and molecules in close proximity. They interact through forces that include the **Coulomb force**. Action at a distance is a force between objects that are not close enough for their atoms to “touch.” That is, they are separated by more than a few atomic diameters.

For example, a charged rubber comb attracts neutral bits of paper from a distance via the Coulomb force. It is very useful to think of an object being surrounded in space by a **force field**. The force field carries the force to another object (called a test object) some distance away.

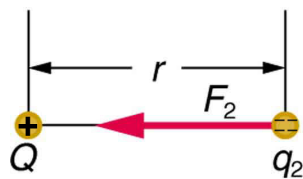
Concept of a Field

A field is a way of conceptualizing and mapping the force that surrounds any object and acts on another object at a distance without apparent physical connection. For example, the gravitational field surrounding the earth (and all other masses) represents the gravitational force that would be experienced if another mass were placed at a given point within the field.

In the same way, the Coulomb force field surrounding any charge extends throughout space. Using Coulomb's law, $F = k|q_1q_2|/r^2$, its magnitude is given by the equation $F = k|qQ|/r^2$, for a **point charge** (a particle having a charge Q) acting on a **test charge** q at a distance r (see [\[link\]](#)). Both the magnitude and direction of the Coulomb force field depend on Q and the test charge q .



(a)



(b)

The Coulomb force field due to a positive charge Q is shown acting on two different charges. Both charges are the same distance from Q . (a) Since q_1 is positive, the force F_1 acting on it is repulsive. (b) The charge q_2 is negative and greater in magnitude than q_1 , and so the force F_2 acting on it is attractive and stronger than F_1 . The Coulomb force field is thus not unique at any point in space, because it depends on the test charges q_1 and q_2

as well as the
charge Q .

To simplify things, we would prefer to have a field that depends only on Q and not on the test charge q . The electric field is defined in such a manner that it represents only the charge creating it and is unique at every point in space. Specifically, the electric field E is defined to be the ratio of the Coulomb force to the test charge:

Equation:

$$\mathbf{E} = \frac{\mathbf{F}}{q},$$

where \mathbf{F} is the electrostatic force (or Coulomb force) exerted on a positive test charge q . It is understood that \mathbf{E} is in the same direction as \mathbf{F} . It is also assumed that q is so small that it does not alter the charge distribution creating the electric field. The units of electric field are newtons per coulomb (N/C). If the electric field is known, then the electrostatic force on any charge q is simply obtained by multiplying charge times electric field, or $\mathbf{F} = q\mathbf{E}$. Consider the electric field due to a point charge Q . According to Coulomb's law, the force it exerts on a test charge q is $F = k|qQ|/r^2$. Thus the magnitude of the electric field, E , for a point charge is

Equation:

$$E = \left| \frac{F}{q} \right| = k \left| \frac{qQ}{qr^2} \right| = k \frac{|Q|}{r^2}.$$

Since the test charge cancels, we see that

Equation:

$$E = k \frac{|Q|}{r^2}.$$

The electric field is thus seen to depend only on the charge Q and the distance r ; it is completely independent of the test charge q .

Example:

Calculating the Electric Field of a Point Charge

Calculate the strength and direction of the electric field E due to a point charge of 2.00 nC (nano-Coulombs) at a distance of 5.00 mm from the charge.

Strategy

We can find the electric field created by a point charge by using the equation $E = kQ/r^2$.

Solution

Here $Q = 2.00 \times 10^{-9} \text{ C}$ and $r = 5.00 \times 10^{-3} \text{ m}$. Entering those values into the above equation gives

Equation:

$$\begin{aligned} E &= k \frac{Q}{r^2} \\ &= (8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2) \times \frac{(2.00 \times 10^{-9} \text{ C})}{(5.00 \times 10^{-3} \text{ m})^2} \\ &= 7.19 \times 10^5 \text{ N/C}. \end{aligned}$$

Discussion

This **electric field strength** is the same at any point 5.00 mm away from the charge Q that creates the field. It is positive, meaning that it has a direction pointing away from the charge Q .

Example:

Calculating the Force Exerted on a Point Charge by an Electric Field

What force does the electric field found in the previous example exert on a point charge of $-0.250 \mu\text{C}$?

Strategy

Since we know the electric field strength and the charge in the field, the force on that charge can be calculated using the definition of electric field

$\mathbf{E} = \mathbf{F}/q$ rearranged to $\mathbf{F} = q\mathbf{E}$.

Solution

The magnitude of the force on a charge $q = -0.250 \mu\text{C}$ exerted by a field of strength $E = 7.20 \times 10^5 \text{ N/C}$ is thus,

Equation:

$$\begin{aligned} F &= -qE \\ &= (0.250 \times 10^{-6} \text{ C})(7.20 \times 10^5 \text{ N/C}) \\ &= 0.180 \text{ N.} \end{aligned}$$

Because q is negative, the force is directed opposite to the direction of the field.

Discussion

The force is attractive, as expected for unlike charges. (The field was created by a positive charge and here acts on a negative charge.) The charges in this example are typical of common static electricity, and the modest attractive force obtained is similar to forces experienced in static cling and similar situations.

Note:

PhET Explorations: Electric Field of Dreams

Play ball! Add charges to the Field of Dreams and see how they react to the electric field. Turn on a background electric field and adjust the direction and magnitude.

[Electri](#)
[c Field](#)
[of](#)
[Dream](#)
[s](#)

Test Prep for AP Courses

Exercise:

Problem:

Two particles with charges $+2q$ and $+q$ are separated by a distance r . The $+2q$ particle has an electric field E at distance r and exerts a force F on the $+q$ particle. Use this information to answer questions 31–32.

What is the electric field of the $+q$ particle at the same distance and what force does it exert on the $+2q$ particle?

- a. $E/2, F/2$
- b. $E, F/2$
- c. $E/2, F$
- d. E, F

Solution:

(c)

Exercise:

Problem:

When the $+q$ particle is replaced by a $+3q$ particle, what will be the electric field and force from the $+2q$ particle experienced by the $+3q$ particle?

- a. $E/3, 3F$
- b. $E, 3F$
- c. $E/3, F$
- d. E, F

Exercise:

Problem: The direction of the electric field of a negative charge is

- a. inward for both positive and negative charges.
 - b. outward for both positive and negative charges.
 - c. inward for other positive charges and outward for other negative charges.
 - d. outward for other positive charges and inward for other negative charges.
-

Solution:

(a)

Exercise:

Problem: The force responsible for holding an atom together is

- a. frictional
- b. electric
- c. gravitational
- d. magnetic

Exercise:

Problem:

When a positively charged particle exerts an inward force on another particle P , what will be the charge of P ?

- a. positive
 - b. negative
 - c. neutral
 - d. cannot be determined
-

Solution:

(b)

Exercise:

Problem:

Find the force exerted due to a particle having a charge of $3.2 \times 10^{-19} \text{ C}$ on another identical particle 5 cm away.

Exercise:**Problem:**

Suppose that the force exerted on an electron is $5.6 \times 10^{-17} \text{ N}$, directed to the east.

- Find the magnitude of the electric field that exerts the force.
- What will be the direction of the electric field?
- If the electron is replaced by a proton, what will be the magnitude of force exerted?
- What will be the direction of force on the proton?

Solution:

(a) 350 N/C , (b) west, (c) $5.6 \times 10^{-17} \text{ N}$, (d) west.

Section Summary

- The electrostatic force field surrounding a charged object extends out into space in all directions.
- The electrostatic force exerted by a point charge on a test charge at a distance r depends on the charge of both charges, as well as the distance between the two.
- The electric field **E** is defined to be

Equation:

$$\mathbf{E} = \frac{\mathbf{F}}{q},$$

where \mathbf{F} is the Coulomb or electrostatic force exerted on a small positive test charge q . \mathbf{E} has units of N/C.

- The magnitude of the electric field \mathbf{E} created by a point charge Q is

Equation:

$$\mathbf{E} = k \frac{|Q|}{r^2}.$$

where r is the distance from Q . The electric field \mathbf{E} is a vector and fields due to multiple charges add like vectors.

Conceptual Questions

Exercise:

Problem:

Why must the test charge q in the definition of the electric field be vanishingly small?

Exercise:

Problem:

Are the direction and magnitude of the Coulomb force unique at a given point in space? What about the electric field?

Problem Exercises

Exercise:

Problem:

What is the magnitude and direction of an electric field that exerts a 2.00×10^{-5} N upward force on a $-1.75 \mu\text{C}$ charge?

Exercise:

Problem:

What is the magnitude and direction of the force exerted on a $3.50\ \mu\text{C}$ charge by a $250\ \text{N/C}$ electric field that points due east?

Solution:

$$8.75 \times 10^{-4}\ \text{N}$$

Exercise:**Problem:**

Calculate the magnitude of the electric field $2.00\ \text{m}$ from a point charge of $5.00\ \text{mC}$ (such as found on the terminal of a Van de Graaff).

Exercise:**Problem:**

(a) What magnitude point charge creates a $10,000\ \text{N/C}$ electric field at a distance of $0.250\ \text{m}$? (b) How large is the field at $10.0\ \text{m}$?

Solution:

(a) $6.94 \times 10^{-8}\ \text{C}$

(b) $6.25\ \text{N/C}$

Exercise:**Problem:**

Calculate the initial (from rest) acceleration of a proton in a $5.00 \times 10^6\ \text{N/C}$ electric field (such as created by a research Van de Graaff). Explicitly show how you follow the steps in the Problem-Solving Strategy for electrostatics.

Exercise:

Problem:

(a) Find the direction and magnitude of an electric field that exerts a $4.80 \times 10^{-17} \text{ N}$ westward force on an electron. (b) What magnitude and direction force does this field exert on a proton?

Solution:

(a) 300 N/C (east)

(b) $4.80 \times 10^{-17} \text{ N}$ (east)

Glossary

field

a map of the amount and direction of a force acting on other objects, extending out into space

point charge

A charged particle, designated Q , generating an electric field

test charge

A particle (designated q) with either a positive or negative charge set down within an electric field generated by a point charge

Electric Field Lines: Multiple Charges

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Calculate the total force (magnitude and direction) exerted on a test charge from more than one charge.
- Describe an electric field diagram of a positive point charge and of a negative point charge with twice the magnitude of the positive charge.
- Draw the electric field lines between two points of the same charge and between two points of opposite charge.

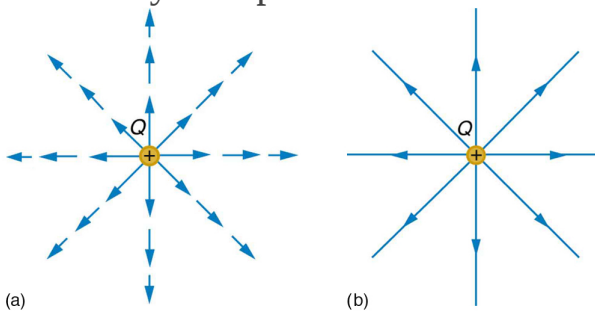
The information presented in this section supports the following AP® learning objectives and science practices:

- **2.C.1.2** The student is able to calculate any one of the variables – electric force, electric charge, and electric field – at a point given the values and sign or direction of the other two quantities.
- **2.C.2.1** The student is able to qualitatively and semiquantitatively apply the vector relationship between the electric field and the net electric charge creating that field.
- **2.C.4.1** The student is able to distinguish the characteristics that differ between monopole fields (gravitational field of spherical mass and electrical field due to single point charge) and dipole fields (electric dipole field and magnetic field) and make claims about the spatial behavior of the fields using qualitative or semiquantitative arguments based on vector addition of fields due to each point source, including identifying the locations and signs of sources from a vector diagram of the field. (**S.P. 2.2, 6.4, 7.2**)

- **2.C.4.2** The student is able to apply mathematical routines to determine the magnitude and direction of the electric field at specified points in the vicinity of a small set (2-4) of point charges, and express the results in terms of magnitude and direction of the field in a visual representation by drawing field vectors of appropriate length and direction at the specified points. **(S.P. 1.4, 2.2)**
- **3.C.2.3** The student is able to use mathematics to describe the electric force that results from the interaction of several separated point charges (generally 2-4 point charges, though more are permitted in situations of high symmetry). **(S.P. 2.2)**

Drawings using lines to represent **electric fields** around charged objects are very useful in visualizing field strength and direction. Since the electric field has both magnitude and direction, it is a vector. Like all **vectors**, the electric field can be represented by an arrow that has length proportional to its magnitude and that points in the correct direction. (We have used arrows extensively to represent force vectors, for example.)

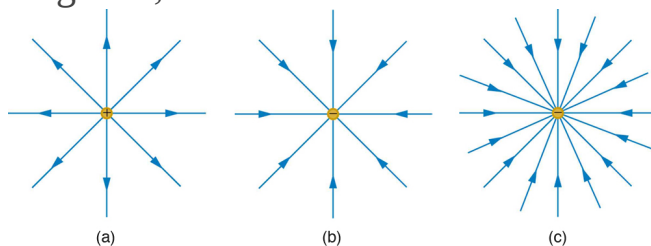
[\[link\]](#) shows two pictorial representations of the same electric field created by a positive point charge Q . [\[link\]](#) (b) shows the standard representation using continuous lines. [\[link\]](#) (a) shows numerous individual arrows with each arrow representing the force on a test charge q . Field lines are essentially a map of infinitesimal force vectors.



Two equivalent representations

of the electric field due to a positive charge Q . (a) Arrows representing the electric field's magnitude and direction. (b) In the standard representation, the arrows are replaced by continuous field lines having the same direction at any point as the electric field. The closeness of the lines is directly related to the strength of the electric field. A test charge placed anywhere will feel a force in the direction of the field line; this force will have a strength proportional to the density of the lines (being greater near the charge, for example).

Note that the electric field is defined for a positive test charge q , so that the field lines point away from a positive charge and toward a negative charge. (See [\[link\]](#).) The electric field strength is exactly proportional to the number of field lines per unit area, since the magnitude of the electric field for a point charge is $E = k|Q|/r^2$ and area is proportional to r^2 . This pictorial representation, in which field lines represent the direction and their closeness (that is, their areal density or the number of lines crossing a unit area) represents strength, is used for all fields: electrostatic, gravitational, magnetic, and others.



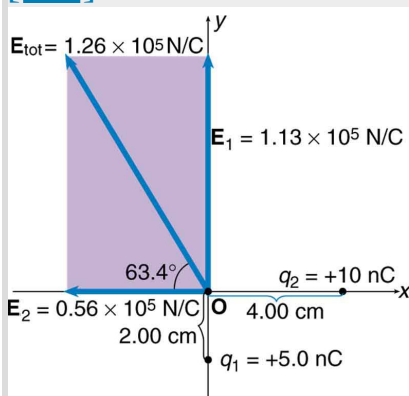
The electric field surrounding three different point charges. (a) A positive charge. (b) A negative charge of equal magnitude. (c) A larger negative charge.

In many situations, there are multiple charges. The total electric field created by multiple charges is the vector sum of the individual fields created by each charge. The following example shows how to add electric field vectors.

Example:

Adding Electric Fields

Find the magnitude and direction of the total electric field due to the two point charges, q_1 and q_2 , at the origin of the coordinate system as shown in [\[link\]](#).



The electric fields \mathbf{E}_1 and \mathbf{E}_2 at the origin O add to \mathbf{E}_{tot} .

Strategy

Since the electric field is a vector (having magnitude and direction), we add electric fields with the same vector techniques used for other types of vectors. We first must find the electric field due to each charge at the point of interest, which is the origin of the coordinate system (O) in this instance. We pretend that there is a positive test charge, q , at point O, which allows us to determine the direction of the fields \mathbf{E}_1 and \mathbf{E}_2 . Once those fields are found, the total field can be determined using **vector addition**.

Solution

The electric field strength at the origin due to q_1 is labeled E_1 and is calculated:

Equation:

$$E_1 = k \frac{q_1}{r_1^2} = \left(8.99 \times 10^9 \text{ N} \cdot \text{m}^2 / \text{C}^2 \right) \frac{(5.00 \times 10^{-9} \text{ C})}{(2.00 \times 10^{-2} \text{ m})^2}$$
$$E_1 = 1.124 \times 10^5 \text{ N/C}.$$

Similarly, E_2 is

Equation:

$$E_2 = k \frac{q_2}{r_2^2} = \left(8.99 \times 10^9 \text{ N} \cdot \text{m}^2 / \text{C}^2 \right) \frac{(10.0 \times 10^{-9} \text{ C})}{(4.00 \times 10^{-2} \text{ m})^2}$$
$$E_2 = 0.5619 \times 10^5 \text{ N/C}.$$

Four digits have been retained in this solution to illustrate that E_1 is exactly twice the magnitude of E_2 . Now arrows are drawn to represent the magnitudes and directions of \mathbf{E}_1 and \mathbf{E}_2 . (See [\[link\]](#).) The direction of the electric field is that of the force on a positive charge so both arrows point directly away from the positive charges that create them. The arrow for \mathbf{E}_1 is exactly twice the length of that for \mathbf{E}_2 . The arrows form a right triangle in this case and can be added using the Pythagorean theorem. The magnitude of the total field E_{tot} is

Equation:

$$\begin{aligned} E_{\text{tot}} &= (E_1^2 + E_2^2)^{1/2} \\ &= \{(1.124 \times 10^5 \text{ N/C})^2 + (0.5619 \times 10^5 \text{ N/C})^2\}^{1/2} \\ &= 1.26 \times 10^5 \text{ N/C}. \end{aligned}$$

The direction is

Equation:

$$\begin{aligned}\theta &= \tan^{-1}\left(\frac{E_1}{E_2}\right) \\ &= \tan^{-1}\left(\frac{1.124 \times 10^5 \text{ N/C}}{0.5619 \times 10^5 \text{ N/C}}\right) \\ &= 63.4^\circ,\end{aligned}$$

or 63.4° above the x -axis.

Discussion

In cases where the electric field vectors to be added are not perpendicular, vector components or graphical techniques can be used. The total electric field found in this example is the total electric field at only one point in space. To find the total electric field due to these two charges over an entire region, the same technique must be repeated for each point in the region. This impossibly lengthy task (there are an infinite number of points in space) can be avoided by calculating the total field at representative points and using some of the unifying features noted next.

[\[link\]](#) shows how the electric field from two point charges can be drawn by finding the total field at representative points and drawing electric field lines consistent with those points. While the electric fields from multiple charges are more complex than those of single charges, some simple features are easily noticed.

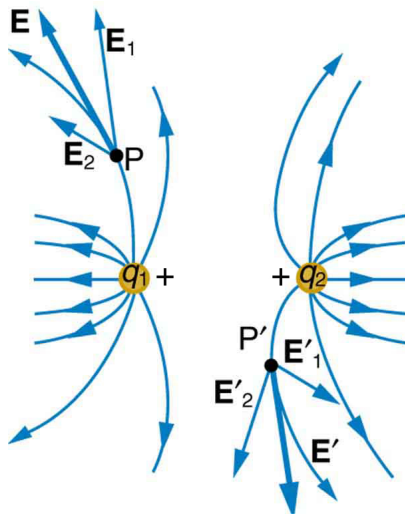
For example, the field is weaker between like charges, as shown by the lines being farther apart in that region. (This is because the fields from each charge exert opposing forces on any charge placed between them.) (See [\[link\]](#) and [\[link\]](#)(a).) Furthermore, at a great distance from two like charges, the field becomes identical to the field from a single, larger charge.

[\[link\]](#)(b) shows the electric field of two unlike charges.

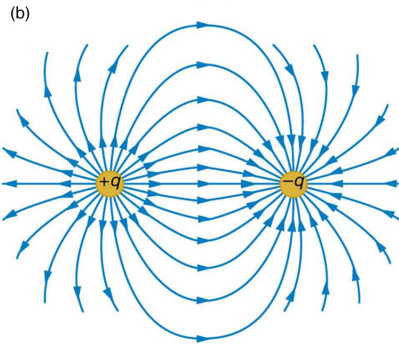
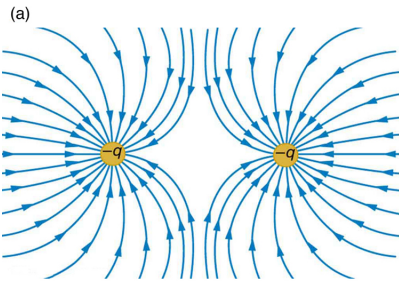
Note:**Making Connections: Electric Dipole**

As the two unlike charges are also equal in magnitude, the pair of charges is also known as an electric dipole.

The field is stronger between the charges. In that region, the fields from each charge are in the same direction, and so their strengths add. The field of two unlike charges is weak at large distances, because the fields of the individual charges are in opposite directions and so their strengths subtract. At very large distances, the field of two unlike charges looks like that of a smaller single charge.



Two positive point charges q_1 and q_2 produce the resultant electric field shown. The field is calculated at representative points and then smooth field lines drawn following the rules outlined in the text.



(a) Two negative charges produce the fields shown. It is very similar to the field produced by two positive charges, except that the directions are reversed. The field is clearly weaker between the charges. The individual forces on a test charge in that region are in opposite directions.

(b) Two opposite charges produce the field shown, which is stronger in the

region between the charges.

We use electric field lines to visualize and analyze electric fields (the lines are a pictorial tool, not a physical entity in themselves). The properties of electric field lines for any charge distribution can be summarized as follows:

1. Field lines must begin on positive charges and terminate on negative charges, or at infinity in the hypothetical case of isolated charges.
2. The number of field lines leaving a positive charge or entering a negative charge is proportional to the magnitude of the charge.
3. The strength of the field is proportional to the closeness of the field lines—more precisely, it is proportional to the number of lines per unit area perpendicular to the lines.
4. The direction of the electric field is tangent to the field line at any point in space.
5. Field lines can never cross.

The last property means that the field is unique at any point. The field line represents the direction of the field; so if they crossed, the field would have two directions at that location (an impossibility if the field is unique).

Note:

PhET Explorations: Charges and Fields

Move point charges around on the playing field and then view the electric field, voltages, equipotential lines, and more. It's colorful, it's dynamic, it's free.

[Charge
s and
Fields](#)

Test Prep for AP Courses

Exercise:

Problem:

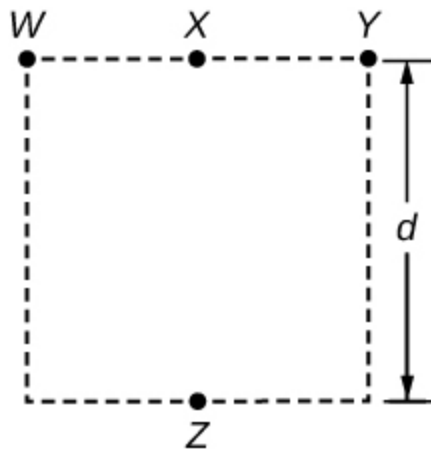


An electric dipole (with $+2q$ and $-2q$ as the two charges) is shown in the figure above. A third charge, $-q$ is placed equidistant from the dipole charges. What will be the direction of the net force on the third charge?

- a. \rightarrow
- b. \leftarrow
- c. \downarrow
- d. \uparrow

Exercise:

Problem:



Four objects, each with charge $+q$, are held fixed on a square with sides of length d , as shown in the figure. Objects X and Z are at the midpoints of the sides of the square. The electrostatic force exerted by object W on object X is F . Use this information to answer questions 39–40.

What is the magnitude of force exerted by object W on Z ?

- a. $F/7$
- b. $F/5$
- c. $F/3$
- d. $F/2$

Solution:

(b)

Exercise:

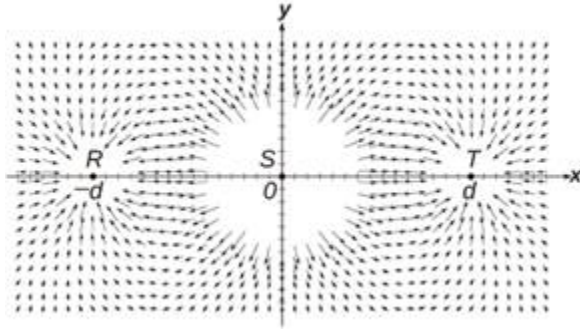
Problem:

What is the magnitude of the net force exerted on object X by objects W , Y , and Z ?

- a. $F/4$
- b. $F/2$
- c. $9F/4$
- d. $3F$

Exercise:

Problem:



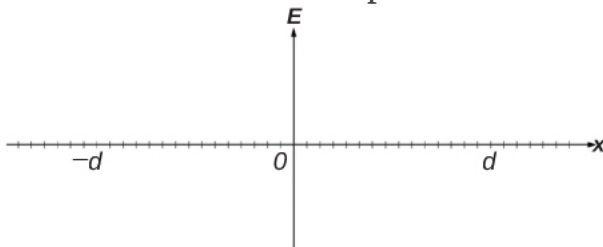
Electric field with three charged objects.

The figure above represents the electric field in the vicinity of three small charged objects, R , S , and T . The objects have charges $-q$, $+2q$, and $-q$, respectively, and are located on the x -axis at $-d$, 0 , and d . Field vectors of very large magnitude are omitted for clarity.

- (a) i) Briefly describe the characteristics of the field diagram that indicate that the sign of the charges of objects R and T is negative and that the sign of the charge of object S is positive.
- ii) Briefly describe the characteristics of the field diagram that indicate that the magnitudes of the charges of objects R and T are equal and that the magnitude of the charge of object S is about twice that of objects R and T .

For the following parts, an electric field directed to the right is defined to be positive.

- (b) On the axes below, sketch a graph of the electric field E along the x -axis as a function of position x .



An Electric field (E) axis and
Position (x) axis.

(c) Write an expression for the electric field E along the x -axis as a function of position x in the region between objects S and T in terms of q , d , and fundamental constants, as appropriate.

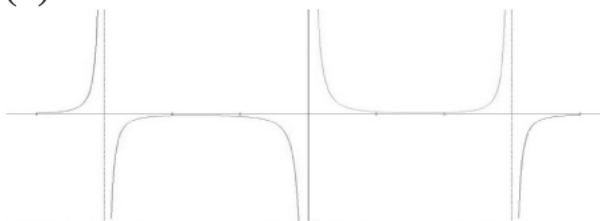
(d) Your classmate tells you there is a point between S and T where the electric field is zero. Determine whether this statement is true, and explain your reasoning using two of the representations from parts (a), (b), or (c).

Solution:

(a) i) Field vectors near objects point toward negatively charged objects and away from positively charged objects.

(a) ii) The vectors closest to R and T are about the same length and start at about the same distance. We have that $q_R/d^2 = q_T/d^2$, so the charge on R is about the same as the charge on T . The closest vectors around S are about the same length as those around R and T . The vectors near S start at about 6 units away, while vectors near R and T start at about 4 units. We have that $q_R/d^2 = q_S/D^2$, so $q_S/q_R = D^2/d^2 = 36/16 = 2.25$, and so the charge on S is about twice that on R and T .

(b)



A vector diagram.

(c)

$$E = k \left[-\frac{q}{(d+x)^2} + \frac{2q}{(x)^2} + \frac{q}{(d-x)^2} \right]$$

(d) The statement is not true. The vector diagram shows field vectors in this region with nonzero length, and the vectors not shown have even greater lengths. The equation in part (c) shows that, when $0 < x < d$, the denominator of the negative term is always greater than the denominator of the third term, but the numerator is the same. So the negative term always has a smaller magnitude than the third term and since the second term is positive the sum of the terms is always positive.

Section Summary

- Drawings of electric field lines are useful visual tools. The properties of electric field lines for any charge distribution are that:
- Field lines must begin on positive charges and terminate on negative charges, or at infinity in the hypothetical case of isolated charges.
- The number of field lines leaving a positive charge or entering a negative charge is proportional to the magnitude of the charge.
- The strength of the field is proportional to the closeness of the field lines—more precisely, it is proportional to the number of lines per unit area perpendicular to the lines.
- The direction of the electric field is tangent to the field line at any point in space.
- Field lines can never cross.

Conceptual Questions

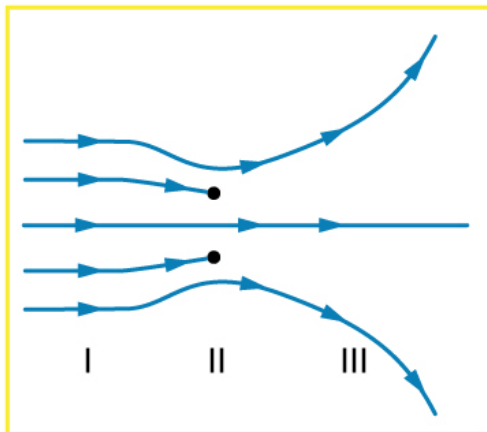
Exercise:

Problem:

Compare and contrast the Coulomb force field and the electric field. To do this, make a list of five properties for the Coulomb force field analogous to the five properties listed for electric field lines. Compare each item in your list of Coulomb force field properties with those of the electric field—are they the same or different? (For example, electric field lines cannot cross. Is the same true for Coulomb field lines?)

Exercise:**Problem:**

[\[link\]](#) shows an electric field extending over three regions, labeled I, II, and III. Answer the following questions. (a) Are there any isolated charges? If so, in what region and what are their signs? (b) Where is the field strongest? (c) Where is it weakest? (d) Where is the field the most uniform?

**Problem Exercises****Exercise:****Problem:**

(a) Sketch the electric field lines near a point charge $+q$. (b) Do the same for a point charge $-3.00q$.

Exercise:

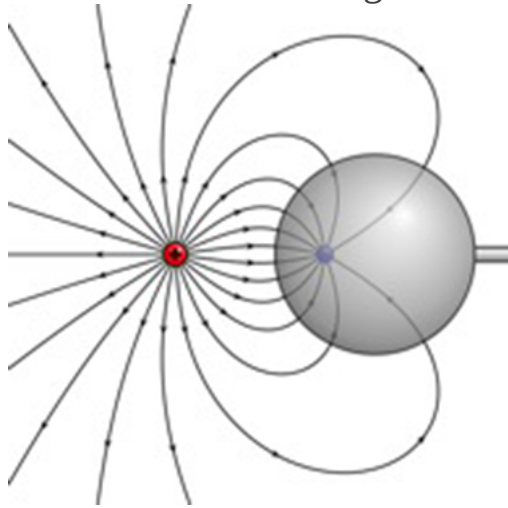
Problem:

Sketch the electric field lines a long distance from the charge distributions shown in [\[link\]](#) (a) and (b)

Exercise:

Problem:

[\[link\]](#) shows the electric field lines near two charges q_1 and q_2 . What is the ratio of their magnitudes? (b) Sketch the electric field lines a long distance from the charges shown in the figure.



The electric field near two charges.

Exercise:

Problem:

Sketch the electric field lines in the vicinity of two opposite charges, where the negative charge is three times greater in magnitude than the positive. (See [\[link\]](#) for a similar situation).

Glossary

electric field

a three-dimensional map of the electric force extended out into space from a point charge

electric field lines

a series of lines drawn from a point charge representing the magnitude and direction of force exerted by that charge

vector

a quantity with both magnitude and direction

vector addition

mathematical combination of two or more vectors, including their magnitudes, directions, and positions

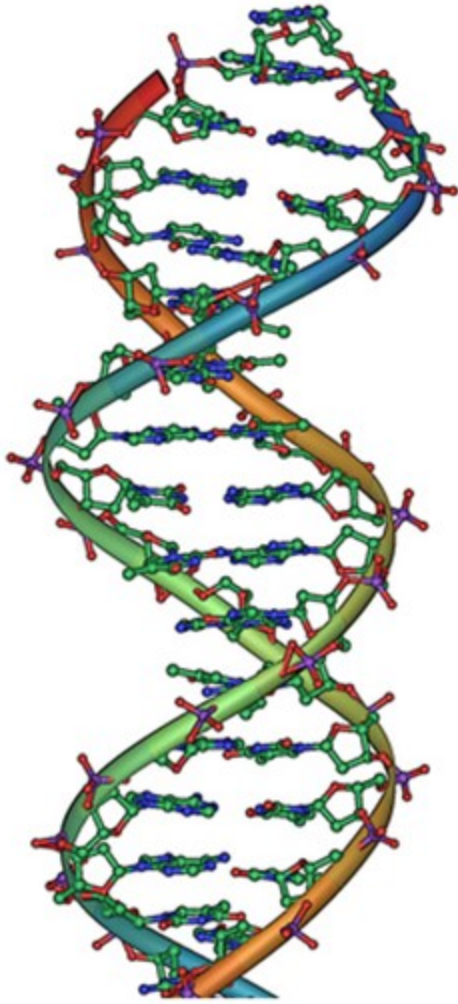
Electric Forces in Biology

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Describe how a water molecule is polar.
- Explain electrostatic screening by a water molecule within a living cell.

Classical electrostatics has an important role to play in modern molecular biology. Large molecules such as proteins, nucleic acids, and so on—so important to life—are usually electrically charged. DNA itself is highly charged; it is the electrostatic force that not only holds the molecule together but gives the molecule structure and strength. [\[link\]](#) is a schematic of the DNA double helix.



DNA is a highly charged molecule. The DNA double helix shows the two coiled strands each containing a row of nitrogenous bases, which “code” the genetic information needed by a living organism. The strands are connected by bonds between pairs of bases. While pairing combinations between certain bases are fixed (C-

G and A-T), the sequence of nucleotides in the strand varies. (credit: Jerome Walker)

The four nucleotide bases are given the symbols A (adenine), C (cytosine), G (guanine), and T (thymine). The order of the four bases varies in each strand, but the pairing between bases is always the same. C and G are always paired and A and T are always paired, which helps to preserve the order of bases in cell division (mitosis) so as to pass on the correct genetic information. Since the Coulomb force drops with distance ($F \propto 1/r^2$), the distances between the base pairs must be small enough that the electrostatic force is sufficient to hold them together.

DNA is a highly charged molecule, with about $2q_e$ (fundamental charge) per 0.3×10^{-9} m. The distance separating the two strands that make up the DNA structure is about 1 nm, while the distance separating the individual atoms within each base is about 0.3 nm.

One might wonder why electrostatic forces do not play a larger role in biology than they do if we have so many charged molecules. The reason is that the electrostatic force is “diluted” due to **screening** between molecules. This is due to the presence of other charges in the cell.

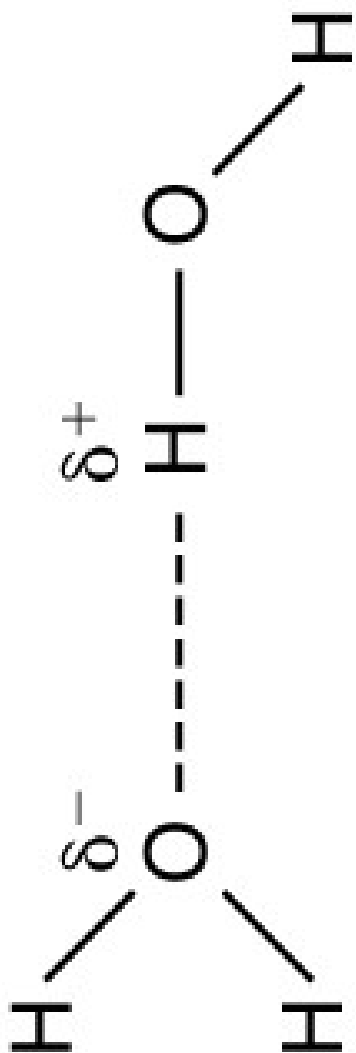
Polarity of Water Molecules

The best example of this charge screening is the water molecule, represented as H_2O . Water is a strongly **polar molecule**. Its 10 electrons (8 from the oxygen atom and 2 from the two hydrogen atoms) tend to remain closer to the oxygen nucleus than the hydrogen nuclei. This creates two centers of equal and opposite charges—what is called a **dipole**, as illustrated in [\[link\]](#). The magnitude of the dipole is called the dipole moment.

These two centers of charge will terminate some of the electric field lines coming from a free charge, as on a DNA molecule. This results in a reduction in the strength of the **Coulomb interaction**. One might say that screening makes the Coulomb force a short range force rather than long range.

Other ions of importance in biology that can reduce or screen Coulomb interactions are Na^+ , and K^+ , and Cl^- . These ions are located both inside and outside of living cells. The movement of these ions through cell membranes is crucial to the motion of nerve impulses through nerve axons.

Recent studies of electrostatics in biology seem to show that electric fields in cells can be extended over larger distances, in spite of screening, by “microtubules” within the cell. These microtubules are hollow tubes composed of proteins that guide the movement of chromosomes when cells divide, the motion of other organisms within the cell, and provide mechanisms for motion of some cells (as motors).



This schematic shows water (H_2O) as a polar molecule. Unequal sharing of electrons between the oxygen (O) and hydrogen (H) atoms leads to a net separation of positive and negative charge—forming a dipole. The symbols δ^-

and δ^+ indicate that the oxygen side of the H_2O molecule tends to be more negative, while the hydrogen ends tend to be more positive.

This leads to an attraction of opposite charges between molecules.

Section Summary

- Many molecules in living organisms, such as DNA, carry a charge.
- An uneven distribution of the positive and negative charges within a polar molecule produces a dipole.
- The effect of a Coulomb field generated by a charged object may be reduced or blocked by other nearby charged objects.
- Biological systems contain water, and because water molecules are polar, they have a strong effect on other molecules in living systems.

Conceptual Question

Exercise:

Problem:

A cell membrane is a thin layer enveloping a cell. The thickness of the membrane is much less than the size of the cell. In a static situation the membrane has a charge distribution of $-2.5 \times 10^{-6} \text{C/m}^2$ on its inner surface and $+2.5 \times 10^{-6} \text{C/m}^2$ on its outer surface. Draw a diagram of the cell and the surrounding cell membrane. Include on this diagram the charge distribution and the corresponding electric field. Is there any electric field inside the cell? Is there any electric field outside the cell?

Glossary**dipole**

a molecule's lack of symmetrical charge distribution, causing one side to be more positive and another to be more negative

polar molecule

a molecule with an asymmetrical distribution of positive and negative charge

screening

the dilution or blocking of an electrostatic force on a charged object by the presence of other charges nearby

Coulomb interaction

the interaction between two charged particles generated by the Coulomb forces they exert on one another

Applications of Electrostatics

LEARNING OBJECTIVES

By the end of this section, you will be able to:

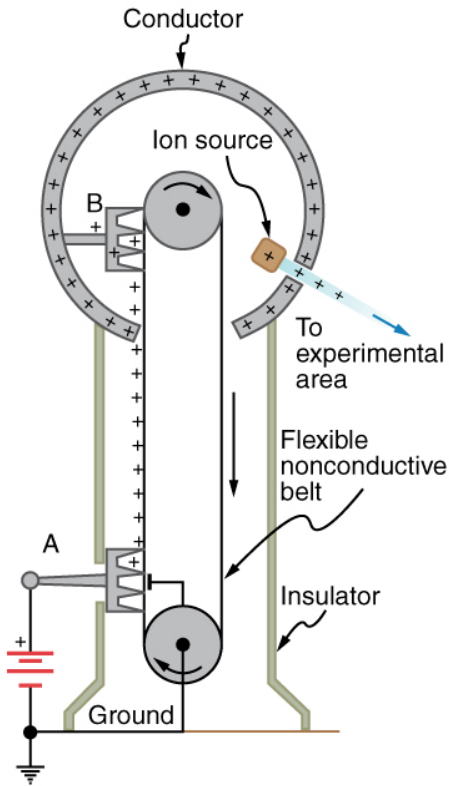
- Name several real-world applications of the study of electrostatics.

The study of **electrostatics** has proven useful in many areas. This module covers just a few of the many applications of electrostatics.

The Van de Graaff Generator

Van de Graaff generators (or Van de Graaffs) are not only spectacular devices used to demonstrate high voltage due to static electricity—they are also used for serious research. The first was built by Robert Van de Graaff in 1931 (based on original suggestions by Lord Kelvin) for use in nuclear physics research. [\[link\]](#) shows a schematic of a large research version. Van de Graaffs utilize both smooth and pointed surfaces, and conductors and insulators to generate large static charges and, hence, large voltages.

A very large excess charge can be deposited on the sphere, because it moves quickly to the outer surface. Practical limits arise because the large electric fields polarize and eventually ionize surrounding materials, creating free charges that neutralize excess charge or allow it to escape. Nevertheless, voltages of 15 million volts are well within practical limits.



Schematic of Van de Graaff generator. A battery (A) supplies excess positive charge to a pointed conductor, the points of which spray the charge onto a moving insulating belt near the bottom. The pointed conductor (B) on top in the large sphere picks up the charge. (The induced electric field at the points is so large that it removes the charge from the belt.) This can be done because the charge does not

remain inside the conducting sphere but moves to its outside surface. An ion source inside the sphere produces positive ions, which are accelerated away from the positive sphere to high velocities.

Note:

Take-Home Experiment: Electrostatics and Humidity

Rub a comb through your hair and use it to lift pieces of paper. It may help to tear the pieces of paper rather than cut them neatly. Repeat the exercise in your bathroom after you have had a long shower and the air in the bathroom is moist. Is it easier to get electrostatic effects in dry or moist air? Why would torn paper be more attractive to the comb than cut paper? Explain your observations.

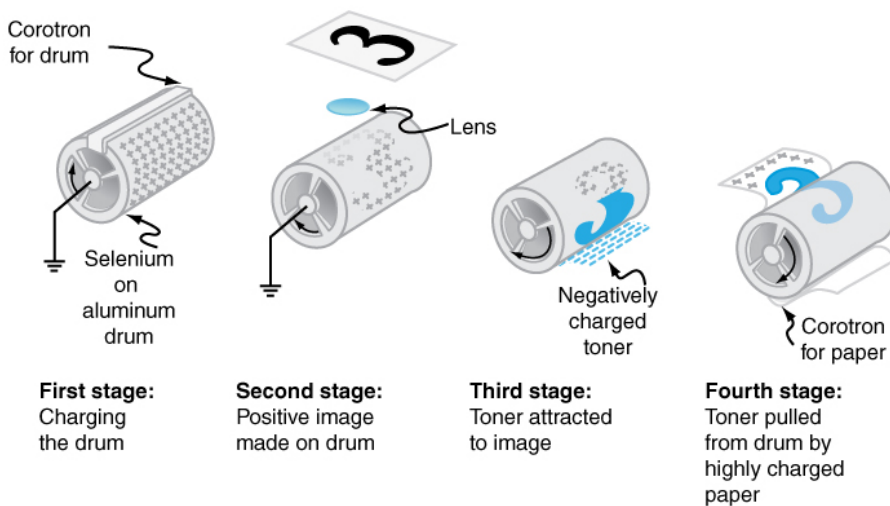
Xerography

Most copy machines use an electrostatic process called **xerography**—a word coined from the Greek words *xeros* for dry and *graphos* for writing. The heart of the process is shown in simplified form in [\[link\]](#).

A selenium-coated aluminum drum is sprayed with positive charge from points on a device called a corotron. Selenium is a substance with an interesting property—it is a **photoconductor**. That is, selenium is an insulator when in the dark and a conductor when exposed to light.

In the first stage of the xerography process, the conducting aluminum drum is **grounded** so that a negative charge is induced under the thin layer of uniformly positively charged selenium. In the second stage, the surface of the drum is exposed to the image of whatever is to be copied. Where the image is light, the selenium becomes conducting, and the positive charge is neutralized. In dark areas, the positive charge remains, and so the image has been transferred to the drum.

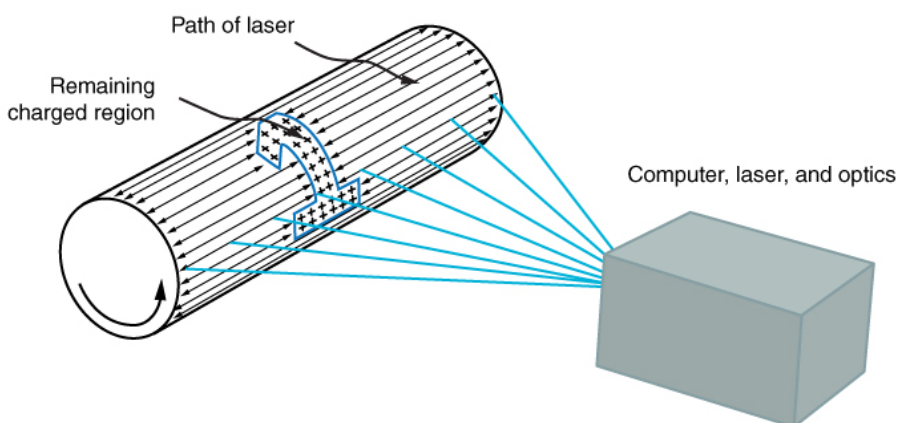
The third stage takes a dry black powder, called toner, and sprays it with a negative charge so that it will be attracted to the positive regions of the drum. Next, a blank piece of paper is given a greater positive charge than on the drum so that it will pull the toner from the drum. Finally, the paper and electrostatically held toner are passed through heated pressure rollers, which melt and permanently adhere the toner within the fibers of the paper.



Xerography is a dry copying process based on electrostatics. The major steps in the process are the charging of the photoconducting drum, transfer of an image creating a positive charge duplicate, attraction of toner to the charged parts of the drum, and transfer of toner to the paper. Not shown are heat treatment of the paper and cleansing of the drum for the next copy.

Laser Printers

Laser printers use the xerographic process to make high-quality images on paper, employing a laser to produce an image on the photoconducting drum as shown in [\[link\]](#). In its most common application, the laser printer receives output from a computer, and it can achieve high-quality output because of the precision with which laser light can be controlled. Many laser printers do significant information processing, such as making sophisticated letters or fonts, and may contain a computer more powerful than the one giving them the raw data to be printed.

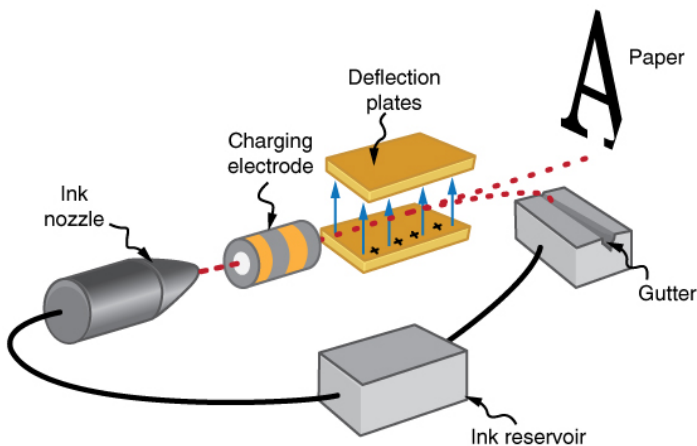


In a laser printer, a laser beam is scanned across a photoconducting drum, leaving a positive charge image. The other steps for charging the drum and transferring the image to paper are the same as in xerography. Laser light can be very precisely controlled, enabling laser printers to produce high-quality images.

Ink Jet Printers and Electrostatic Painting

The **ink jet printer**, commonly used to print computer-generated text and graphics, also employs electrostatics. A nozzle makes a fine spray of tiny ink droplets, which are then given an electrostatic charge. (See [\[link\]](#).)

Once charged, the droplets can be directed, using pairs of charged plates, with great precision to form letters and images on paper. Ink jet printers can produce color images by using a black jet and three other jets with primary colors, usually cyan, magenta, and yellow, much as a color television produces color. (This is more difficult with xerography, requiring multiple drums and toners.)



The nozzle of an ink-jet printer produces small ink droplets, which are sprayed with electrostatic charge. Various computer-driven devices are then used to direct the droplets to the correct positions on a page.

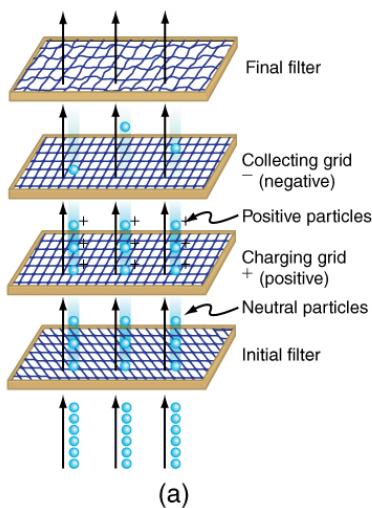
Electrostatic painting employs electrostatic charge to spray paint onto odd-shaped surfaces. Mutual repulsion of like charges causes the paint to fly away from its source. Surface tension forms drops, which are then attracted by unlike charges to the surface to be painted. Electrostatic painting can reach those hard-to-get at places, applying an even coat in a controlled

manner. If the object is a conductor, the electric field is perpendicular to the surface, tending to bring the drops in perpendicularly. Corners and points on conductors will receive extra paint. Felt can similarly be applied.

Smoke Precipitators and Electrostatic Air Cleaning

Another important application of electrostatics is found in air cleaners, both large and small. The electrostatic part of the process places excess (usually positive) charge on smoke, dust, pollen, and other particles in the air and then passes the air through an oppositely charged grid that attracts and retains the charged particles. (See [\[link\]](#).)

Large **electrostatic precipitators** are used industrially to remove over 99% of the particles from stack gas emissions associated with the burning of coal and oil. Home precipitators, often in conjunction with the home heating and air conditioning system, are very effective in removing polluting particles, irritants, and allergens.



(a)



(b)

- (a) Schematic of an electrostatic precipitator. Air is passed through grids of opposite charge. The first grid charges airborne particles, while the second attracts and collects them.
- (b) The dramatic effect of electrostatic precipitators is seen by

the absence of smoke from this power plant. (credit: Cmdalgleish, Wikimedia Commons)

Note:

Problem-Solving Strategies for Electrostatics

1. Examine the situation to determine if static electricity is involved. This may concern separated stationary charges, the forces among them, and the electric fields they create.
2. Identify the system of interest. This includes noting the number, locations, and types of charges involved.
3. Identify exactly what needs to be determined in the problem (identify the unknowns). A written list is useful. Determine whether the Coulomb force is to be considered directly—if so, it may be useful to draw a free-body diagram, using electric field lines.
4. Make a list of what is given or can be inferred from the problem as stated (identify the knowns). It is important to distinguish the Coulomb force F from the electric field E , for example.
5. Solve the appropriate equation for the quantity to be determined (the unknown) or draw the field lines as requested.
6. Examine the answer to see if it is reasonable: Does it make sense? Are units correct and the numbers involved reasonable?

Integrated Concepts

The Integrated Concepts exercises for this module involve concepts such as electric charges, electric fields, and several other topics. Physics is most interesting when applied to general situations involving more than a narrow set of physical principles. The electric field exerts force on charges, for example, and hence the relevance of [Dynamics: Force and Newton's Laws of Motion](#). The following topics are involved in some or all of the problems labeled “Integrated Concepts”:

- [Kinematics](#)
- [Two-Dimensional Kinematics](#)
- [Dynamics: Force and Newton's Laws of Motion](#)
- [Uniform Circular Motion and Gravitation](#)
- [Statics and Torque](#)
- [Fluid Statics](#)

The following worked example illustrates how this strategy is applied to an Integrated Concept problem:

Example:

Acceleration of a Charged Drop of Gasoline

If steps are not taken to ground a gasoline pump, static electricity can be placed on gasoline when filling your car's tank. Suppose a tiny drop of gasoline has a mass of 4.00×10^{-15} kg and is given a positive charge of 3.20×10^{-19} C. (a) Find the weight of the drop. (b) Calculate the electric force on the drop if there is an upward electric field of strength 3.00×10^5 N/C due to other static electricity in the vicinity. (c) Calculate the drop's acceleration.

Strategy

To solve an integrated concept problem, we must first identify the physical principles involved and identify the chapters in which they are found. Part (a) of this example asks for weight. This is a topic of dynamics and is defined in [Dynamics: Force and Newton's Laws of Motion](#). Part (b) deals with electric force on a charge, a topic of [Electric Charge and Electric Field](#). Part (c) asks for acceleration, knowing forces and mass. These are part of Newton's laws, also found in [Dynamics: Force and Newton's Laws of Motion](#).

The following solutions to each part of the example illustrate how the specific problem-solving strategies are applied. These involve identifying knowns and unknowns, checking to see if the answer is reasonable, and so on.

Solution for (a)

Weight is mass times the acceleration due to gravity, as first expressed in

Equation:

$$w = mg.$$

Entering the given mass and the average acceleration due to gravity yields

Equation:

$$w = (4.00 \times 10^{-15} \text{ kg})(9.80 \text{ m/s}^2) = 3.92 \times 10^{-14} \text{ N}.$$

Discussion for (a)

This is a small weight, consistent with the small mass of the drop.

Solution for (b)

The force an electric field exerts on a charge is given by rearranging the following equation:

Equation:

$$F = qE.$$

Here we are given the charge ($3.20 \times 10^{-19} \text{ C}$ is twice the fundamental unit of charge) and the electric field strength, and so the electric force is found to be

Equation:

$$F = (3.20 \times 10^{-19} \text{ C})(3.00 \times 10^5 \text{ N/C}) = 9.60 \times 10^{-14} \text{ N}.$$

Discussion for (b)

While this is a small force, it is greater than the weight of the drop.

Solution for (c)

The acceleration can be found using Newton's second law, provided we can identify all of the external forces acting on the drop. We assume only the drop's weight and the electric force are significant. Since the drop has a positive charge and the electric field is given to be upward, the electric force is upward. We thus have a one-dimensional (vertical direction) problem, and we can state Newton's second law as

Equation:

$$a = \frac{F_{\text{net}}}{m}.$$

where $F_{\text{net}} = F - w$. Entering this and the known values into the expression for Newton's second law yields

Equation:

$$\begin{aligned} a &= \frac{F-w}{m} \\ &= \frac{9.60 \times 10^{-14} \text{ N} - 3.92 \times 10^{-14} \text{ N}}{4.00 \times 10^{-15} \text{ kg}} \\ &= 14.2 \text{ m/s}^2. \end{aligned}$$

Discussion for (c)

This is an upward acceleration great enough to carry the drop to places where you might not wish to have gasoline.

This worked example illustrates how to apply problem-solving strategies to situations that include topics in different chapters. The first step is to identify the physical principles involved in the problem. The second step is to solve for the unknown using familiar problem-solving strategies. These are found throughout the text, and many worked examples show how to use them for single topics. In this integrated concepts example, you can see how to apply them across several topics. You will find these techniques useful in applications of physics outside a physics course, such as in your profession, in other science disciplines, and in everyday life. The following problems will build your skills in the broad application of physical principles.

Note:

Unreasonable Results

The Unreasonable Results exercises for this module have results that are unreasonable because some premise is unreasonable or because certain of the premises are inconsistent with one another. Physical principles applied correctly then produce unreasonable results. The purpose of these problems is to give practice in assessing whether nature is being accurately described, and if it is not to trace the source of difficulty.

Note:**Problem-Solving Strategy**

To determine if an answer is reasonable, and to determine the cause if it is not, do the following.

1. Solve the problem using strategies as outlined above. Use the format followed in the worked examples in the text to solve the problem as usual.
2. Check to see if the answer is reasonable. Is it too large or too small, or does it have the wrong sign, improper units, and so on?
3. If the answer is unreasonable, look for what specifically could cause the identified difficulty. Usually, the manner in which the answer is unreasonable is an indication of the difficulty. For example, an extremely large Coulomb force could be due to the assumption of an excessively large separated charge.

Section Summary

- Electrostatics is the study of electric fields in static equilibrium.
- In addition to research using equipment such as a Van de Graaff generator, many practical applications of electrostatics exist, including photocopiers, laser printers, ink-jet printers and electrostatic air filters.

Problems & Exercises**Exercise:**

Problem:

(a) What is the electric field 5.00 m from the center of the terminal of a Van de Graaff with a 3.00 mC charge, noting that the field is equivalent to that of a point charge at the center of the terminal? (b) At this distance, what force does the field exert on a $2.00\ \mu\text{C}$ charge on the Van de Graaff's belt?

Exercise:**Problem:**

(a) What is the direction and magnitude of an electric field that supports the weight of a free electron near the surface of Earth? (b) Discuss what the small value for this field implies regarding the relative strength of the gravitational and electrostatic forces.

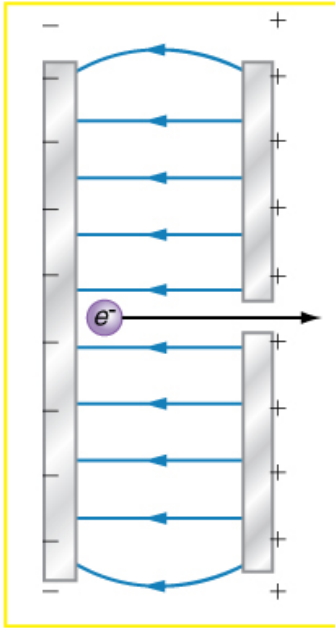
Solution:

(a) $5.58 \times 10^{-11}\ \text{N/C}$

(b) the coulomb force is extraordinarily stronger than gravity

Exercise:**Problem:**

A simple and common technique for accelerating electrons is shown in [\[link\]](#), where there is a uniform electric field between two plates. Electrons are released, usually from a hot filament, near the negative plate, and there is a small hole in the positive plate that allows the electrons to continue moving. (a) Calculate the acceleration of the electron if the field strength is $2.50 \times 10^4\ \text{N/C}$. (b) Explain why the electron will not be pulled back to the positive plate once it moves through the hole.



Parallel
conducting
plates with
opposite charges
on them create a
relatively
uniform electric
field used to
accelerate
electrons to the
right. Those that
go through the
hole can be used
to make a TV or
computer screen
glow or to
produce X-rays.

Exercise:

Problem:

Earth has a net charge that produces an electric field of approximately 150 N/C downward at its surface. (a) What is the magnitude and sign of the excess charge, noting the electric field of a conducting sphere is equivalent to a point charge at its center? (b) What acceleration will the field produce on a free electron near Earth's surface? (c) What mass object with a single extra electron will have its weight supported by this field?

Solution:

(a) $-6.76 \times 10^5 \text{ C}$

(b) $2.63 \times 10^{13} \text{ m/s}^2$ (upward)

(c) $2.45 \times 10^{-18} \text{ kg}$

Exercise:**Problem:**

Point charges of $25.0 \mu\text{C}$ and $45.0 \mu\text{C}$ are placed 0.500 m apart. (a) At what point along the line between them is the electric field zero? (b) What is the electric field halfway between them?

Exercise:**Problem:**

What can you say about two charges q_1 and q_2 , if the electric field one-fourth of the way from q_1 to q_2 is zero?

Solution:

The charge q_2 is 9 times greater than q_1 .

Exercise:**Problem: Integrated Concepts**

Calculate the angular velocity ω of an electron orbiting a proton in the hydrogen atom, given the radius of the orbit is 0.530×10^{-10} m. You may assume that the proton is stationary and the centripetal force is supplied by Coulomb attraction.

Exercise:

Problem: Integrated Concepts

An electron has an initial velocity of 5.00×10^6 m/s in a uniform 2.00×10^5 N/C strength electric field. The field accelerates the electron in the direction opposite to its initial velocity. (a) What is the direction of the electric field? (b) How far does the electron travel before coming to rest? (c) How long does it take the electron to come to rest? (d) What is the electron's velocity when it returns to its starting point?

Exercise:

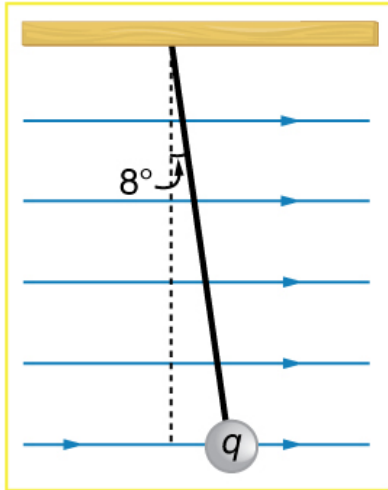
Problem: Integrated Concepts

The practical limit to an electric field in air is about 3.00×10^6 N/C. Above this strength, sparking takes place because air begins to ionize and charges flow, reducing the field. (a) Calculate the distance a free proton must travel in this field to reach 3.00% of the speed of light, starting from rest. (b) Is this practical in air, or must it occur in a vacuum?

Exercise:

Problem: Integrated Concepts

A 5.00 g charged insulating ball hangs on a 30.0 cm long string in a uniform horizontal electric field as shown in [\[link\]](#). Given the charge on the ball is $1.00 \mu\text{C}$, find the strength of the field.

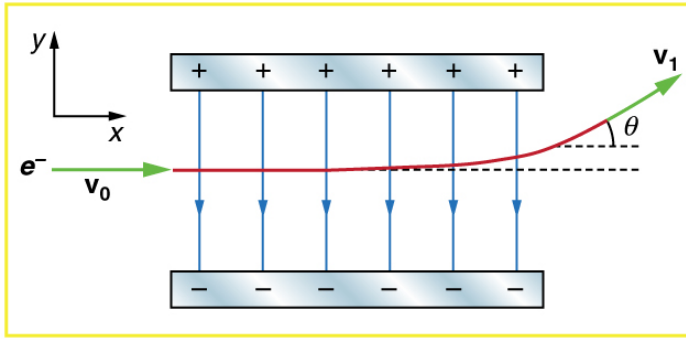


A horizontal electric field causes the charged ball to hang at an angle of 8.00° .

Exercise:

Problem: Integrated Concepts

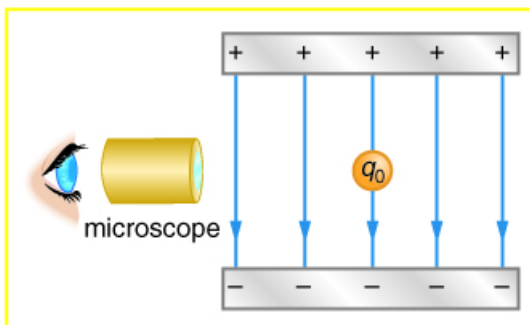
[\[link\]](#) shows an electron passing between two charged metal plates that create an 100 N/C vertical electric field perpendicular to the electron's original horizontal velocity. (These can be used to change the electron's direction, such as in an oscilloscope.) The initial speed of the electron is $3.00 \times 10^6 \text{ m/s}$, and the horizontal distance it travels in the uniform field is 4.00 cm . (a) What is its vertical deflection? (b) What is the vertical component of its final velocity? (c) At what angle does it exit? Neglect any edge effects.



Exercise:

Problem: Integrated Concepts

The classic Millikan oil drop experiment was the first to obtain an accurate measurement of the charge on an electron. In it, oil drops were suspended against the gravitational force by a vertical electric field. (See [\[link\]](#).) Given the oil drop to be $1.00 \mu\text{m}$ in radius and have a density of 920 kg/m^3 : (a) Find the weight of the drop. (b) If the drop has a single excess electron, find the electric field strength needed to balance its weight.



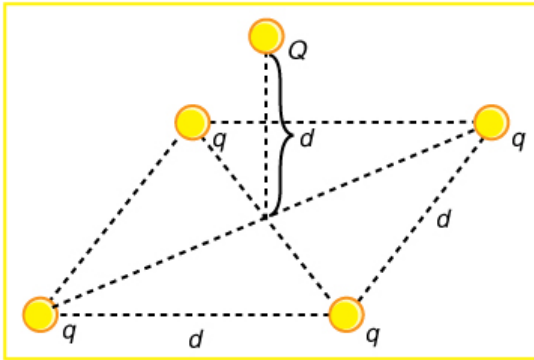
In the Millikan oil drop experiment, small drops can be suspended in an electric field by the force exerted on a single excess electron. Classically, this experiment was used to determine the electron charge q_e by

measuring the electric field
and mass of the drop.

Exercise:

Problem: Integrated Concepts

(a) In [\[link\]](#), four equal charges q lie on the corners of a square. A fifth charge Q is on a mass m directly above the center of the square, at a height equal to the length d of one side of the square. Determine the magnitude of q in terms of Q , m , and d , if the Coulomb force is to equal the weight of m . (b) Is this equilibrium stable or unstable? Discuss.



Four equal charges on the
corners of a horizontal
square support the weight of
a fifth charge located
directly above the center of
the square.

Exercise:

Problem: Unreasonable Results

(a) Calculate the electric field strength near a 10.0 cm diameter
conducting sphere that has 1.00 C of excess charge on it. (b) What is

unreasonable about this result? (c) Which assumptions are responsible?

Exercise:

Problem: Unreasonable Results

(a) Two 0.500 g raindrops in a thunderhead are 1.00 cm apart when they each acquire 1.00 mC charges. Find their acceleration. (b) What is unreasonable about this result? (c) Which premise or assumption is responsible?

Exercise:

Problem: Unreasonable Results

A wrecking yard inventor wants to pick up cars by charging a 0.400 m diameter ball and inducing an equal and opposite charge on the car. If a car has a 1000 kg mass and the ball is to be able to lift it from a distance of 1.00 m: (a) What minimum charge must be used? (b) What is the electric field near the surface of the ball? (c) Why are these results unreasonable? (d) Which premise or assumption is responsible?

Exercise:

Problem: Construct Your Own Problem

Consider two insulating balls with evenly distributed equal and opposite charges on their surfaces, held with a certain distance between the centers of the balls. Construct a problem in which you calculate the electric field (magnitude and direction) due to the balls at various points along a line running through the centers of the balls and extending to infinity on either side. Choose interesting points and comment on the meaning of the field at those points. For example, at what points might the field be just that due to one ball and where does the field become negligibly small? Among the things to be considered are the magnitudes of the charges and the distance between the centers of the balls. Your instructor may wish for you to consider the electric

field off axis or for a more complex array of charges, such as those in a water molecule.

Exercise:

Problem: Construct Your Own Problem

Consider identical spherical conducting space ships in deep space where gravitational fields from other bodies are negligible compared to the gravitational attraction between the ships. Construct a problem in which you place identical excess charges on the space ships to exactly counter their gravitational attraction. Calculate the amount of excess charge needed. Examine whether that charge depends on the distance between the centers of the ships, the masses of the ships, or any other factors. Discuss whether this would be an easy, difficult, or even impossible thing to do in practice.

Glossary

Van de Graaff generator

a machine that produces a large amount of excess charge, used for experiments with high voltage

electrostatics

the study of electric forces that are static or slow-moving

photoconductor

a substance that is an insulator until it is exposed to light, when it becomes a conductor

xerography

a dry copying process based on electrostatics

grounded

connected to the ground with a conductor, so that charge flows freely to and from the Earth to the grounded object

laser printer

uses a laser to create a photoconductive image on a drum, which attracts dry ink particles that are then rolled onto a sheet of paper to print a high-quality copy of the image

ink-jet printer

small ink droplets sprayed with an electric charge are controlled by electrostatic plates to create images on paper

electrostatic precipitators

filters that apply charges to particles in the air, then attract those charges to a filter, removing them from the airstream

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In Electric Charge and Electric Field, we just scratched the surface (or at least rubbed it) of electrical phenomena. Two of the most familiar aspects of

electricity are its energy and *voltage*. We know, for example, that great amounts of electrical energy can be stored in batteries, are transmitted cross-country through power lines, and may jump from clouds to explode the sap of trees. In a similar manner, at molecular levels, *ions* cross cell membranes and transfer information. We also know about voltages associated with electricity. Batteries are typically a few volts, the outlets in your home produce 120 volts, and power lines can be as high as hundreds of thousands of volts. But energy and voltage are not the same thing. A motorcycle battery, for example, is small and would not be very successful in replacing the much larger battery in a car, yet each has the same voltage. In this chapter, we shall examine the relationship between voltage and electrical energy and begin to explore some of the many applications of electricity. We do so by introducing the concept of electric potential and describing the relationship between electric field and electric potential.

This chapter presents the concept of equipotential lines (lines of equal potential) as a way to visualize the electric field (Enduring Understanding 2.E, Essential Knowledge 2.E.2). An analogy between the isolines on topographic maps for gravitational field and equipotential lines for the electric field is used to develop a conceptual understanding of equipotential lines (Essential Knowledge 2.E.1). The relationship between the magnitude of an electric field, change in electric potential, and displacement is stated for a uniform field and extended for the more general case using the concept of the “average value” of the electric field (Essential Knowledge 2.E.3).

The concept that an electric field is caused by charged objects (Enduring Understanding 2.C) supports Big Idea 2, that fields exist in space and can be used to explain interactions. The relationship between the electric field, electric charge, and electric force (Essential Knowledge 2.C.1) is used to describe the behavior of charged particles. The uniformity of the electric field between two oppositely charged parallel plates with uniformly distributed electric charge (Essential Knowledge 2.C.5), as well as the properties of materials and their geometry, are used to develop understanding of the capacitance of a capacitor (Essential Knowledge 4.E.4).

This chapter also supports Big Idea 4, that interactions between systems result in changes in those systems. This idea is applied to electric properties of various systems of charged objects, demonstrating the effect of electric interactions on electric properties of systems (Enduring Understanding 4.E). This fact in turn supports Big Idea 5, that changes due to interactions are governed by conservation laws. In particular, the energy of a system is conserved (Enduring Understanding 5.B). Any system that has internal structure can have internal energy. For a system of charged objects, internal energy can change as a result of changes in the arrangement of charges and their geometric configuration as long as work is done on, or by, the system (Essential Knowledge 5.B.2). When objects within the system interact with conservative forces, such as electric forces, the internal energy is defined by the potential energy of that interaction (Essential Knowledge 5.B.3). In general, the internal energy of a system is the sum of the kinetic energies of all its objects and the potential energy of interaction between the objects within the system (Essential Knowledge 5.B.4).

The concepts in this chapter support:

Big Idea 1 Objects and systems have properties such as mass and charge. Systems may have internal structure.

Enduring Understanding 1.E Materials have many macroscopic properties that result from the arrangement and interactions of the atoms and molecules that make up the material.

Essential Knowledge 1.E.4 Matter has a property called electric permittivity.

Big Idea 2 Fields existing in space can be used to explain interactions.

Enduring Understanding 2.C An electric field is caused by an object with electric charge.

Essential Knowledge 2.C.1 The magnitude of the electric force F exerted on an object with electric charge q by an electric field $F = qE$. The direction of the force is determined by the direction of the field and the sign of the charge, with positively charged objects accelerating in the direction of the

field and negatively charged objects accelerating in the direction opposite the field. This should include a vector field map for positive point charges, negative point charges, spherically symmetric charge distribution, and uniformly charged parallel plates.

Essential Knowledge 2.C.5 Between two oppositely charged parallel plates with uniformly distributed electric charge, at points far from the edges of the plates, the electric field is perpendicular to the plates and is constant in both magnitude and direction.

Enduring Understanding 2.E Physicists often construct a map of isolines connecting points of equal value for some quantity related to a field and use these maps to help visualize the field.

Essential Knowledge 2.E.1 Isolines on a topographic (elevation) map describe lines of approximately equal gravitational potential energy per unit mass (gravitational equipotential). As the distance between two different isolines decreases, the steepness of the surface increases. [Contour lines on topographic maps are useful teaching tools for introducing the concept of equipotential lines. Students are encouraged to use the analogy in their answers when explaining gravitational and electrical potential and potential differences.]

Essential Knowledge 2.E.2 Isolines in a region where an electric field exists represent lines of equal electric potential, referred to as equipotential lines.

Essential Knowledge 2.E.3 The average value of the electric field in a region equals the change in electric potential across that region divided by the change in position (displacement) in the relevant direction.

Big Idea 4 Interactions between systems can result in changes in those systems.

Enduring Understanding 4.E The electric and magnetic properties of a system can change in response to the presence of, or changes in, other objects or systems.

Essential Knowledge 4.E.4 The resistance of a resistor, and the capacitance of a capacitor, can be understood from the basic properties of electric fields and forces, as well as the properties of materials and their geometry.

Big Idea 5 Changes that occur as a result of interactions are constrained by conservation laws.

Enduring Understanding 5.B The energy of a system is conserved.

Essential Knowledge 5.B.2 A system with internal structure can have internal energy, and changes in a system's internal structure can result in changes in internal energy. [Physics 1: includes mass–spring oscillators and simple pendulums. Physics 2: charged object in electric fields and examining changes in internal energy with changes in configuration.]

Essential Knowledge 5.B.3 A system with internal structure can have potential energy. Potential energy exists within a system if the objects within that system interact with conservative forces.

Essential Knowledge 5.B.4 The internal energy of a system includes the kinetic energy of the objects that make up the system and the potential energy of the configuration of the objects that make up the system.

Electric Potential Energy: Potential Difference

LEARNING OBJECTIVES

By the end of this section you will be able to:

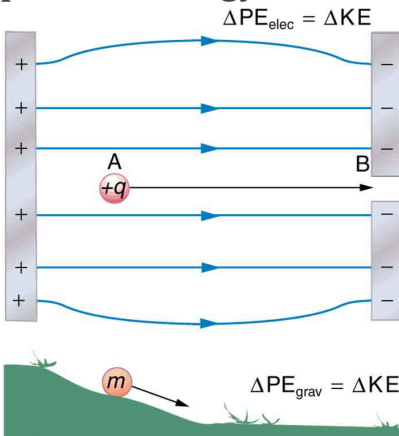
- Define electric potential and electric potential energy.
- Describe the relationship between potential difference and electrical potential energy.
- Explain electron volt and its usage in submicroscopic processes.
- Determine electric potential energy given potential difference and amount of charge.

The information presented in this section supports the following AP® learning objectives and science practices:

- **2.C.1.1** The student is able to predict the direction and the magnitude of the force exerted on an object with an electric charge q placed in an electric field \mathbf{E} using the mathematical model of the relation between an electric force and an electric field: $\mathbf{F} = q\mathbf{E}$; a vector relation. **(S.P. 6.4, 7.2)**
- **2.C.1.2** The student is able to calculate any one of the variables—electric force, electric charge, and electric field—at a point given the values and sign or direction of the other two quantities. **(S.P. 2.2)**
- **5.B.2.1** The student is able to calculate the expected behavior of a system using the object model (i.e., by ignoring changes in internal structure) to analyze a situation. Then, when the model fails, the student can justify the use of conservation of energy principles to calculate the change in internal energy due to changes in internal structure because the object is actually a system. **(S.P. 1.4, 2.1)**

- **5.B.3.1** The student is able to describe and make qualitative and/or quantitative predictions about everyday examples of systems with internal potential energy. (S.P. 2.2, 6.4, 7.2)
- **5.B.3.2** The student is able to make quantitative calculations of the internal potential energy of a system from a description or diagram of that system. (S.P. 1.4, 2.2)
- **5.B.3.3** The student is able to apply mathematical reasoning to create a description of the internal potential energy of a system from a description or diagram of the objects and interactions in that system. (S.P. 1.4, 2.2)
- **5.B.4.1** The student is able to describe and make predictions about the internal energy of systems. (S.P. 6.4, 7.2)
- **5.B.4.2** The student is able to calculate changes in kinetic energy and potential energy of a system, using information from representations of that system. (S.P. 1.4, 2.1, 2.2)

When a free positive charge q is accelerated by an electric field, such as shown in [\[link\]](#), it is given kinetic energy. The process is analogous to an object being accelerated by a gravitational field. It is as if the charge is going down an electrical hill where its electric potential energy is converted to kinetic energy. Let us explore the work done on a charge q by the electric field in this process, so that we may develop a definition of electric potential energy.



A charge accelerated by an electric field is analogous to a mass going down a hill. In both cases potential energy is converted to another form. Work is done by a force, but since this force is conservative, we can write

$$W = -\Delta PE.$$

The electrostatic or Coulomb force is conservative, which means that the work done on q is independent of the path taken. This is exactly analogous to the gravitational force in the absence of dissipative forces such as friction. When a force is conservative, it is possible to define a potential energy associated with the force, and it is usually easier to deal with the potential energy (because it depends only on position) than to calculate the work directly.

We use the letters PE to denote electric potential energy, which has units of joules (J). The change in potential energy, ΔPE , is crucial, since the work done by a conservative force is the negative of the change in potential energy; that is, $W = -\Delta PE$. For example, work W done to accelerate a positive charge from rest is positive and results from a loss in PE, or a negative ΔPE . There must be a minus sign in front of ΔPE to make W positive. PE can be found at any point by taking one point as a reference and calculating the work needed to move a charge to the other point.

Note:

Potential Energy

$W = -\Delta PE$. For example, work W done to accelerate a positive charge from rest is positive and results from a loss in PE, or a negative ΔPE . There must be a minus sign in front of ΔPE to make W positive. PE can be found at any point by taking one point as a reference and calculating the work needed to move a charge to the other point.

Gravitational potential energy and electric potential energy are quite analogous. Potential energy accounts for work done by a conservative force and gives added insight regarding energy and energy transformation without the necessity of dealing with the force directly. It is much more common, for example, to use the concept of voltage (related to electric potential energy) than to deal with the Coulomb force directly.

Calculating the work directly is generally difficult, since $W = Fd \cos \theta$ and the direction and magnitude of F can be complex for multiple charges, for odd-shaped objects, and along arbitrary paths. But we do know that, since $F = qE$, the work, and hence ΔPE , is proportional to the test charge q . To have a physical quantity that is independent of test charge, we define **electric potential** V (or simply potential, since electric is understood) to be the potential energy per unit charge:

Equation:

$$V = \frac{PE}{q}.$$

Note:

Electric Potential

This is the electric potential energy per unit charge.

Equation:

$$V = \frac{PE}{q}$$

Since PE is proportional to q , the dependence on q cancels. Thus V does not depend on q . The change in potential energy ΔPE is crucial, and so we are concerned with the difference in potential or potential difference ΔV between two points, where

Equation:

$$\Delta V = V_B - V_A = \frac{\Delta\text{PE}}{q}.$$

The **potential difference** between points A and B, $V_B - V_A$, is thus defined to be the change in potential energy of a charge q moved from A to B, divided by the charge. Units of potential difference are joules per coulomb, given the name volt (V) after Alessandro Volta.

Equation:

$$1 \text{ V} = 1 \frac{\text{J}}{\text{C}}$$

Note:

Potential Difference

The potential difference between points A and B, $V_B - V_A$, is defined to be the change in potential energy of a charge q moved from A to B, divided by the charge. Units of potential difference are joules per coulomb, given the name volt (V) after Alessandro Volta.

Equation:

$$1 \text{ V} = 1 \frac{\text{J}}{\text{C}}$$

The familiar term **voltage** is the common name for potential difference. Keep in mind that whenever a voltage is quoted, it is understood to be the potential difference between two points. For example, every battery has two terminals, and its voltage is the potential difference between them. More fundamentally, the point you choose to be zero volts is arbitrary. This is analogous to the fact that gravitational potential energy has an arbitrary zero, such as sea level or perhaps a lecture hall floor.

In summary, the relationship between potential difference (or voltage) and electrical potential energy is given by

Equation:

$$\Delta V = \frac{\Delta \text{PE}}{q} \text{ and } \Delta \text{PE} = q\Delta V.$$

Note:

Potential Difference and Electrical Potential Energy

The relationship between potential difference (or voltage) and electrical potential energy is given by

Equation:

$$\Delta V = \frac{\Delta \text{PE}}{q} \text{ and } \Delta \text{PE} = q\Delta V.$$

The second equation is equivalent to the first.

Note:

Real World Connections: Electric Potential in Electronic Devices

You probably use devices with stored electric potential daily. Do you own or use any electronic devices that do not have to be attached to a wall socket? What happens if you use these items long enough? Do they cease functioning? What do you do in that case? Choose one of these types of electronic devices and determine how much electric potential (measured in

volts) the item requires for proper functioning. Then estimate the amount of time between replenishments of potential. Describe how the time between replenishments of potential depends on use.

Note:

Answer

Ready examples include calculators and cell phones. The former will either be solar powered, or have replaceable batteries, probably four 1.5 V for a total of 6 V. The latter will need to be recharged with a specialized charger, which probably puts out 5 V. Times will be highly dependent on which item is used, but should be less with more intense use.

Voltage is not the same as energy. Voltage is the energy per unit charge. Thus a motorcycle battery and a car battery can both have the same voltage (more precisely, the same potential difference between battery terminals), yet one stores much more energy than the other since $\Delta PE = q\Delta V$. The car battery can move more charge than the motorcycle battery, although both are 12 V batteries.

Example:

Calculating Energy

Suppose you have a 12.0 V motorcycle battery that can move 5000 C of charge, and a 12.0 V car battery that can move 60,000 C of charge. How much energy does each deliver? (Assume that the numerical value of each charge is accurate to three significant figures.)

Strategy

To say we have a 12.0 V battery means that its terminals have a 12.0 V potential difference. When such a battery moves charge, it puts the charge

through a potential difference of 12.0 V, and the charge is given a change in potential energy equal to $\Delta PE = q\Delta V$.

So to find the energy output, we multiply the charge moved by the potential difference.

Solution

For the motorcycle battery, $q = 5000 \text{ C}$ and $\Delta V = 12.0 \text{ V}$. The total energy delivered by the motorcycle battery is

Equation:

$$\begin{aligned}\Delta PE_{\text{cycle}} &= (5000 \text{ C})(12.0 \text{ V}) \\ &= (5000 \text{ C})(12.0 \text{ J/C}) \\ &= 6.00 \times 10^4 \text{ J}.\end{aligned}$$

Similarly, for the car battery, $q = 60,000 \text{ C}$ and

Equation:

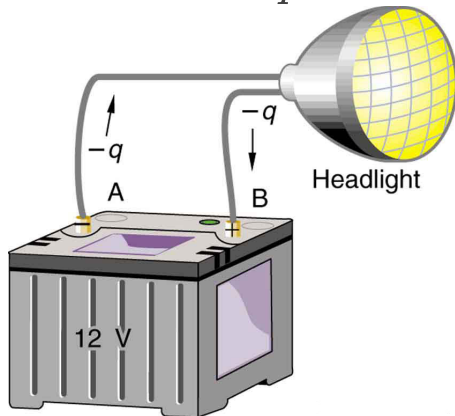
$$\begin{aligned}\Delta PE_{\text{car}} &= (60,000 \text{ C})(12.0 \text{ V}) \\ &= 7.20 \times 10^5 \text{ J}.\end{aligned}$$

Discussion

While voltage and energy are related, they are not the same thing. The voltages of the batteries are identical, but the energy supplied by each is quite different. Note also that as a battery is discharged, some of its energy is used internally and its terminal voltage drops, such as when headlights dim because of a low car battery. The energy supplied by the battery is still calculated as in this example, but not all of the energy is available for external use.

Note that the energies calculated in the previous example are absolute values. The change in potential energy for the battery is negative, since it loses energy. These batteries, like many electrical systems, actually move negative charge—electrons in particular. The batteries repel electrons from their negative terminals (A) through whatever circuitry is involved and attract them to their positive terminals (B) as shown in [\[link\]](#). The change in potential is $\Delta V = V_B - V_A = +12 \text{ V}$ and the charge q is negative, so that

$\Delta PE = q\Delta V$ is negative, meaning the potential energy of the battery has decreased when q has moved from A to B.



A battery moves negative charge from its negative terminal through a headlight to its positive terminal.

Appropriate combinations of chemicals in the battery separate charges so that the negative terminal has an excess of negative charge, which is repelled by it and attracted to the excess positive charge on the other terminal. In terms of potential, the positive terminal is at a higher voltage than the negative. Inside the battery, both positive and negative charges move.

Note:**Making Connections: Potential Energy in a Battery**

The previous example stated that the potential energy of a battery decreased with each electron it pushed out. However, shouldn't this reduced internal energy reduce the potential, as well? Yes, it should. So why don't we notice this?

Part of the answer is that the amount of energy taken by any one electron is extremely small, and therefore it doesn't reduce the potential much. But the main reason is that the energy is stored in the battery as a chemical reaction waiting to happen, not as electric potential. This reaction only runs when a load is attached to both terminals of the battery. Any one set of chemical reactants has a certain maximum potential that it can provide; this is why larger batteries consist of cells attached in series, so that the overall potential increases additively. As these reactants get used up, each cell gives less potential to the electrons it is moving; eventually this potential falls below a useful threshold. Then the battery either needs to be charged, which reverses the chemical reaction and reconstitutes the original reactants; or changed.

Example:**How Many Electrons Move through a Headlight Each Second?**

When a 12.0 V car battery runs a single 30.0 W headlight, how many electrons pass through it each second?

Strategy

To find the number of electrons, we must first find the charge that moved in 1.00 s. The charge moved is related to voltage and energy through the equation $\Delta PE = q\Delta V$. A 30.0 W lamp uses 30.0 joules per second. Since the battery loses energy, we have $\Delta PE = -30.0 \text{ J}$ and, since the electrons are going from the negative terminal to the positive, we see that $\Delta V = +12.0 \text{ V}$.

Solution

To find the charge q moved, we solve the equation $\Delta PE = q\Delta V$:

Equation:

$$q = \frac{\Delta PE}{\Delta V}.$$

Entering the values for ΔPE and ΔV , we get

Equation:

$$q = \frac{-30.0 \text{ J}}{+12.0 \text{ V}} = \frac{-30.0 \text{ J}}{+12.0 \text{ J/C}} = -2.50 \text{ C}.$$

The number of electrons n_e is the total charge divided by the charge per electron. That is,

Equation:

$$n_e = \frac{-2.50 \text{ C}}{-1.60 \times 10^{-19} \text{ C/e}^-} = 1.56 \times 10^{19} \text{ electrons}.$$

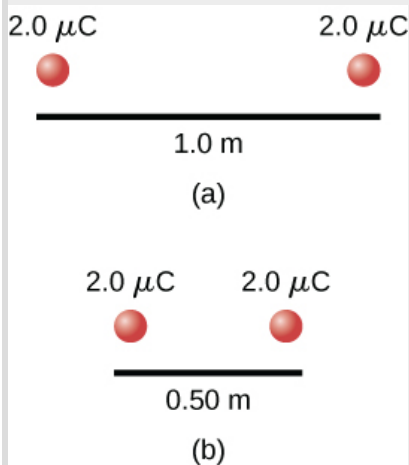
Discussion

This is a very large number. It is no wonder that we do not ordinarily observe individual electrons with so many being present in ordinary systems. In fact, electricity had been in use for many decades before it was determined that the moving charges in many circumstances were negative. Positive charge moving in the opposite direction of negative charge often produces identical effects; this makes it difficult to determine which is moving or whether both are moving.

Note:**Applying the Science Practices: Work and Potential Energy in Point Charges**

Consider a system consisting of two positive point charges, each $2.0 \mu\text{C}$, placed 1.0 m away from each other. We can calculate the potential (i.e. internal) energy of this configuration by computing the potential due to one of the charges, and then calculating the potential energy of the second charge in the potential of the first. Applying Equations (19.38) and (19.2)

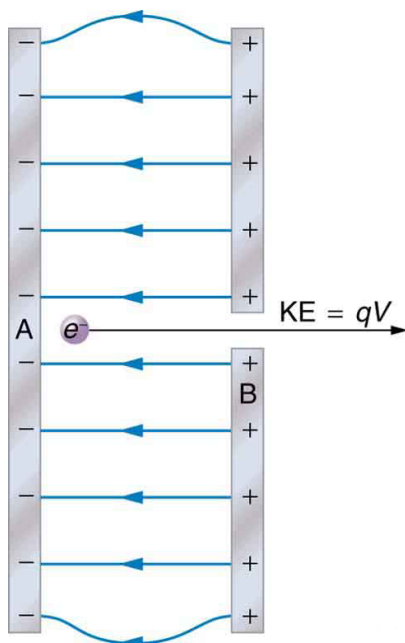
gives us a potential energy of 3.6×10^{-2} J. If we move the charges closer to each other, say, to 0.50 m apart, the potential energy doubles. Note that, to create this second case, some outside force would have had to do work on this system to change the configuration, and hence it was not a closed system. However, because the electric force is conservative, we can use the work-energy theorem to state that, since there was no change in kinetic energy, all of the work done went into increasing the internal energy of the system. Also note that if the point charges had different signs they would be attracted to each other, so they would be capable of doing work on an outside system when the distance between them decreased. As work is done on the outside system, the internal energy in the two-charge system decreases.



Work is done by moving two charges with the same sign closer to each other, increasing the internal energy of the two-charge system.

The Electron Volt

The energy per electron is very small in macroscopic situations like that in the previous example—a tiny fraction of a joule. But on a submicroscopic scale, such energy per particle (electron, proton, or ion) can be of great importance. For example, even a tiny fraction of a joule can be great enough for these particles to destroy organic molecules and harm living tissue. The particle may do its damage by direct collision, or it may create harmful x rays, which can also inflict damage. It is useful to have an energy unit related to submicroscopic effects. [\[link\]](#) shows a situation related to the definition of such an energy unit. An electron is accelerated between two charged metal plates as it might be in an old-model television tube or oscilloscope. The electron is given kinetic energy that is later converted to another form—light in the television tube, for example. (Note that downhill for the electron is uphill for a positive charge.) Since energy is related to voltage by $\Delta PE = q\Delta V$, we can think of the joule as a coulomb-volt.



A typical electron
gun accelerates
electrons using a
potential difference

between two metal plates. The energy of the electron in electron volts is numerically the same as the voltage between the plates.

For example, a 5000 V potential difference produces 5000 eV electrons.

On the submicroscopic scale, it is more convenient to define an energy unit called the **electron volt** (eV), which is the energy given to a fundamental charge accelerated through a potential difference of 1 V. In equation form,

Equation:

$$\begin{aligned} 1 \text{ eV} &= (1.60 \times 10^{-19} \text{ C})(1 \text{ V}) = (1.60 \times 10^{-19} \text{ C})(1 \text{ J/C}) \\ &= 1.60 \times 10^{-19} \text{ J.} \end{aligned}$$

Note:

Electron Volt

On the submicroscopic scale, it is more convenient to define an energy unit called the electron volt (eV), which is the energy given to a fundamental charge accelerated through a potential difference of 1 V. In equation form,

Equation:

$$\begin{aligned} 1 \text{ eV} &= (1.60 \times 10^{-19} \text{ C})(1 \text{ V}) = (1.60 \times 10^{-19} \text{ C})(1 \text{ J/C}) \\ &= 1.60 \times 10^{-19} \text{ J.} \end{aligned}$$

An electron accelerated through a potential difference of 1 V is given an energy of 1 eV. It follows that an electron accelerated through 50 V is given 50 eV. A potential difference of 100,000 V (100 kV) will give an electron an energy of 100,000 eV (100 keV), and so on. Similarly, an ion with a double positive charge accelerated through 100 V will be given 200 eV of energy. These simple relationships between accelerating voltage and particle charges make the electron volt a simple and convenient energy unit in such circumstances.

Note:**Connections: Energy Units**

The electron volt (eV) is the most common energy unit for submicroscopic processes. This will be particularly noticeable in the chapters on modern physics. Energy is so important to so many subjects that there is a tendency to define a special energy unit for each major topic. There are, for example, calories for food energy, kilowatt-hours for electrical energy, and therms for natural gas energy.

The electron volt is commonly employed in submicroscopic processes—chemical valence energies and molecular and nuclear binding energies are among the quantities often expressed in electron volts. For example, about 5 eV of energy is required to break up certain organic molecules. If a proton is accelerated from rest through a potential difference of 30 kV, it is given an energy of 30 keV (30,000 eV) and it can break up as many as 6000 of these molecules ($30,000 \text{ eV} \div 5 \text{ eV per molecule} = 6000 \text{ molecules}$). Nuclear decay energies are on the order of 1 MeV (1,000,000 eV) per event and can, thus, produce significant biological damage.

Conservation of Energy

The total energy of a system is conserved if there is no net addition (or subtraction) of work or heat transfer. For conservative forces, such as the

electrostatic force, conservation of energy states that mechanical energy is a constant.

Mechanical energy is the sum of the kinetic energy and potential energy of a system; that is, $KE + PE = \text{constant}$. A loss of PE of a charged particle becomes an increase in its KE. Here PE is the electric potential energy. Conservation of energy is stated in equation form as

Equation:

$$KE + PE = \text{constant}$$

or

Equation:

$$KE_i + PE_i = KE_f + PE_f,$$

where i and f stand for initial and final conditions. As we have found many times before, considering energy can give us insights and facilitate problem solving.

Example:

Electrical Potential Energy Converted to Kinetic Energy

Calculate the final speed of a free electron accelerated from rest through a potential difference of 100 V. (Assume that this numerical value is accurate to three significant figures.)

Strategy

We have a system with only conservative forces. Assuming the electron is accelerated in a vacuum, and neglecting the gravitational force (we will check on this assumption later), all of the electrical potential energy is converted into kinetic energy. We can identify the initial and final forms of energy to be $KE_i = 0$, $KE_f = \frac{1}{2}mv^2$, $PE_i = qV$, and $PE_f = 0$.

Solution

Conservation of energy states that

Equation:

$$KE_i + PE_i = KE_f + PE_f.$$

Entering the forms identified above, we obtain

Equation:

$$qV = \frac{mv^2}{2}.$$

We solve this for v :

Equation:

$$v = \sqrt{\frac{2qV}{m}}.$$

Entering values for q , V , and m gives

Equation:

$$\begin{aligned} v &= \sqrt{\frac{2(-1.60 \times 10^{-19} \text{ C})(-100 \text{ J/C})}{9.11 \times 10^{-31} \text{ kg}}} \\ &= 5.93 \times 10^6 \text{ m/s.} \end{aligned}$$

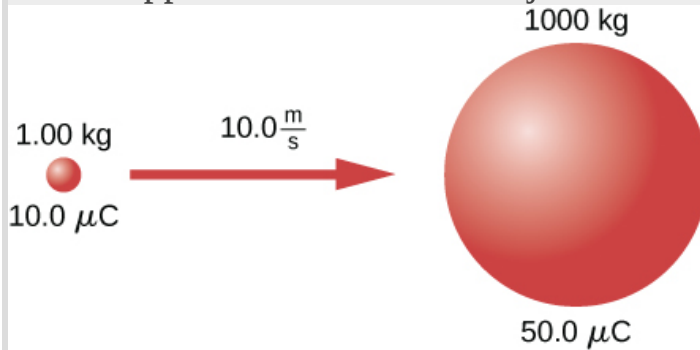
Discussion

Note that both the charge and the initial voltage are negative, as in [\[link\]](#). From the discussions in [Electric Charge and Electric Field](#), we know that electrostatic forces on small particles are generally very large compared with the gravitational force. The large final speed confirms that the gravitational force is indeed negligible here. The large speed also indicates how easy it is to accelerate electrons with small voltages because of their very small mass. Voltages much higher than the 100 V in this problem are typically used in electron guns. Those higher voltages produce electron speeds so great that relativistic effects must be taken into account. That is why a low voltage is considered (accurately) in this example.

Note:

Making Connections: Kinetic and Potential Energy in Point Charges

Now consider another system of two point charges. One has a mass of 1000 kg and a charge of $50.0\ \mu\text{C}$, and is initially stationary. The other has a mass of 1.00 kg, a charge of $10.0\ \mu\text{C}$, and is initially traveling directly at the first point charge at $10.0\ \text{m/s}$ from very far away. What will be the closest approach of these two objects to each other?



A system consisting of two point charges initially has the smaller charge moving toward the larger charge

Note that the internal energy of this two-charge system will not change, due to an absence of external forces acting on the system. Initially, the internal energy is equal to the kinetic energy of the smaller charge, and the potential energy is effectively zero due to the enormous distance between the two objects. Conservation of energy tells us that the internal energy of this system will not change. Hence the distance of closest approach will be when the internal energy is equal to the potential energy between the two charges, and there is no kinetic energy in this system.

The initial kinetic energy may be calculated as 50.0 J. Applying Equations (19.38) and (19.2), we find a distance of 9.00 cm. After this, the mutual repulsion will send the lighter object off to infinity again. Note that we did not include potential energy due to gravity, as the masses concerned are so small compared to the charges that the result will never come close to showing up in significant digits. Furthermore, the first object is much more massive than the second. As a result, any motion induced in it will also be too small to show up in the significant digits.

Test Prep for AP Courses

Exercise:

Problem:

An electron is placed in an electric field of 12.0 N/C to the right. What is the resulting force on the electron?

- a. $1.33 \times 10^{-20} \text{ N}$ right
 - b. $1.33 \times 10^{-20} \text{ N}$ left
 - c. $1.92 \times 10^{-18} \text{ N}$ right
 - d. $1.92 \times 10^{-18} \text{ N}$ left
-

Solution:

(a)

Exercise:

Problem:

A positively charged object in a certain electric field is currently being pushed west by the resulting force. How will the force change if the charge grows? What if it becomes negative?

Exercise:

Problem:

A -5.0 C charge is being forced south by a 60 N force. What are the magnitude and direction of the local electric field?

- a. 12 N/C south
 - b. 12 N/C north
 - c. 300 N/C south
 - d. 300 N/C north
-

Solution:

(b)

Exercise:

Problem:

A charged object has a net force of 100 N east acting on it due to an electric field of 50 N/C pointing north. How is this possible? If not, why not?

Exercise:

Problem:

How many electrons have to be moved by a car battery containing 7.20×10^5 J at 12 V to reduce the energy by 1%?

- a. 4.80×10^{27}
- b. 4.00×10^{26}
- c. 3.75×10^{21}
- d. 3.13×10^{20}

Solution:

(c)

Exercise:

Problem:

Most of the electricity in the power grid is generated by powerful turbines spinning around. Why don't these turbines slow down from the work they do moving electrons?

Exercise:

Problem:

A typical AAA battery can move 2000 C of charge at 1.5 V. How long will this run a 50 mW LED?

- a. 1000 minutes

- b. 120,000 seconds
- c. 15 hours
- d. 250 minutes

Solution:

(a)

Exercise:

Problem:

Find an example car (or other vehicle) battery, and compute how many of the AAA batteries in the previous problem it would take to equal the energy stored in it. Which is more compact?

Exercise:

Problem:

What is the internal energy of a system consisting of two point charges, one $2.0\ \mu\text{C}$, and the other $-3.0\ \mu\text{C}$, placed $1.2\ \text{m}$ away from each other?

- a. $-3.8 \times 10^{-2}\ \text{J}$
- b. $-4.5 \times 10^{-2}\ \text{J}$
- c. $4.5 \times 10^{-2}\ \text{J}$
- d. $3.8 \times 10^{-2}\ \text{J}$

Solution:

(b)

Exercise:

Problem:

A system of three point charges has a $1.00\ \mu\text{C}$ charge at the origin, a $-2.00\ \mu\text{C}$ charge at $x=30\ \text{cm}$, and a $3.00\ \mu\text{C}$ charge at $x=70\ \text{cm}$. What is the total stored potential energy of this configuration?

Exercise:**Problem:**

A system has $2.00\ \mu\text{C}$ charges at $(50\ \text{cm}, 0)$ and $(-50\ \text{cm}, 0)$ and a $-1.00\ \mu\text{C}$ charge at $(0, 70\ \text{cm})$. As the y -coordinate of the $-1.00\ \mu\text{C}$ charge increases, the potential energy _____. As the y -coordinate of the $-1.00\ \mu\text{C}$ charge decreases, the potential energy _____.

- a. increases, increases
- b. increases, decreases
- c. decreases, increases
- d. decreases, decreases

Solution:

(b)

Exercise:**Problem:**

A system of three point charges has a $1.00\ \mu\text{C}$ charge at the origin, a $-2.00\ \mu\text{C}$ charge at $x=30\ \text{cm}$, and a $3.00\ \mu\text{C}$ charge at $x=70\ \text{cm}$. What happens to the total potential energy of this system if the $-2.00\ \mu\text{C}$ charge and the $3.00\ \mu\text{C}$ charge trade places?

Exercise:**Problem:**

Take a square configuration of point charges, two positive and two negative, all of the same magnitude, with like charges sharing diagonals. What will happen to the internal energy of this system if one of the negative charges becomes a positive charge of the same magnitude?

- a. increase
- b. decrease

- c. no change
 - d. not enough information
-

Solution:

(a)

Exercise:

Problem:

Take a square configuration of point charges, two positive and two negative, all of the same magnitude, with like charges sharing diagonals. What will happen to the internal energy of this system if the sides of the square decrease in length?

Exercise:

Problem:

A system has $2.00\ \mu\text{C}$ charges at $(50\ \text{cm}, 0)$ and $(-50\ \text{cm}, 0)$ and a $-1.00\ \mu\text{C}$ charge at $(0, 70\ \text{cm})$, with a velocity in the $-y$ -direction. When the $-1.00\ \mu\text{C}$ charge is at $(0, 0)$ the potential energy is at a ____ and the kinetic energy is ____.

- a. maximum, maximum
 - b. maximum, minimum
 - c. minimum, maximum
 - d. minimum, minimum
-

Solution:

(c)

Exercise:

Problem:

What is the velocity of an electron that goes through a $10\ \text{V}$ potential after initially being at rest?

Section Summary

- Electric potential is potential energy per unit charge.
- The potential difference between points A and B, $V_B - V_A$, defined to be the change in potential energy of a charge q moved from A to B, is equal to the change in potential energy divided by the charge, Potential difference is commonly called voltage, represented by the symbol ΔV .

Equation:

$$\Delta V = \frac{\Delta PE}{q} \text{ and } \Delta PE = q\Delta V.$$

- An electron volt is the energy given to a fundamental charge accelerated through a potential difference of 1 V. In equation form,

Equation:

$$\begin{aligned} 1 \text{ eV} &= (1.60 \times 10^{-19} \text{ C})(1 \text{ V}) = (1.60 \times 10^{-19} \text{ C})(1 \text{ J/C}) \\ &= 1.60 \times 10^{-19} \text{ J.} \end{aligned}$$

- Mechanical energy is the sum of the kinetic energy and potential energy of a system, that is, $KE + PE$. This sum is a constant.

Conceptual Questions

Exercise:

Problem:

Voltage is the common word for potential difference. Which term is more descriptive, voltage or potential difference?

Exercise:

Problem:

If the voltage between two points is zero, can a test charge be moved between them with zero net work being done? Can this necessarily be done without exerting a force? Explain.

Exercise:**Problem:**

What is the relationship between voltage and energy? More precisely, what is the relationship between potential difference and electric potential energy?

Exercise:

Problem: Voltages are always measured between two points. Why?

Exercise:**Problem:**

How are units of volts and electron volts related? How do they differ?

Problems & Exercises**Exercise:****Problem:**

Find the ratio of speeds of an electron and a negative hydrogen ion (one having an extra electron) accelerated through the same voltage, assuming non-relativistic final speeds. Take the mass of the hydrogen ion to be 1.67×10^{-27} kg.

Solution:

42.8

Exercise:

Problem:

An evacuated tube uses an accelerating voltage of 40 kV to accelerate electrons to hit a copper plate and produce x rays. Non-relativistically, what would be the maximum speed of these electrons?

Exercise:**Problem:**

A bare helium nucleus has two positive charges and a mass of 6.64×10^{-27} kg. (a) Calculate its kinetic energy in joules at 2.00% of the speed of light. (b) What is this in electron volts? (c) What voltage would be needed to obtain this energy?

Exercise:**Problem: Integrated Concepts**

Singly charged gas ions are accelerated from rest through a voltage of 13.0 V. At what temperature will the average kinetic energy of gas molecules be the same as that given these ions?

Solution:

$$1.00 \times 10^5 \text{ K}$$

Exercise:**Problem: Integrated Concepts**

The temperature near the center of the Sun is thought to be 15 million degrees Celsius (1.5×10^7 °C). Through what voltage must a singly charged ion be accelerated to have the same energy as the average kinetic energy of ions at this temperature?

Exercise:**Problem: Integrated Concepts**

(a) What is the average power output of a heart defibrillator that dissipates 400 J of energy in 10.0 ms? (b) Considering the high-power output, why doesn't the defibrillator produce serious burns?

Solution:

(a) 4×10^4 W

(b) A defibrillator does not cause serious burns because the skin conducts electricity well at high voltages, like those used in defibrillators. The gel used aids in the transfer of energy to the body, and the skin doesn't absorb the energy, but rather lets it pass through to the heart.

Exercise:

Problem: Integrated Concepts

A lightning bolt strikes a tree, moving 20.0 C of charge through a potential difference of 1.00×10^2 MV. (a) What energy was dissipated? (b) What mass of water could be raised from 15°C to the boiling point and then boiled by this energy? (c) Discuss the damage that could be caused to the tree by the expansion of the boiling steam.

Exercise:

Problem: Integrated Concepts

A 12.0 V battery-operated bottle warmer heats 50.0 g of glass, 2.50×10^2 g of baby formula, and 2.00×10^2 g of aluminum from 20.0°C to 90.0°C. (a) How much charge is moved by the battery? (b) How many electrons per second flow if it takes 5.00 min to warm the formula? (Hint: Assume that the specific heat of baby formula is about the same as the specific heat of water.)

Solution:

(a) 7.40×10^3 C

(b) 1.54×10^{20} electrons per second

Exercise:

Problem: Integrated Concepts

A battery-operated car utilizes a 12.0 V system. Find the charge the batteries must be able to move in order to accelerate the 750 kg car from rest to 25.0 m/s, make it climb a 2.00×10^2 m high hill, and then cause it to travel at a constant 25.0 m/s by exerting a 5.00×10^2 N force for an hour.

Solution:

3.89×10^6 C

Exercise:

Problem: Integrated Concepts

Fusion probability is greatly enhanced when appropriate nuclei are brought close together, but mutual Coulomb repulsion must be overcome. This can be done using the kinetic energy of high-temperature gas ions or by accelerating the nuclei toward one another. (a) Calculate the potential energy of two singly charged nuclei separated by 1.00×10^{-12} m by finding the voltage of one at that distance and multiplying by the charge of the other. (b) At what temperature will atoms of a gas have an average kinetic energy equal to this needed electrical potential energy?

Exercise:

Problem: Unreasonable Results

(a) Find the voltage near a 10.0 cm diameter metal sphere that has 8.00 C of excess positive charge on it. (b) What is unreasonable about this result? (c) Which assumptions are responsible?

Solution:

(a) $1.44 \times 10^{12} \text{ V}$

(b) This voltage is very high. A 10.0 cm diameter sphere could never maintain this voltage; it would discharge.

(c) An 8.00 C charge is more charge than can reasonably be accumulated on a sphere of that size.

Exercise:**Problem: Construct Your Own Problem**

Consider a battery used to supply energy to a cellular phone. Construct a problem in which you determine the energy that must be supplied by the battery, and then calculate the amount of charge it must be able to move in order to supply this energy. Among the things to be considered are the energy needs and battery voltage. You may need to look ahead to interpret manufacturer's battery ratings in ampere-hours as energy in joules.

Glossary

electric potential

potential energy per unit charge

potential difference (or voltage)

change in potential energy of a charge moved from one point to another, divided by the charge; units of potential difference are joules per coulomb, known as volt

electron volt

the energy given to a fundamental charge accelerated through a potential difference of one volt

mechanical energy

sum of the kinetic energy and potential energy of a system; this sum is a constant

Electric Potential in a Uniform Electric Field

LEARNING OBJECTIVES

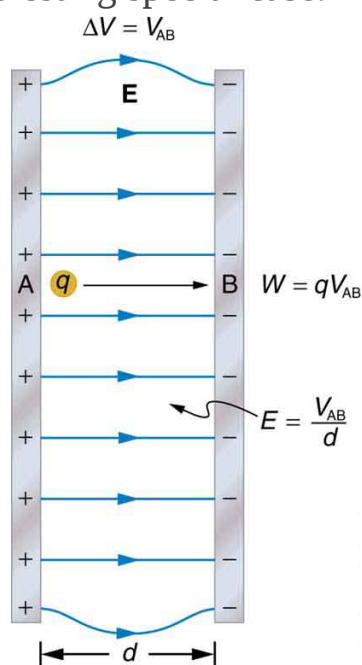
By the end of this section, you will be able to:

- Describe the relationship between voltage and electric field.
- Derive an expression for the electric potential and electric field.
- Calculate electric field strength given distance and voltage.

The information presented in this section supports the following AP[®] learning objectives and science practices:

- **2.C.5.2** The student is able to calculate the magnitude and determine the direction of the electric field between two electrically charged parallel plates, given the charge of each plate, or the electric potential difference and plate separation. **(S.P. 2.2)**
- **2.C.5.3** The student is able to represent the motion of an electrically charged particle in the uniform field between two oppositely charged plates and express the connection of this motion to projectile motion of an object with mass in the Earth's gravitational field. **(S.P. 1.1, 2.2, 7.1)**
- **2.E.3.1** The student is able to apply mathematical routines to calculate the average value of the magnitude of the electric field in a region from a description of the electric potential in that region using the displacement along the line on which the difference in potential is evaluated. **(S.P. 2.2)**
- **2.E.3.2** The student is able to apply the concept of the isoline representation of electric potential for a given electric charge distribution to predict the average value of the electric field in the region. **(S.P. 1.4, 6.4)**

In the previous section, we explored the relationship between voltage and energy. In this section, we will explore the relationship between voltage and electric field. For example, a uniform electric field \mathbf{E} is produced by placing a potential difference (or voltage) ΔV across two parallel metal plates, labeled A and B. (See [\[link\]](#).) Examining this will tell us what voltage is needed to produce a certain electric field strength; it will also reveal a more fundamental relationship between electric potential and electric field. From a physicist's point of view, either ΔV or \mathbf{E} can be used to describe any charge distribution. ΔV is most closely tied to energy, whereas \mathbf{E} is most closely related to force. ΔV is a **scalar** quantity and has no direction, while \mathbf{E} is a **vector** quantity, having both magnitude and direction. (Note that the magnitude of the electric field strength, a scalar quantity, is represented by E below.) The relationship between ΔV and \mathbf{E} is revealed by calculating the work done by the force in moving a charge from point A to point B. But, as noted in [Electric Potential Energy: Potential Difference](#), this is complex for arbitrary charge distributions, requiring calculus. We therefore look at a uniform electric field as an interesting special case.



The relationship between V and E for parallel conducting plates is $E = V/d$. (Note that $\Delta V = V_{AB}$ in magnitude. For a charge that is moved from plate A at higher potential to plate B at lower potential, a minus sign needs to be included as follows:
 $-\Delta V = V_A - V_B = V_{AB}$
 . See the text for details.)

The work done by the electric field in [\[link\]](#) to move a positive charge q from A, the positive plate, higher potential, to B, the negative plate, lower potential, is

Equation:

$$W = -\Delta PE = -q\Delta V.$$

The potential difference between points A and B is

Equation:

$$-\Delta V = -(V_B - V_A) = V_A - V_B = V_{AB}.$$

Entering this into the expression for work yields

Equation:

$$W = qV_{AB}.$$

Work is $W = Fd \cos \theta$; here $\cos \theta = 1$, since the path is parallel to the field, and so $W = Fd$. Since $F = qE$, we see that $W = qEd$. Substituting this expression for work into the previous equation gives

Equation:

$$qEd = qV_{AB}.$$

The charge cancels, and so the voltage between points A and B is seen to be

Equation:

$$\left. \begin{array}{l} V_{AB} = Ed \\ E = \frac{V_{AB}}{d} \end{array} \right\} (\text{uniform } E - \text{field only}),$$

where d is the distance from A to B, or the distance between the plates in [\[link\]](#). Note that the above equation implies the units for electric field are volts per meter. We already know the units for electric field are newtons per coulomb; thus the following relation among units is valid:

Equation:

$$1 \text{ N/C} = 1 \text{ V/m}.$$

Note:

Voltage between Points A and B

Equation:

$$\left. \begin{array}{l} V_{AB} = Ed \\ E = \frac{V_{AB}}{d} \end{array} \right\} (\text{uniform } E - \text{field only}),$$

where d is the distance from A to B, or the distance between the plates.

Example:**What Is the Highest Voltage Possible between Two Plates?**

Dry air will support a maximum electric field strength of about $3.0 \times 10^6 \text{ V/m}$. Above that value, the field creates enough ionization in the air to make the air a conductor. This allows a discharge or spark that reduces the field. What, then, is the maximum voltage between two parallel conducting plates separated by 2.5 cm of dry air?

Strategy

We are given the maximum electric field E between the plates and the distance d between them. The equation $V_{AB} = Ed$ can thus be used to calculate the maximum voltage.

Solution

The potential difference or voltage between the plates is

Equation:

$$V_{AB} = Ed.$$

Entering the given values for E and d gives

Equation:

$$V_{AB} = (3.0 \times 10^6 \text{ V/m})(0.025 \text{ m}) = 7.5 \times 10^4 \text{ V}$$

or

Equation:

$$V_{AB} = 75 \text{ kV}.$$

(The answer is quoted to only two digits, since the maximum field strength is approximate.)

Discussion

One of the implications of this result is that it takes about 75 kV to make a spark jump across a 2.5 cm (1 in.) gap, or 150 kV for a 5 cm spark. This limits the voltages that can exist between conductors, perhaps on a power transmission line. A smaller voltage will cause a spark if there are points on the surface, since points create greater fields than smooth surfaces. Humid air breaks down at a lower field strength, meaning that a smaller voltage will make a spark jump through humid air. The largest voltages can be built up, say with static electricity, on dry days.

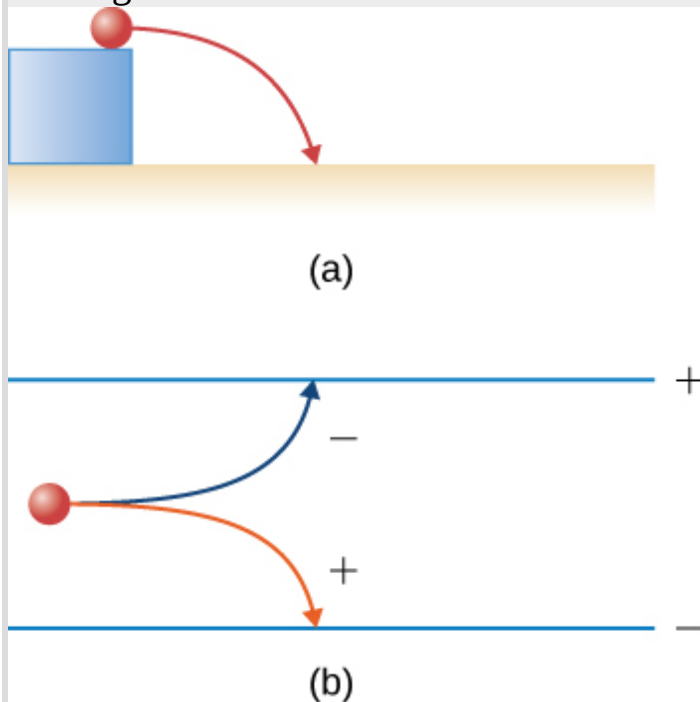


A spark chamber is used to trace the paths of high-energy particles. Ionization created by the particles as they pass through the gas between the plates allows a spark to jump. The sparks are perpendicular to the plates, following electric field lines between them. The potential difference between adjacent plates is not high enough to cause sparks without the ionization produced by particles from accelerator experiments (or

cosmic rays).
(credit: Daderot,
Wikimedia
Commons)

Note:

Making Connections: Uniform Fields



(a) A massive particle launched horizontally in a downward gravitational field will fall to the ground. (b) A positively charged particle launched horizontally in a downward electric field will fall toward the negative potential; a negatively charged particle will move in the opposite direction.

Recall from Projectile Motion ([link](#)) that a massive projectile launched horizontally (for example, from a cliff) in a uniform downward gravitational field (as we find near the surface of the Earth) will follow a parabolic trajectory downward until it hits the ground, as shown in [link](#) (a).

An identical outcome occurs for a positively charged particle in a uniform electric field ([link](#)(b)); it follows the electric field “downhill” until it runs into something. The difference between the two cases is that the gravitational force is always attractive; the electric force has two kinds of charges, and therefore may be either attractive or repulsive. Therefore, a negatively charged particle launched into the same field will fall “uphill”.

Example:

Field and Force inside an Electron Gun

(a) An electron gun has parallel plates separated by 4.00 cm and gives electrons 25.0 keV of energy. What is the electric field strength between the plates? (b) What force would this field exert on a piece of plastic with a 0.500 μC charge that gets between the plates?

Strategy

Since the voltage and plate separation are given, the electric field strength can be calculated directly from the expression $E = \frac{V_{AB}}{d}$. Once the electric field strength is known, the force on a charge is found using $\mathbf{F} = q \mathbf{E}$. Since the electric field is in only one direction, we can write this equation in terms of the magnitudes, $F = q E$.

Solution for (a)

The expression for the magnitude of the electric field between two uniform metal plates is

Equation:

$$E = \frac{V_{AB}}{d}.$$

Since the electron is a single charge and is given 25.0 keV of energy, the potential difference must be 25.0 kV. Entering this value for V_{AB} and the

plate separation of 0.0400 m, we obtain

Equation:

$$E = \frac{25.0 \text{ kV}}{0.0400 \text{ m}} = 6.25 \times 10^5 \text{ V/m}.$$

Solution for (b)

The magnitude of the force on a charge in an electric field is obtained from the equation

Equation:

$$F = qE.$$

Substituting known values gives

Equation:

$$F = (0.500 \times 10^{-6} \text{ C})(6.25 \times 10^5 \text{ V/m}) = 0.313 \text{ N}.$$

Discussion

Note that the units are newtons, since $1 \text{ V/m} = 1 \text{ N/C}$. The force on the charge is the same no matter where the charge is located between the plates. This is because the electric field is uniform between the plates.

In more general situations, regardless of whether the electric field is uniform, it points in the direction of decreasing potential, because the force on a positive charge is in the direction of \mathbf{E} and also in the direction of lower potential V . Furthermore, the magnitude of \mathbf{E} equals the rate of decrease of V with distance. The faster V decreases over distance, the greater the electric field. In equation form, the general relationship between voltage and electric field is

Equation:

$$E = -\frac{\Delta V}{\Delta s},$$

where Δs is the distance over which the change in potential, ΔV , takes place. The minus sign tells us that \mathbf{E} points in the direction of decreasing potential. The electric field is said to be the *gradient* (as in grade or slope) of the electric potential.

Note:

Relationship between Voltage and Electric Field

In equation form, the general relationship between voltage and electric field is

Equation:

$$E = - \frac{\Delta V}{\Delta s},$$

where Δs is the distance over which the change in potential, ΔV , takes place. The minus sign tells us that \mathbf{E} points in the direction of decreasing potential. The electric field is said to be the *gradient* (as in grade or slope) of the electric potential.

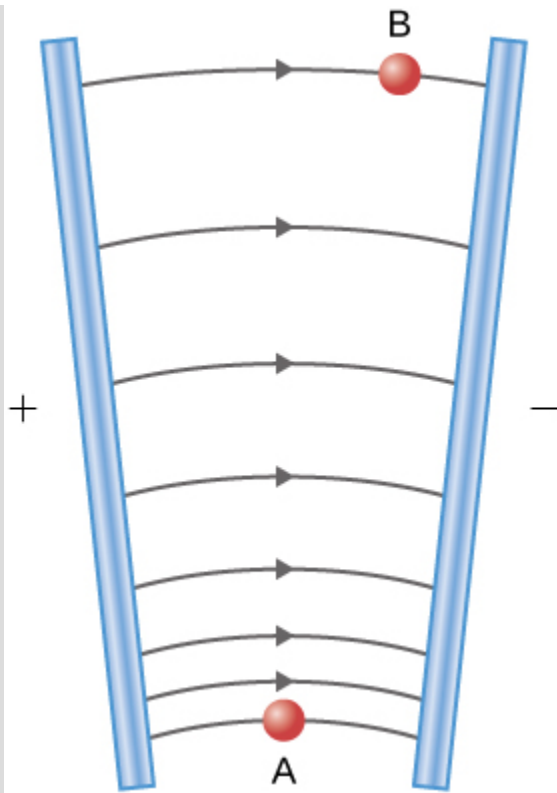
Note that Equation (19.36) is defining the average electric field over the given region.

For continually changing potentials, ΔV and Δs become infinitesimals and differential calculus must be employed to determine the electric field.

Note:

Making Connections: Non-Parallel Conducting Plates

Consider two conducting plates, placed as shown in [\[link\]](#). If the plates are held at a fixed potential difference ΔV , the average electric field is strongest between the near edges of the plates, and weakest between the two far edges of the plates.



Two nonparallel plates, held at a fixed potential difference.

Now assume that the potential difference is 60 V. If the arc length along the field line labeled by A is 10 cm, what is the electric field at point A? How about point B, if the arc length along that field line is 17 cm? How would the density of the electric potential isolines, if they were drawn on the figure, compare at these two points? Can you use this concept to estimate what the electric field strength would be at a point midway between A and B?

Note:

Answer

Applying Equation. 19.36, we find that the electric fields at A and B are 600 V/m and 350 V/m respectively. The isolines would be denser at A

than at B, and would spread out evenly from A to B. Therefore, the electric field at a point halfway between the two would have an arc length of 13.5 cm and be approximately 440 V/m.

Test Prep for AP Courses

Exercise:

Problem:

A negatively charged massive particle is dropped from above the two plates in [\[link\]](#) into the space between them. Which best describes the trajectory it takes?

- a. A rightward-curving parabola
- b. A leftward-curving parabola
- c. A rightward-curving section of a circle
- d. A leftward-curving section of a circle

Solution:

(b)

Exercise:

Problem:

Two massive particles with identical charge are launched into the uniform field between two plates from the same launch point with the same velocity. They both impact the positively charged plate, but the second one does so four times as far as the first. What sign is the charge? What physical difference would give them different impact points (quantify as a relative percent)? How does this compare to the gravitational projectile motion case?

Exercise:**Problem:**

Two plates are lying horizontally, but stacked with one 10.0 cm above the other. If the upper plate is held at +100 V, what is the magnitude and direction of the electric field between the plates if the lower is held at +50.0 V? -50.0 V?

- a. 500 V/m, 1500 V/m, down
 - b. 500 V/m, 1500 V/m, up
 - c. 1500 V/m, 500 V/m, down
 - d. 1500 V/m, 500 V/m, up
-

Solution:

(a)

Exercise:**Problem:**

Two parallel conducting plates are 15 cm apart, each with an area of 0.75 m^2 . The left one has a charge of -0.225 C placed on it, while the right has a charge of 0.225 C. What is the magnitude and direction of the electric field between the two?

Exercise:**Problem:**

Consider three parallel conducting plates, with a space of 3.0 cm between them. The leftmost one is at a potential of +45 V, the middle one is held at ground, and the rightmost is at a potential of -75 V. What is the magnitude of the average electric field on an electron traveling between the plates? (Assume that the middle one has holes for the electron to go through.)

- a. 1500 V/m
- b. 2500 V/m

- c. 4000 V/m
 - d. 2000 V/m
-

Solution:

(d)

Exercise:

Problem:

A new kind of electron gun has a rear plate at -25.0 kV, a grounded plate 2.00 cm in front of that, and a $+25.0$ kV plate 4.00 cm in front of that. What is the magnitude of the average electric field?

Exercise:

Problem:

A certain electric potential isoline graph has isolines every 5.0 V. If six of these lines cross a 40 cm path drawn between two points of interest, what is the (magnitude of the average) electric field along this path?

- a. 750 V/m
 - b. 150 V/m
 - c. 38 V/m
 - d. 75 V/m
-

Solution:

(d)

Exercise:

Problem:

Given a system of two parallel conducting plates held at a fixed potential difference, describe what happens to the isolines of the electric potential between them as the distance between them is changed. How does this relate to the electric field strength?

Section Summary

- The voltage between points A and B is

Equation:

$$\left. \begin{aligned} V_{AB} &= Ed \\ E &= \frac{V_{AB}}{d} \end{aligned} \right\} (\text{uniform } E - \text{field only}),$$

where d is the distance from A to B, or the distance between the plates.

- In equation form, the general relationship between voltage and electric field is

Equation:

$$E = - \frac{\Delta V}{\Delta s},$$

where Δs is the distance over which the change in potential, ΔV , takes place. The minus sign tells us that **E** points in the direction of decreasing potential.) The electric field is said to be the *gradient* (as in grade or slope) of the electric potential.

Conceptual Questions

Exercise:

Problem:

Discuss how potential difference and electric field strength are related. Give an example.

Exercise:

Problem:

What is the strength of the electric field in a region where the electric potential is constant?

Exercise:**Problem:**

Will a negative charge, initially at rest, move toward higher or lower potential? Explain why.

Problems & Exercises**Exercise:****Problem:**

Show that units of V/m and N/C for electric field strength are indeed equivalent.

Exercise:**Problem:**

What is the strength of the electric field between two parallel conducting plates separated by 1.00 cm and having a potential difference (voltage) between them of 1.50×10^4 V?

Exercise:**Problem:**

The electric field strength between two parallel conducting plates separated by 4.00 cm is 7.50×10^4 V/m. (a) What is the potential difference between the plates? (b) The plate with the lowest potential is taken to be at zero volts. What is the potential 1.00 cm from that plate (and 3.00 cm from the other)?

Solution:

(a) 3.00 kV

(b) 750 V

Exercise:

Problem:

How far apart are two conducting plates that have an electric field strength of $4.50 \times 10^3 \text{ V/m}$ between them, if their potential difference is 15.0 kV?

Exercise:**Problem:**

(a) Will the electric field strength between two parallel conducting plates exceed the breakdown strength for air ($3.0 \times 10^6 \text{ V/m}$) if the plates are separated by 2.00 mm and a potential difference of $5.0 \times 10^3 \text{ V}$ is applied? (b) How close together can the plates be with this applied voltage?

Solution:

(a) No. The electric field strength between the plates is $2.5 \times 10^6 \text{ V/m}$, which is lower than the breakdown strength for air ($3.0 \times 10^6 \text{ V/m}$).

(b) 1.7 mm

Exercise:**Problem:**

The voltage across a membrane forming a cell wall is 80.0 mV and the membrane is 9.00 nm thick. What is the electric field strength? (The value is surprisingly large, but correct. Membranes are discussed in [Capacitors and Dielectrics](#) and [Nerve Conduction—Electrocardiograms](#).) You may assume a uniform electric field.

Exercise:

Problem:

Membrane walls of living cells have surprisingly large electric fields across them due to separation of ions. (Membranes are discussed in some detail in [Nerve Conduction—Electrocardiograms](#).) What is the voltage across an 8.00 nm-thick membrane if the electric field strength across it is 5.50 MV/m? You may assume a uniform electric field.

Solution:

44.0 mV

Exercise:**Problem:**

Two parallel conducting plates are separated by 10.0 cm, and one of them is taken to be at zero volts. (a) What is the electric field strength between them, if the potential 8.00 cm from the zero volt plate (and 2.00 cm from the other) is 450 V? (b) What is the voltage between the plates?

Exercise:**Problem:**

Find the maximum potential difference between two parallel conducting plates separated by 0.500 cm of air, given the maximum sustainable electric field strength in air to be 3.0×10^6 V/m.

Solution:

15 kV

Exercise:**Problem:**

A doubly charged ion is accelerated to an energy of 32.0 keV by the electric field between two parallel conducting plates separated by 2.00 cm. What is the electric field strength between the plates?

Exercise:**Problem:**

An electron is to be accelerated in a uniform electric field having a strength of $2.00 \times 10^6 \text{ V/m}$. (a) What energy in keV is given to the electron if it is accelerated through 0.400 m? (b) Over what distance would it have to be accelerated to increase its energy by 50.0 GeV?

Solution:

(a) 800 KeV

(b) 25.0 km

Glossary

scalar

physical quantity with magnitude but no direction

vector

physical quantity with both magnitude and direction

Electrical Potential Due to a Point Charge

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain point charges and express the equation for electric potential of a point charge.
- Distinguish between electric potential and electric field.
- Determine the electric potential of a point charge given charge and distance.

Point charges, such as electrons, are among the fundamental building blocks of matter. Furthermore, spherical charge distributions (like on a metal sphere) create external electric fields exactly like a point charge. The electric potential due to a point charge is, thus, a case we need to consider. Using calculus to find the work needed to move a test charge q from a large distance away to a distance of r from a point charge Q , and noting the connection between work and potential ($W = -q\Delta V$), it can be shown that the *electric potential V of a point charge* is

Equation:

$$V = \frac{kQ}{r} \text{ (Point Charge),}$$

where k is a constant equal to $9.0 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2$.

Note:

Electric Potential V of a Point Charge

The electric potential V of a point charge is given by

Equation:

$$V = \frac{kQ}{r} \text{ (Point Charge).}$$

The potential at infinity is chosen to be zero. Thus V for a point charge decreases with distance, whereas \mathbf{E} for a point charge decreases with distance squared:

Equation:

$$E = \frac{F}{q} = \frac{kQ}{r^2}.$$

Recall that the electric potential V is a scalar and has no direction, whereas the electric field \mathbf{E} is a vector. To find the voltage due to a combination of point charges, you add the individual voltages as numbers. To find the total electric field, you must add the individual fields as *vectors*, taking magnitude and direction into account. This is consistent with the fact that V is closely associated with energy, a scalar, whereas \mathbf{E} is closely associated with force, a vector.

Example:**What Voltage Is Produced by a Small Charge on a Metal Sphere?**

Charges in static electricity are typically in the nanocoulomb (nC) to microcoulomb (μC) range. What is the voltage 5.00 cm away from the center of a 1-cm diameter metal sphere that has a -3.00 nC static charge?

Strategy

As we have discussed in [Electric Charge and Electric Field](#), charge on a metal sphere spreads out uniformly and produces a field like that of a point

charge located at its center. Thus we can find the voltage using the equation $V = kQ/r$.

Solution

Entering known values into the expression for the potential of a point charge, we obtain

Equation:

$$\begin{aligned} V &= k \frac{Q}{r} \\ &= (8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2) \left(\frac{-3.00 \times 10^{-9} \text{ C}}{5.00 \times 10^{-2} \text{ m}} \right) \\ &= -539 \text{ V}. \end{aligned}$$

Discussion

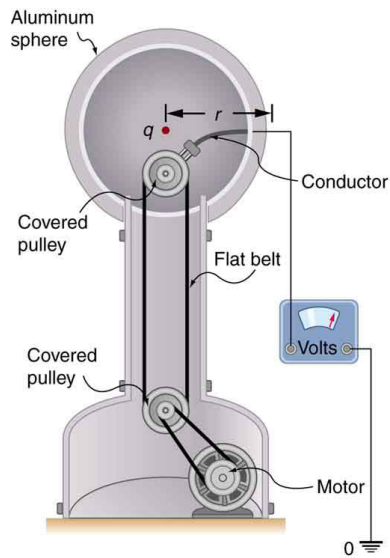
The negative value for voltage means a positive charge would be attracted from a larger distance, since the potential is lower (more negative) than at larger distances. Conversely, a negative charge would be repelled, as expected.

Example:

What Is the Excess Charge on a Van de Graaff Generator

A demonstration Van de Graaff generator has a 25.0 cm diameter metal sphere that produces a voltage of 100 kV near its surface. (See [\[link\]](#).)

What excess charge resides on the sphere? (Assume that each numerical value here is shown with three significant figures.)



The voltage of this demonstration Van de Graaff generator is measured between the charged sphere and ground. Earth's potential is taken to be zero as a reference. The potential of the charged conducting sphere is the same as that of an equal point charge at its center.

Strategy

The potential on the surface will be the same as that of a point charge at the center of the sphere, 12.5 cm away. (The radius of the sphere is 12.5 cm.) We can thus determine the excess charge using the equation

Equation:

$$V = \frac{kQ}{r}.$$

Solution

Solving for Q and entering known values gives

Equation:

$$\begin{aligned} Q &= \frac{rV}{k} \\ &= \frac{(0.125 \text{ m})(100 \times 10^3 \text{ V})}{8.99 \times 10^9 \text{ N}\cdot\text{m}^2/\text{C}^2} \\ &= 1.39 \times 10^{-6} \text{ C} = 1.39 \text{ }\mu\text{C}. \end{aligned}$$

Discussion

This is a relatively small charge, but it produces a rather large voltage. We have another indication here that it is difficult to store isolated charges.

The voltages in both of these examples could be measured with a meter that compares the measured potential with ground potential. Ground potential is often taken to be zero (instead of taking the potential at infinity to be zero). It is the potential difference between two points that is of importance, and very often there is a tacit assumption that some reference point, such as Earth or a very distant point, is at zero potential. As noted in [Electric Potential Energy: Potential Difference](#), this is analogous to taking sea level as $h = 0$ when considering gravitational potential energy, $\text{PE}_g = mgh$.

Section Summary

- Electric potential of a point charge is $V = kQ/r$.
- Electric potential is a scalar, and electric field is a vector. Addition of voltages as numbers gives the voltage due to a combination of point charges, whereas addition of individual fields as vectors gives the total electric field.

Conceptual Questions

Exercise:**Problem:**

In what region of space is the potential due to a uniformly charged sphere the same as that of a point charge? In what region does it differ from that of a point charge?

Exercise:**Problem:**

Can the potential of a non-uniformly charged sphere be the same as that of a point charge? Explain.

Problems & Exercises**Exercise:****Problem:**

A 0.500 cm diameter plastic sphere, used in a static electricity demonstration, has a uniformly distributed 40.0 pC charge on its surface. What is the potential near its surface?

Solution:

144 V

Exercise:**Problem:**

What is the potential 0.530×10^{-10} m from a proton (the average distance between the proton and electron in a hydrogen atom)?

Exercise:

Problem:

(a) A sphere has a surface uniformly charged with 1.00 C. At what distance from its center is the potential 5.00 MV? (b) What does your answer imply about the practical aspect of isolating such a large charge?

Solution:

(a) 1.80 km

(b) A charge of 1 C is a very large amount of charge; a sphere of radius 1.80 km is not practical.

Exercise:**Problem:**

How far from a 1.00 μC point charge will the potential be 100 V? At what distance will it be 2.00×10^2 V?

Exercise:**Problem:**

What are the sign and magnitude of a point charge that produces a potential of -2.00 V at a distance of 1.00 mm?

Solution:

-2.22×10^{-13} C

Exercise:**Problem:**

If the potential due to a point charge is 5.00×10^2 V at a distance of 15.0 m, what are the sign and magnitude of the charge?

Exercise:

Problem:

In nuclear fission, a nucleus splits roughly in half. (a) What is the potential 2.00×10^{-14} m from a fragment that has 46 protons in it? (b) What is the potential energy in MeV of a similarly charged fragment at this distance?

Solution:

(a) 3.31×10^6 V

(b) 152 MeV

Exercise:**Problem:**

A research Van de Graaff generator has a 2.00-m-diameter metal sphere with a charge of 5.00 mC on it. (a) What is the potential near its surface? (b) At what distance from its center is the potential 1.00 MV? (c) An oxygen atom with three missing electrons is released near the Van de Graaff generator. What is its energy in MeV at this distance?

Exercise:**Problem:**

An electrostatic paint sprayer has a 0.200-m-diameter metal sphere at a potential of 25.0 kV that repels paint droplets onto a grounded object. (a) What charge is on the sphere? (b) What charge must a 0.100-mg drop of paint have to arrive at the object with a speed of 10.0 m/s?

Solution:

(a) 2.78×10^{-7} C

(b) 2.00×10^{-10} C

Exercise:

Problem:

In one of the classic nuclear physics experiments at the beginning of the 20th century, an alpha particle was accelerated toward a gold nucleus, and its path was substantially deflected by the Coulomb interaction. If the energy of the doubly charged alpha nucleus was 5.00 MeV, how close to the gold nucleus (79 protons) could it come before being deflected?

Exercise:**Problem:**

(a) What is the potential between two points situated 10 cm and 20 cm from a $3.0 \mu\text{C}$ point charge? (b) To what location should the point at 20 cm be moved to increase this potential difference by a factor of two?

Exercise:**Problem: Unreasonable Results**

- (a) What is the final speed of an electron accelerated from rest through a voltage of 25.0 MV by a negatively charged Van de Graaff terminal?
- (b) What is unreasonable about this result?
- (c) Which assumptions are responsible?

Solution:

- (a) $2.96 \times 10^9 \text{ m/s}$
- (b) This velocity is far too great. It is faster than the speed of light.
- (c) The assumption that the speed of the electron is far less than that of light and that the problem does not require a relativistic treatment produces an answer greater than the speed of light.

Equipotential Lines

LEARNING OBJECTIVES

By the end of this section, you will be able to:

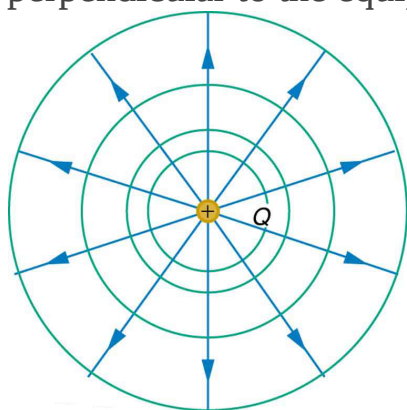
- Explain equipotential lines (also called isolines of electric potential) and equipotential surfaces.
- Describe the action of grounding an electrical appliance.
- Compare electric field and equipotential lines.

The information presented in this section supports the following AP[®] learning objectives and science practices:

- **2.E.2.1** The student is able to determine the structure of isolines of electric potential by constructing them in a given electric field. **(S.P. 6.4, 7.2)**
- **2.E.2.2** The student is able to predict the structure of isolines of electric potential by constructing them in a given electric field and make connections between these isolines and those found in a gravitational field. **(S.P. 6.4, 7.2)**
- **2.E.2.3** The student is able to qualitatively use the concept of isolines to construct isolines of electric potential in an electric field and determine the effect of that field on electrically charged objects. **(S.P. 1.4)**

We can represent electric potentials (voltages) pictorially, just as we drew pictures to illustrate electric fields. Of course, the two are related. Consider [\[link\]](#), which shows an isolated positive point charge and its electric field

lines. Electric field lines radiate out from a positive charge and terminate on negative charges. While we use blue arrows to represent the magnitude and direction of the electric field, we use green lines to represent places where the electric potential is constant. These are called **equipotential lines** in two dimensions, or *equipotential surfaces* in three dimensions. The term *equipotential* is also used as a noun, referring to an equipotential line or surface. The potential for a point charge is the same anywhere on an imaginary sphere of radius r surrounding the charge. This is true since the potential for a point charge is given by $V = kQ/r$ and, thus, has the same value at any point that is a given distance r from the charge. An equipotential sphere is a circle in the two-dimensional view of [\[link\]](#). Since the electric field lines point radially away from the charge, they are perpendicular to the equipotential lines.



An isolated point charge Q with its electric field lines in blue and equipotential lines in green. The potential is the same along each equipotential line, meaning that no work is required to move a charge anywhere along one of those lines.

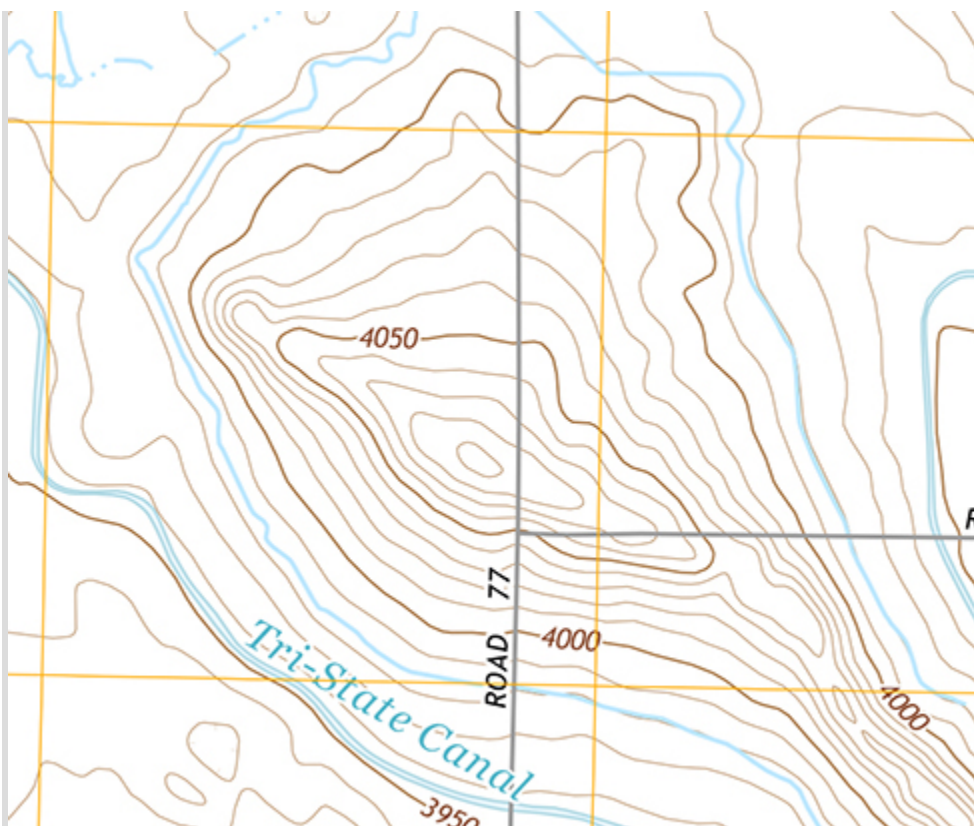
Work is needed to
move a charge
from one
equipotential line to
another.

Equipotential lines
are perpendicular to
electric field lines
in every case.

Note:

Applying the Science Practices: Electric Potential and Peaks

Starting with [\[link\]](#) as a rough example, draw diagrams of isolines for both positive and negative isolated point charges. Be sure to take care with what happens to the spacing of the isolines as you get closer to the charge. Then copy both of these sets of lines, but relabel them as gravitational equipotential lines. Then try to draw the sort of hill or hole or other shape that would have equipotential lines of this form. Does this shape exist in nature?



An example of a topographical map.

You should notice that the lines get closer together the closer you get to the point charge. The hill (or sinkhole, for the equivalent from a negative charge) should have a $1/r$ sort of form, which is not a very common topographical feature.

It is important to note that *equipotential lines are always perpendicular to electric field lines*. No work is required to move a charge along an equipotential, since $\Delta V = 0$. Thus the work is

Equation:

$$W = -\Delta PE = -q\Delta V = 0.$$

Work is zero if force is perpendicular to motion. Force is in the same direction as \mathbf{E} , so that motion along an equipotential must be perpendicular to \mathbf{E} . More precisely, work is related to the electric field by

Equation:

$$W = Fd \cos \theta = qEd \cos \theta = 0.$$

Note that in the above equation, E and F symbolize the magnitudes of the electric field strength and force, respectively. Neither q nor \mathbf{E} nor d is zero, and so $\cos \theta$ must be 0, meaning θ must be 90° . In other words, motion along an equipotential is perpendicular to \mathbf{E} .

One of the rules for static electric fields and conductors is that the electric field must be perpendicular to the surface of any conductor. This implies that a *conductor is an equipotential surface in static situations*. There can be no voltage difference across the surface of a conductor, or charges will flow. One of the uses of this fact is that a conductor can be fixed at zero volts by connecting it to the earth with a good conductor—a process called **grounding**. Grounding can be a useful safety tool. For example, grounding the metal case of an electrical appliance ensures that it is at zero volts relative to the earth.

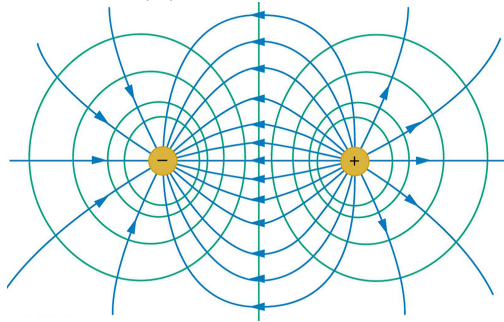
Note:

Grounding

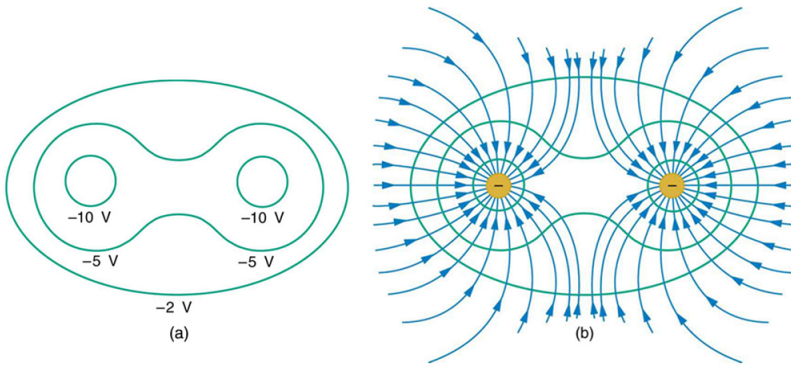
A conductor can be fixed at zero volts by connecting it to the earth with a good conductor—a process called grounding.

Because a conductor is an equipotential, it can replace any equipotential surface. For example, in [\[link\]](#) a charged spherical conductor can replace the point charge, and the electric field and potential surfaces outside of it will be unchanged, confirming the contention that a spherical charge distribution is equivalent to a point charge at its center.

[\[link\]](#) shows the electric field and equipotential lines for two equal and opposite charges. Given the electric field lines, the equipotential lines can be drawn simply by making them perpendicular to the electric field lines. Conversely, given the equipotential lines, as in [\[link\]](#)(a), the electric field lines can be drawn by making them perpendicular to the equipotentials, as in [\[link\]](#)(b).

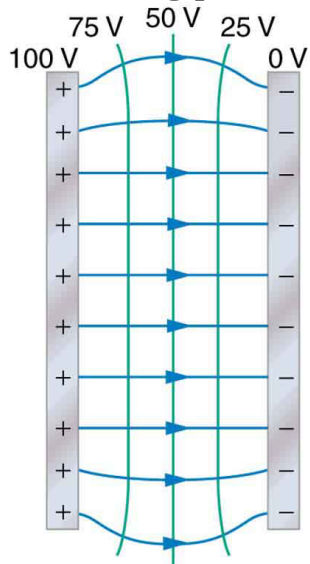


The electric field lines and equipotential lines for two equal but opposite charges. The equipotential lines can be drawn by making them perpendicular to the electric field lines, if those are known. Note that the potential is greatest (most positive) near the positive charge and least (most negative) near the negative charge.



(a) These equipotential lines might be measured with a voltmeter in a laboratory experiment. (b) The corresponding electric field lines are found by drawing them perpendicular to the equipotentials. Note that these fields are consistent with two equal negative charges.

One of the most important cases is that of the familiar parallel conducting plates shown in [\[link\]](#). Between the plates, the equipotentials are evenly spaced and parallel. The same field could be maintained by placing conducting plates at the equipotential lines at the potentials shown.



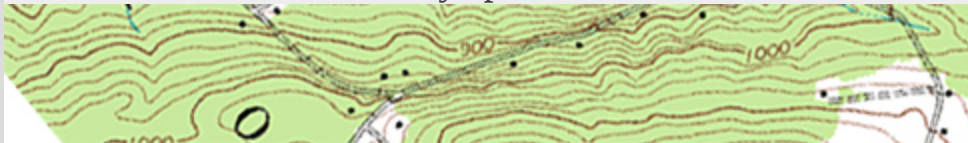
The electric

field and
equipotential
lines between
two metal
plates.

Note:

Making Connections: Slopes and Parallel Plates

Consider the parallel plates in [\[link\]](#). These have equipotential lines that are parallel to the plates in the space between, and evenly spaced. An example of this (with sample values) is given in [\[link\]](#). One could draw a similar set of equipotential isolines for gravity on the hill shown in [\[link\]](#). If the hill has any extent at the same slope, the isolines along that extent would be parallel to each other. Furthermore, in regions of constant slope, the isolines would be evenly spaced.



Note that a topographical map along a ridge has roughly parallel elevation lines, similar to the equipotential lines in [\[link\]](#).

An important application of electric fields and equipotential lines involves the heart. The heart relies on electrical signals to maintain its rhythm. The movement of electrical signals causes the chambers of the heart to contract and relax. When a person has a heart attack, the movement of these electrical signals may be disturbed. An artificial pacemaker and a defibrillator can be used to initiate the rhythm of electrical signals. The

equipotential lines around the heart, the thoracic region, and the axis of the heart are useful ways of monitoring the structure and functions of the heart. An electrocardiogram (ECG) measures the small electric signals being generated during the activity of the heart. More about the relationship between electric fields and the heart is discussed in [Energy Stored in Capacitors](#).

Note:

PhET Explorations: Charges and Fields

Move point charges around on the playing field and then view the electric field, voltages, equipotential lines, and more. It's colorful, it's dynamic, it's free.

[Charge
s and
Fields](#)

Test Prep for AP Courses

Exercise:

Problem:

How would [\[link\]](#) be different with two positive charges replacing the two negative charges?

- a. The equipotential lines would have positive values.
 - b. It would actually resemble [\[link\]](#).
 - c. no change
 - d. not enough information
-

Solution:

(a)

Exercise:**Problem:**

Consider two conducting plates, placed on adjacent sides of a square, but with a 1-m space between the corner of the square and the plate. These plates are not touching, not centered on each other, but are at right angles. Each plate is 1 m wide. If the plates are held at a fixed potential difference ΔV , draw the equipotential lines for this system.

Exercise:**Problem:**

As isolines of electric potential get closer together, the electric field gets stronger. What shape would a hill have as the isolines of gravitational potential get closer together?

- a. constant slope
- b. steeper slope
- c. shallower slope
- d. a U-shape

Solution:

(b)

Exercise:**Problem:**

Between [\[link\]](#) and [\[link\]](#), which more closely resembles the gravitational field between two equal masses, and why?

Exercise:

Problem:

How much work is necessary to keep a positive point charge in orbit around a negative point charge?

- a. A lot; this system is unstable.
- b. Just a little; the isolines are far enough apart that crossing them doesn't take much work.
- c. None; we're traveling along an isoline, which requires no work.
- d. There's not enough information to tell.

Solution:

(c)

Exercise:**Problem:**

Consider two conducting plates, placed on adjacent sides of a square, but with a 1-m space between the corner of the square and the plate. These plates are not touching, not centered on each other, but are at right angles. Each plate is 1 m wide. If the plates are held at a fixed potential difference ΔV , sketch the path of both a positively charged object placed between the near ends, and a negatively charged object placed near the open ends.

Section Summary

- An equipotential line is a line along which the electric potential is constant.
- An equipotential surface is a three-dimensional version of equipotential lines.
- Equipotential lines are always perpendicular to electric field lines.
- The process by which a conductor can be fixed at zero volts by connecting it to the earth with a good conductor is called grounding.

Conceptual Questions

Exercise:

Problem:

What is an equipotential line? What is an equipotential surface?

Exercise:

Problem:

Explain in your own words why equipotential lines and surfaces must be perpendicular to electric field lines.

Exercise:

Problem: Can different equipotential lines cross? Explain.

Problems & Exercises

Exercise:

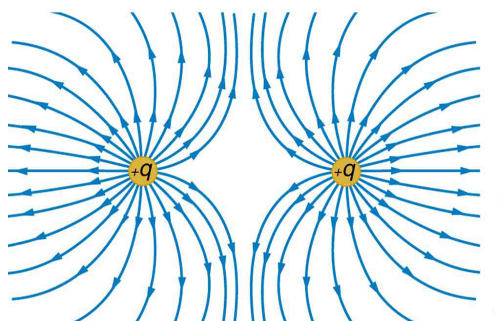
Problem:

(a) Sketch the equipotential lines near a point charge $+q$. Indicate the direction of increasing potential. (b) Do the same for a point charge $-3q$.

Exercise:

Problem:

Sketch the equipotential lines for the two equal positive charges shown in [\[link\]](#). Indicate the direction of increasing potential.



The electric field near two equal positive charges is directed away from each of the charges.

Exercise:

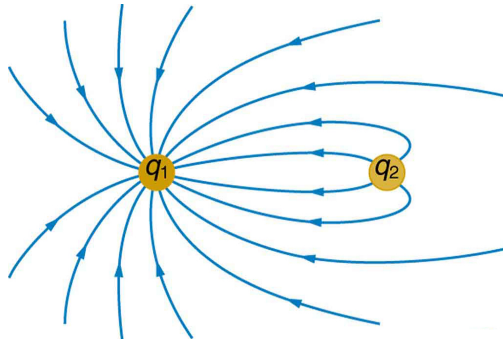
Problem:

[\[link\]](#) shows the electric field lines near two charges q_1 and q_2 , the first having a magnitude four times that of the second. Sketch the equipotential lines for these two charges, and indicate the direction of increasing potential.

Exercise:

Problem:

Sketch the equipotential lines a long distance from the charges shown in [\[link\]](#). Indicate the direction of increasing potential.



The electric field near

two charges.

Exercise:

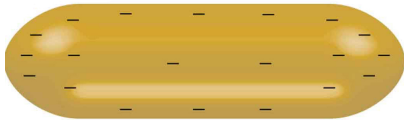
Problem:

Sketch the equipotential lines in the vicinity of two opposite charges, where the negative charge is three times as great in magnitude as the positive. See [\[link\]](#) for a similar situation. Indicate the direction of increasing potential.

Exercise:

Problem:

Sketch the equipotential lines in the vicinity of the negatively charged conductor in [\[link\]](#). How will these equipotentials look a long distance from the object?

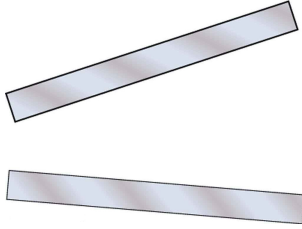


A negatively
charged conductor.

Exercise:

Problem:

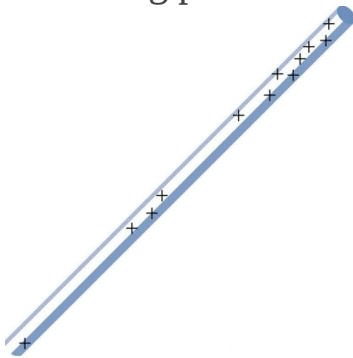
Sketch the equipotential lines surrounding the two conducting plates shown in [\[link\]](#), given the top plate is positive and the bottom plate has an equal amount of negative charge. Be certain to indicate the distribution of charge on the plates. Is the field strongest where the plates are closest? Why should it be?



Exercise:

Problem:

(a) Sketch the electric field lines in the vicinity of the charged insulator in [\[link\]](#). Note its non-uniform charge distribution. (b) Sketch equipotential lines surrounding the insulator. Indicate the direction of increasing potential.



A charged
insulating rod
such as might be
used in a
classroom
demonstration.

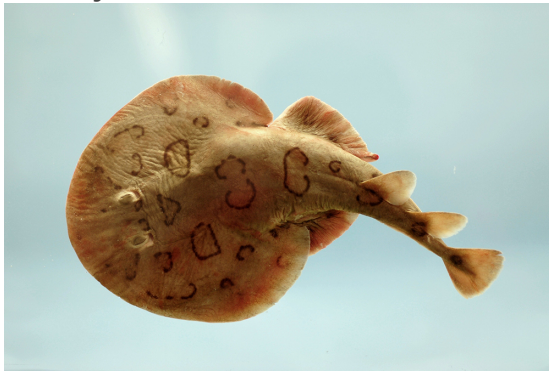
Exercise:

Problem:

The naturally occurring charge on the ground on a fine day out in the open country is -1.00 nC/m^2 . (a) What is the electric field relative to ground at a height of 3.00 m? (b) Calculate the electric potential at this height. (c) Sketch electric field and equipotential lines for this scenario.

Exercise:**Problem:**

The lesser electric ray (*Narcine bancroftii*) maintains an incredible charge on its head and a charge equal in magnitude but opposite in sign on its tail ([\[link\]](#)). (a) Sketch the equipotential lines surrounding the ray. (b) Sketch the equipotentials when the ray is near a ship with a conducting surface. (c) How could this charge distribution be of use to the ray?



Lesser electric ray (*Narcine bancroftii*) (credit: National Oceanic and Atmospheric Administration, NOAA's Fisheries Collection).

Glossary

equipotential line

a line along which the electric potential is constant

grounding

fixing a conductor at zero volts by connecting it to the earth or ground

Capacitors and Dielectrics

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Describe the action of a capacitor and define capacitance.
- Explain parallel plate capacitors and their capacitances.
- Discuss the process of increasing the capacitance of a capacitor with a dielectric.
- Determine capacitance given charge and voltage.

The information presented in this section supports the following AP® learning objectives and science practices:

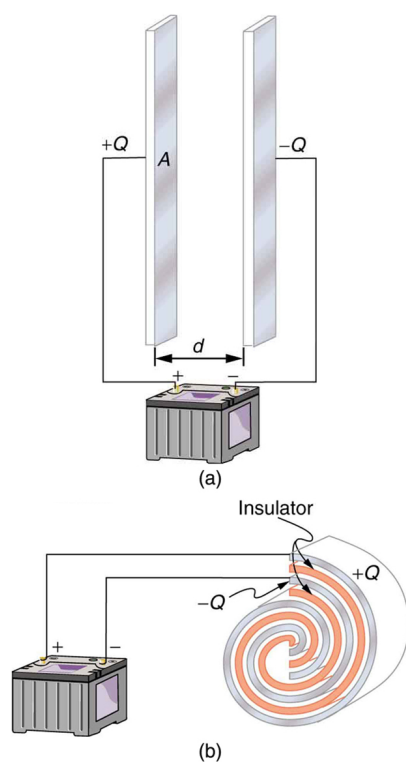
- **4.E.4.1** The student is able to make predictions about the properties of resistors and/or capacitors when placed in a simple circuit based on the geometry of the circuit element and supported by scientific theories and mathematical relationships. **(S.P. 2.2, 6.4)**
- **4.E.4.2** The student is able to design a plan for the collection of data to determine the effect of changing the geometry and/or materials on the resistance or capacitance of a circuit element and relate results to the basic properties of resistors and capacitors. **(S.P. 4.1, 4.2)**
- **4.E.4.3** The student is able to analyze data to determine the effect of changing the geometry and/or materials on the resistance or capacitance of a circuit element and relate results to the basic properties of resistors and capacitors. **(S.P. 5.1)**

A **capacitor** is a device used to store electric charge. Capacitors have applications ranging from filtering static out of radio reception to energy storage in heart defibrillators. Typically, commercial capacitors have two conducting parts close to one another, but not touching, such as those in [\[link\]](#). (Most of the time an insulator is used between the two plates to provide separation—see the discussion on dielectrics below.) When battery terminals are connected to an initially uncharged capacitor, equal amounts of positive and negative charge, $+Q$ and $-Q$, are separated into its two plates. The capacitor remains neutral overall, but we refer to it as storing a charge Q in this circumstance.

Note:

Capacitor

A capacitor is a device used to store electric charge.



Both capacitors

shown here were initially uncharged before being connected to a battery. They now have separated charges of $+Q$ and $-Q$ on their two halves. (a) A parallel plate capacitor. (b) A rolled capacitor with an insulating material between its two conducting sheets.

The amount of charge Q a *capacitor* can store depends on two major factors—the voltage applied and the capacitor’s physical characteristics, such as its size.

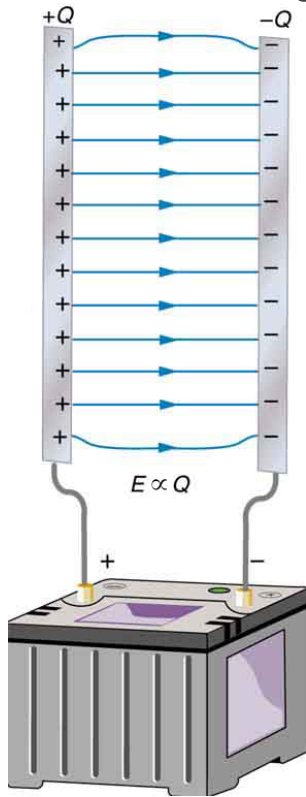
Note:

The Amount of Charge Q a Capacitor Can Store

The amount of charge Q a *capacitor* can store depends on two major factors—the voltage applied and the capacitor’s physical characteristics, such as its size.

A system composed of two identical, parallel conducting plates separated by a distance, as in [\[link\]](#), is called a **parallel plate capacitor**. It is easy to see the relationship between the voltage and the stored charge for a parallel plate capacitor, as shown in [\[link\]](#). Each electric field line starts on an individual positive charge and ends on a negative one, so that there will be

more field lines if there is more charge. (Drawing a single field line per charge is a convenience, only. We can draw many field lines for each charge, but the total number is proportional to the number of charges.) The electric field strength is, thus, directly proportional to Q .



Electric field lines in this parallel plate capacitor, as always, start on positive charges and end on negative charges. Since the electric field strength is proportional to the density

of field lines,
it is also
proportional
to the amount
of charge on
the capacitor.

The field is proportional to the charge:

Equation:

$$E \propto Q,$$

where the symbol \propto means “proportional to.” From the discussion in [Electric Potential in a Uniform Electric Field](#), we know that the voltage across parallel plates is $V = Ed$. Thus,

Equation:

$$V \propto E.$$

It follows, then, that $V \propto Q$, and conversely,

Equation:

$$Q \propto V.$$

This is true in general: The greater the voltage applied to any capacitor, the greater the charge stored in it.

Different capacitors will store different amounts of charge for the same applied voltage, depending on their physical characteristics. We define their **capacitance** C to be such that the charge Q stored in a capacitor is proportional to C . The charge stored in a capacitor is given by

Equation:

$$Q = CV.$$

This equation expresses the two major factors affecting the amount of charge stored. Those factors are the physical characteristics of the capacitor, C , and the voltage, V . Rearranging the equation, we see that *capacitance C is the amount of charge stored per volt*, or

Equation:

$$C = \frac{Q}{V}.$$

Note:

Capacitance

Capacitance C is the amount of charge stored per volt, or

Equation:

$$C = \frac{Q}{V}.$$

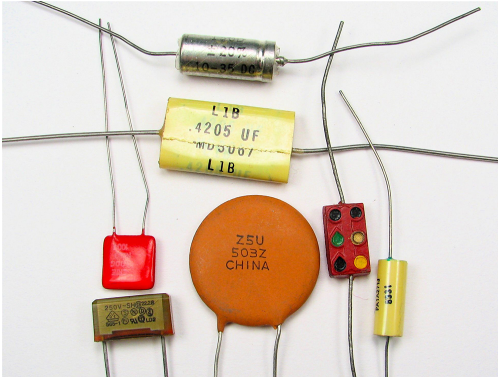
The unit of capacitance is the farad (F), named for Michael Faraday (1791–1867), an English scientist who contributed to the fields of electromagnetism and electrochemistry. Since capacitance is charge per unit voltage, we see that a farad is a coulomb per volt, or

Equation:

$$1 \text{ F} = \frac{1 \text{ C}}{1 \text{ V}}.$$

A 1-farad capacitor would be able to store 1 coulomb (a very large amount of charge) with the application of only 1 volt. One farad is, thus, a very large capacitance. Typical capacitors range from fractions of a picofarad ($1 \text{ pF} = 10^{-12} \text{ F}$) to millifarads ($1 \text{ mF} = 10^{-3} \text{ F}$).

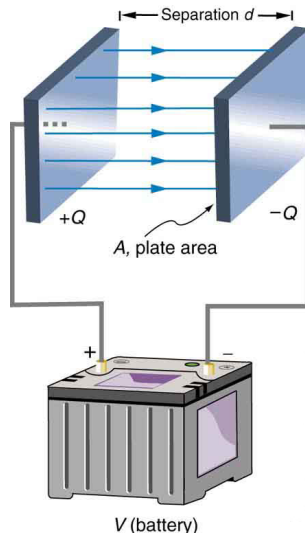
[\[link\]](#) shows some common capacitors. Capacitors are primarily made of ceramic, glass, or plastic, depending upon purpose and size. Insulating materials, called dielectrics, are commonly used in their construction, as discussed below.



Some typical capacitors.
Size and value of
capacitance are not
necessarily related.
(credit: Windell Oskay)

Parallel Plate Capacitor

The parallel plate capacitor shown in [\[link\]](#) has two identical conducting plates, each having a surface area A , separated by a distance d (with no material between the plates). When a voltage V is applied to the capacitor, it stores a charge Q , as shown. We can see how its capacitance depends on A and d by considering the characteristics of the Coulomb force. We know that like charges repel, unlike charges attract, and the force between charges decreases with distance. So it seems quite reasonable that the bigger the plates are, the more charge they can store—because the charges can spread out more. Thus C should be greater for larger A . Similarly, the closer the plates are together, the greater the attraction of the opposite charges on them. So C should be greater for smaller d .



Parallel plate
capacitor
with plates
separated by
a distance d .
Each plate
has an area A

It can be shown that for a parallel plate capacitor there are only two factors (A and d) that affect its capacitance C . The capacitance of a parallel plate capacitor in equation form is given by

Equation:

$$C = \epsilon_0 \frac{A}{d}.$$

Note:

Capacitance of a Parallel Plate Capacitor

Equation:

$$C = \epsilon_0 \frac{A}{d}$$

A is the area of one plate in square meters, and d is the distance between the plates in meters. The constant ϵ_0 is the permittivity of free space; its numerical value in SI units is $\epsilon_0 = 8.85 \times 10^{-12} \text{ F/m}$. The units of F/m are equivalent to $\text{C}^2/\text{N} \cdot \text{m}^2$. The small numerical value of ϵ_0 is related to the large size of the farad. A parallel plate capacitor must have a large area to have a capacitance approaching a farad. (Note that the above equation is valid when the parallel plates are separated by air or free space. When another material is placed between the plates, the equation is modified, as discussed below.)

Example:

Capacitance and Charge Stored in a Parallel Plate Capacitor

(a) What is the capacitance of a parallel plate capacitor with metal plates, each of area 1.00 m^2 , separated by 1.00 mm ? (b) What charge is stored in this capacitor if a voltage of $3.00 \times 10^3 \text{ V}$ is applied to it?

Strategy

Finding the capacitance C is a straightforward application of the equation $C = \epsilon_0 A/d$. Once C is found, the charge stored can be found using the equation $Q = CV$.

Solution for (a)

Entering the given values into the equation for the capacitance of a parallel plate capacitor yields

Equation:

$$\begin{aligned} C &= \epsilon_0 \frac{A}{d} = \left(8.85 \times 10^{-12} \frac{\text{F}}{\text{m}} \right) \frac{1.00 \text{ m}^2}{1.00 \times 10^{-3} \text{ m}} \\ &= 8.85 \times 10^{-9} \text{ F} = 8.85 \text{ nF}. \end{aligned}$$

Discussion for (a)

This small value for the capacitance indicates how difficult it is to make a device with a large capacitance. Special techniques help, such as using very large area thin foils placed close together.

Solution for (b)

The charge stored in any capacitor is given by the equation $Q = CV$. Entering the known values into this equation gives

Equation:

$$\begin{aligned} Q &= CV = (8.85 \times 10^{-9} \text{ F})(3.00 \times 10^3 \text{ V}) \\ &= 26.6 \text{ } \mu\text{C}. \end{aligned}$$

Discussion for (b)

This charge is only slightly greater than those found in typical static electricity. Since air breaks down at about $3.00 \times 10^6 \text{ V/m}$, more charge cannot be stored on this capacitor by increasing the voltage.

Another interesting biological example dealing with electric potential is found in the cell's plasma membrane. The membrane sets a cell off from its surroundings and also allows ions to selectively pass in and out of the cell. There is a potential difference across the membrane of about -70 mV . This is due to the mainly negatively charged ions in the cell and the predominance of positively charged sodium (Na^+) ions outside. Things change when a nerve cell is stimulated. Na^+ ions are allowed to pass through the membrane into the cell, producing a positive membrane potential—the nerve signal. The cell membrane is about 7 to 10 nm thick. An approximate value of the electric field across it is given by

Equation:

$$E = \frac{V}{d} = \frac{-70 \times 10^{-3} \text{ V}}{8 \times 10^{-9} \text{ m}} = -9 \times 10^6 \text{ V/m}.$$

This electric field is enough to cause a breakdown in air.

Dielectric

The previous example highlights the difficulty of storing a large amount of charge in capacitors. If d is made smaller to produce a larger capacitance, then the maximum voltage must be reduced proportionally to avoid breakdown (since $E = V/d$). An important solution to this difficulty is to put an insulating material, called a **dielectric**, between the plates of a capacitor and allow d to be as small as possible. Not only does the smaller d make the capacitance greater, but many insulators can withstand greater electric fields than air before breaking down.

There is another benefit to using a dielectric in a capacitor. Depending on the material used, the capacitance is greater than that given by the equation $C = \epsilon_0 \frac{A}{d}$ by a factor κ , called the *relative permittivity*.[\[footnote\]](#) A parallel plate capacitor with a dielectric between its plates has a capacitance given by

Historically, the term *dielectric constant* was used. However, it is currently deprecated by standards organizations because this term was used for both relative and absolute permittivity, creating unfortunate and unnecessary ambiguity.

Equation:

$$C = \kappa \epsilon_0 \frac{A}{d} \text{ (parallel plate capacitor with dielectric).}$$

Values of the dielectric constant κ for various materials are given in [\[link\]](#). Note that κ for vacuum is exactly 1, and so the above equation is valid in that case, too. If a dielectric is used, perhaps by placing Teflon between the plates of the capacitor in [\[link\]](#), then the capacitance is greater by the factor κ , which for Teflon is 2.1.

Note:

Take-Home Experiment: Building a Capacitor

How large a capacitor can you make using a chewing gum wrapper? The plates will be the aluminum foil, and the separation (dielectric) in between will be the paper.

Material	Dielectric constant κ	Dielectric strength (V/m)
Vacuum	1.00000	—
Air	1.00059	3×10^6
Bakelite	4.9	24×10^6
Fused quartz	3.78	8×10^6
Neoprene rubber	6.7	12×10^6
Nylon	3.4	14×10^6
Paper	3.7	16×10^6
Polystyrene	2.56	24×10^6
Pyrex glass	5.6	14×10^6
Silicon oil	2.5	15×10^6
Strontium titanate	233	8×10^6
Teflon	2.1	60×10^6
Water	80	—

Dielectric Constants and Dielectric Strengths for Various Materials at 20°C

Note also that the dielectric constant for air is very close to 1, so that air-filled capacitors act much like those with vacuum between their plates *except* that the air can become conductive if the electric field strength

becomes too great. (Recall that $E = V/d$ for a parallel plate capacitor.) Also shown in [\[link\]](#) are maximum electric field strengths in V/m, called **dielectric strengths**, for several materials. These are the fields above which the material begins to break down and conduct. The dielectric strength imposes a limit on the voltage that can be applied for a given plate separation. For instance, in [\[link\]](#), the separation is 1.00 mm, and so the voltage limit for air is

Equation:

$$\begin{aligned} V &= E \cdot d \\ &= (3 \times 10^6 \text{ V/m})(1.00 \times 10^{-3} \text{ m}) \\ &= 3000 \text{ V.} \end{aligned}$$

However, the limit for a 1.00 mm separation filled with Teflon is 60,000 V, since the dielectric strength of Teflon is $60 \times 10^6 \text{ V/m}$. So the same capacitor filled with Teflon has a greater capacitance and can be subjected to a much greater voltage. Using the capacitance we calculated in the above example for the air-filled parallel plate capacitor, we find that the Teflon-filled capacitor can store a maximum charge of

Equation:

$$\begin{aligned} Q &= CV \\ &= \kappa C_{\text{air}} V \\ &= (2.1)(8.85 \text{ nF})(6.0 \times 10^4 \text{ V}) \\ &= 1.1 \text{ mC.} \end{aligned}$$

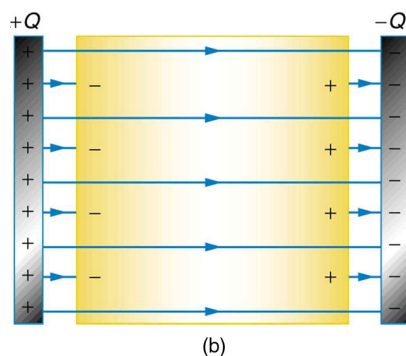
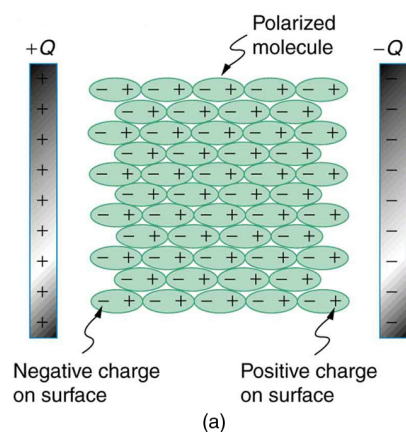
This is 42 times the charge of the same air-filled capacitor.

Note:

Dielectric Strength

The maximum electric field strength above which an insulating material begins to break down and conduct is called its dielectric strength.

Microscopically, how does a dielectric increase capacitance? Polarization of the insulator is responsible. The more easily it is polarized, the greater its dielectric constant κ . Water, for example, is a **polar molecule** because one end of the molecule has a slight positive charge and the other end has a slight negative charge. The polarity of water causes it to have a relatively large dielectric constant of 80. The effect of polarization can be best explained in terms of the characteristics of the Coulomb force. [\[link\]](#) shows the separation of charge schematically in the molecules of a dielectric material placed between the charged plates of a capacitor. The Coulomb force between the closest ends of the molecules and the charge on the plates is attractive and very strong, since they are very close together. This attracts more charge onto the plates than if the space were empty and the opposite charges were a distance d away.



(a) The molecules in the insulating material between

the plates of a capacitor are polarized by the charged plates. This produces a layer of opposite charge on the surface of the dielectric that attracts more charge onto the plate, increasing its capacitance. (b)

The dielectric reduces the electric field strength inside the capacitor, resulting in a smaller voltage between the plates for the same charge. The capacitor stores the same charge for a smaller voltage, implying that it has a larger capacitance because of the dielectric.

Another way to understand how a dielectric increases capacitance is to consider its effect on the electric field inside the capacitor. [\[link\]](#)(b) shows the electric field lines with a dielectric in place. Since the field lines end on charges in the dielectric, there are fewer of them going from one side of the capacitor to the other. So the electric field strength is less than if there were

a vacuum between the plates, even though the same charge is on the plates. The voltage between the plates is $V = Ed$, so it too is reduced by the dielectric. Thus there is a smaller voltage V for the same charge Q ; since $C = Q/V$, the capacitance C is greater.

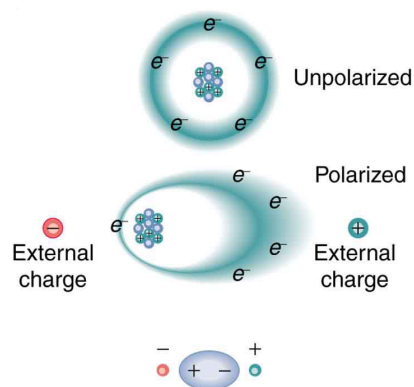
The dielectric constant is generally defined to be $\kappa = E_0/E$, or the ratio of the electric field in a vacuum to that in the dielectric material, and is intimately related to the polarizability of the material.

Note:

Things Great and Small

The Submicroscopic Origin of Polarization

Polarization is a separation of charge within an atom or molecule. As has been noted, the planetary model of the atom pictures it as having a positive nucleus orbited by negative electrons, analogous to the planets orbiting the Sun. Although this model is not completely accurate, it is very helpful in explaining a vast range of phenomena and will be refined elsewhere, such as in [Atomic Physics](#). The submicroscopic origin of polarization can be modeled as shown in [\[link\]](#).



Large-scale view of polarized atom

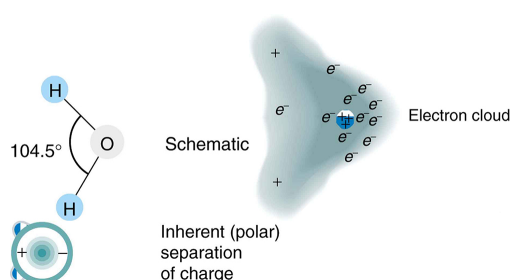
Artist's conception
of a polarized atom.
The orbits of

electrons around the nucleus are shifted slightly by the external charges (shown exaggerated). The resulting separation of charge within the atom means that it is polarized. Note that the unlike charge is now closer to the external charges, causing the polarization.

We will find in [Atomic Physics](#) that the orbits of electrons are more properly viewed as electron clouds with the density of the cloud related to the probability of finding an electron in that location (as opposed to the definite locations and paths of planets in their orbits around the Sun). This cloud is shifted by the Coulomb force so that the atom on average has a separation of charge. Although the atom remains neutral, it can now be the source of a Coulomb force, since a charge brought near the atom will be closer to one type of charge than the other.

Some molecules, such as those of water, have an inherent separation of charge and are thus called polar molecules. [\[link\]](#) illustrates the separation of charge in a water molecule, which has two hydrogen atoms and one oxygen atom (H_2O). The water molecule is not symmetric—the hydrogen atoms are repelled to one side, giving the molecule a boomerang shape. The electrons in a water molecule are more concentrated around the more highly charged oxygen nucleus than around the hydrogen nuclei. This makes the oxygen end of the molecule slightly negative and leaves the hydrogen ends slightly positive. The inherent separation of charge in polar molecules

makes it easier to align them with external fields and charges. Polar molecules therefore exhibit greater polarization effects and have greater dielectric constants. Those who study chemistry will find that the polar nature of water has many effects. For example, water molecules gather ions much more effectively because they have an electric field and a separation of charge to attract charges of both signs. Also, as brought out in the previous chapter, polar water provides a shield or screening of the electric fields in the highly charged molecules of interest in biological systems.



Artist's conception of a water molecule. There is an inherent separation of charge, and so water is a polar molecule. Electrons in the molecule are attracted to the oxygen nucleus and leave an excess of positive charge near the two hydrogen nuclei. (Note that the schematic on the right is a rough illustration of the distribution of electrons in the water molecule. It does not show the actual numbers of protons and electrons involved in the structure.)

Note:**PhET Explorations: Capacitor Lab**

Explore how a capacitor works! Change the size of the plates and add a dielectric to see the effect on capacitance. Change the voltage and see charges built up on the plates. Observe the electric field in the capacitor. Measure the voltage and the electric field.

[Capacitor Lab](#)

Test Prep for AP Courses**Exercise:****Problem:**

Two parallel plate capacitors are otherwise identical, except the second one has twice the distance between the plates of the first. If placed in otherwise identical circuits, how much charge will the second plate have on it compared to the first?

- a. four times as much
- b. twice as much
- c. the same
- d. half as much

Solution:

(d)

Exercise:

Problem:

In a very simple circuit consisting of a battery and a capacitor with an adjustable distance between the plates, how does the voltage vary as the distance is altered?

Exercise:

Problem:

A parallel plate capacitor with adjustable-size square plates is placed in a circuit. How does the charge on the capacitor change as the length of the sides of the plates is increased?

- a. it grows proportional to length^2
- b. it grows proportional to length
- c. it shrinks proportional to length
- d. it shrinks proportional to length^2

Solution:

(a)

Exercise:

Problem:

Design an experiment to test the relative permittivities of various materials, and briefly describe some basic features of the results.

Exercise:

Problem:

A student was changing one of the dimensions of a square parallel plate capacitor and measuring the resultant charge in a circuit with a battery. However, the student forgot which dimension was being varied, and didn't write it or any units down. Given the table, which dimension was it?

Dimension	1.00	1.10	1.20	1.30
Charge(μC)	0.50	0.61	0.71	0.86

- a. The distance between the plates
- b. The area
- c. The length of a side
- d. Both the area and the length of a side

Solution:

(c)

Exercise:**Problem:**

In an experiment in which a circular parallel plate capacitor in a circuit with a battery has the radius and plate separation grow at the same relative rate, what will happen to the total charge on the capacitor?

Section Summary

- A capacitor is a device used to store charge.
- The amount of charge Q a capacitor can store depends on two major factors—the voltage applied and the capacitor’s physical characteristics, such as its size.
- The capacitance C is the amount of charge stored per volt, or

Equation:

$$C = \frac{Q}{V}.$$

- The capacitance of a parallel plate capacitor is $C = \epsilon_0 \frac{A}{d}$, when the plates are separated by air or free space. ϵ_0 is called the permittivity of free space.
- A parallel plate capacitor with a dielectric between its plates has a capacitance given by

Equation:

$$C = \kappa \epsilon_0 \frac{A}{d},$$

where κ is the dielectric constant of the material.

- The maximum electric field strength above which an insulating material begins to break down and conduct is called dielectric strength.

Conceptual Questions

Exercise:

Problem:

Does the capacitance of a device depend on the applied voltage? What about the charge stored in it?

Exercise:

Problem:

Use the characteristics of the Coulomb force to explain why capacitance should be proportional to the plate area of a capacitor. Similarly, explain why capacitance should be inversely proportional to the separation between plates.

Exercise:**Problem:**

Give the reason why a dielectric material increases capacitance compared with what it would be with air between the plates of a capacitor. What is the independent reason that a dielectric material also allows a greater voltage to be applied to a capacitor? (The dielectric thus increases C and permits a greater V .)

Exercise:**Problem:**

How does the polar character of water molecules help to explain water's relatively large dielectric constant? ([link](#))

Exercise:**Problem:**

Sparks will occur between the plates of an air-filled capacitor at lower voltage when the air is humid than when dry. Explain why, considering the polar character of water molecules.

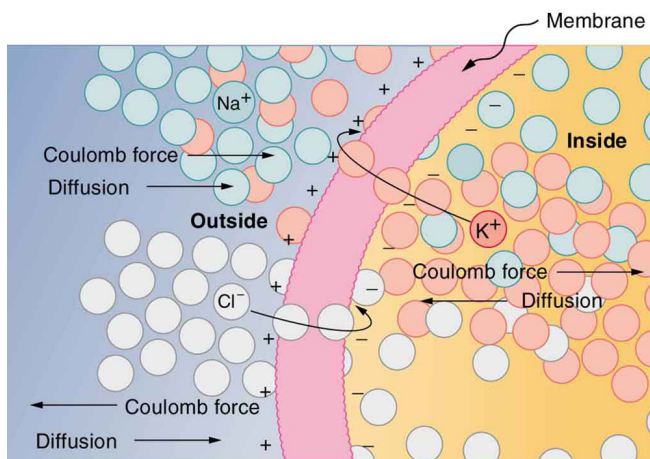
Exercise:**Problem:**

Water has a large dielectric constant, but it is rarely used in capacitors. Explain why.

Exercise:

Problem:

Membranes in living cells, including those in humans, are characterized by a separation of charge across the membrane. Effectively, the membranes are thus charged capacitors with important functions related to the potential difference across the membrane. Is energy required to separate these charges in living membranes and, if so, is its source the metabolism of food energy or some other source?



The semipermeable membrane of a cell has different concentrations of ions inside and out. Diffusion moves the K^+ (potassium) and Cl^- (chloride) ions in the directions shown, until the Coulomb force halts further transfer. This results in a layer of positive charge on the outside, a layer of negative charge on the inside, and thus a voltage across the cell membrane. The membrane is normally impermeable to Na^+ (sodium ions).

Problems & Exercises

Exercise:

Problem:

What charge is stored in a $180\ \mu\text{F}$ capacitor when $120\ \text{V}$ is applied to it?

Solution:

$21.6\ \text{mC}$

Exercise:

Problem:

Find the charge stored when $5.50\ \text{V}$ is applied to an $8.00\ \text{pF}$ capacitor.

Exercise:

Problem: What charge is stored in the capacitor in [\[link\]](#)?

Solution:

$80.0\ \text{mC}$

Exercise:

Problem:

Calculate the voltage applied to a $2.00\ \mu\text{F}$ capacitor when it holds $3.10\ \mu\text{C}$ of charge.

Exercise:

Problem:

What voltage must be applied to an 8.00 nF capacitor to store 0.160 mC of charge?

Solution:

20.0 kV

Exercise:**Problem:**

What capacitance is needed to store 3.00 μC of charge at a voltage of 120 V?

Exercise:**Problem:**

What is the capacitance of a large Van de Graaff generator's terminal, given that it stores 8.00 mC of charge at a voltage of 12.0 MV?

Solution:

667 pF

Exercise:**Problem:**

Find the capacitance of a parallel plate capacitor having plates of area 5.00 m^2 that are separated by 0.100 mm of Teflon.

Exercise:**Problem:**

(a) What is the capacitance of a parallel plate capacitor having plates of area 1.50 m^2 that are separated by 0.0200 mm of neoprene rubber? (b) What charge does it hold when 9.00 V is applied to it?

Solution:

(a) $4.4 \mu\text{F}$

(b) $4.0 \times 10^{-5} \text{ C}$

Exercise:**Problem: Integrated Concepts**

A prankster applies 450 V to an $80.0 \mu\text{F}$ capacitor and then tosses it to an unsuspecting victim. The victim's finger is burned by the discharge of the capacitor through 0.200 g of flesh. What is the temperature increase of the flesh? Is it reasonable to assume no phase change?

Exercise:**Problem: Unreasonable Results**

(a) A certain parallel plate capacitor has plates of area 4.00 m^2 , separated by 0.0100 mm of nylon, and stores 0.170 C of charge. What is the applied voltage? (b) What is unreasonable about this result? (c) Which assumptions are responsible or inconsistent?

Solution:

(a) 14.2 kV

(b) The voltage is unreasonably large, more than 100 times the breakdown voltage of nylon.

(c) The assumed charge is unreasonably large and cannot be stored in a capacitor of these dimensions.

Glossary

capacitor

a device that stores electric charge

capacitance

amount of charge stored per unit volt

dielectric

an insulating material

dielectric strength

the maximum electric field above which an insulating material begins to break down and conduct

parallel plate capacitor

two identical conducting plates separated by a distance

polar molecule

a molecule with inherent separation of charge

Capacitors in Series and Parallel

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Derive expressions for total capacitance in series and in parallel.
- Identify series and parallel parts in the combination of connection of capacitors.
- Calculate the effective capacitance in series and parallel given individual capacitances.

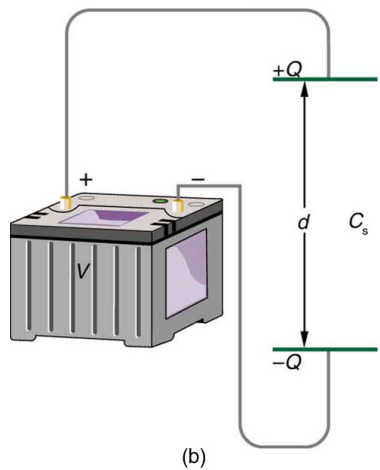
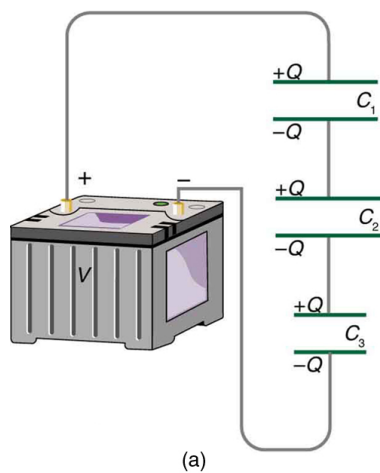
Several capacitors may be connected together in a variety of applications. Multiple connections of capacitors act like a single equivalent capacitor. The total capacitance of this equivalent single capacitor depends both on the individual capacitors and how they are connected. There are two simple and common types of connections, called *series* and *parallel*, for which we can easily calculate the total capacitance. Certain more complicated connections can also be related to combinations of series and parallel.

Capacitance in Series

[\[link\]](#)(a) shows a series connection of three capacitors with a voltage applied. As for any capacitor, the capacitance of the combination is related to charge and voltage by $C = \frac{Q}{V}$.

Note in [\[link\]](#) that opposite charges of magnitude Q flow to either side of the originally uncharged combination of capacitors when the voltage V is

applied. Conservation of charge requires that equal-magnitude charges be created on the plates of the individual capacitors, since charge is only being separated in these originally neutral devices. The end result is that the combination resembles a single capacitor with an effective plate separation greater than that of the individual capacitors alone. (See [link](#)(b).) Larger plate separation means smaller capacitance. It is a general feature of series connections of capacitors that the total capacitance is less than any of the individual capacitances.



(a) Capacitors connected in series. The magnitude of the charge on each

plate is Q . (b) An equivalent capacitor has a larger plate separation d . Series connections produce a total capacitance that is less than that of any of the individual capacitors.

We can find an expression for the total capacitance by considering the voltage across the individual capacitors shown in [\[link\]](#). Solving $C = \frac{Q}{V}$ for V gives $V = \frac{Q}{C}$. The voltages across the individual capacitors are thus $V_1 = \frac{Q}{C_1}$, $V_2 = \frac{Q}{C_2}$, and $V_3 = \frac{Q}{C_3}$. The total voltage is the sum of the individual voltages:

Equation:

$$V = V_1 + V_2 + V_3.$$

Now, calling the total capacitance C_S for series capacitance, consider that

Equation:

$$V = \frac{Q}{C_S} = V_1 + V_2 + V_3.$$

Entering the expressions for V_1 , V_2 , and V_3 , we get

Equation:

$$\frac{Q}{C_S} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3}.$$

Canceling the Q s, we obtain the equation for the total capacitance in series C_S to be

Equation:

$$\frac{1}{C_S} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \dots,$$

where “...” indicates that the expression is valid for any number of capacitors connected in series. An expression of this form always results in a total capacitance C_S that is less than any of the individual capacitances C_1, C_2, \dots , as the next example illustrates.

Note:

Total Capacitance in Series, C_S

Total capacitance in series: $\frac{1}{C_S} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \dots$

Example:

What Is the Series Capacitance?

Find the total capacitance for three capacitors connected in series, given their individual capacitances are 1.000, 5.000, and 8.000 μF .

Strategy

With the given information, the total capacitance can be found using the equation for capacitance in series.

Solution

Entering the given capacitances into the expression for $\frac{1}{C_S}$ gives

$$\frac{1}{C_S} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}.$$

Equation:

$$\frac{1}{C_S} = \frac{1}{1.000 \mu\text{F}} + \frac{1}{5.000 \mu\text{F}} + \frac{1}{8.000 \mu\text{F}} = \frac{1.325}{\mu\text{F}}$$

Inverting to find C_S yields $C_S = \frac{\mu\text{F}}{1.325} = 0.755 \mu\text{F}$.

Discussion

The total series capacitance C_S is less than the smallest individual capacitance, as promised. In series connections of capacitors, the sum is less than the parts. In fact, it is less than any individual. Note that it is sometimes possible, and more convenient, to solve an equation like the above by finding the least common denominator, which in this case (showing only whole-number calculations) is 40. Thus,

Equation:

$$\frac{1}{C_S} = \frac{40}{40 \mu\text{F}} + \frac{8}{40 \mu\text{F}} + \frac{5}{40 \mu\text{F}} = \frac{53}{40 \mu\text{F}},$$

so that

Equation:

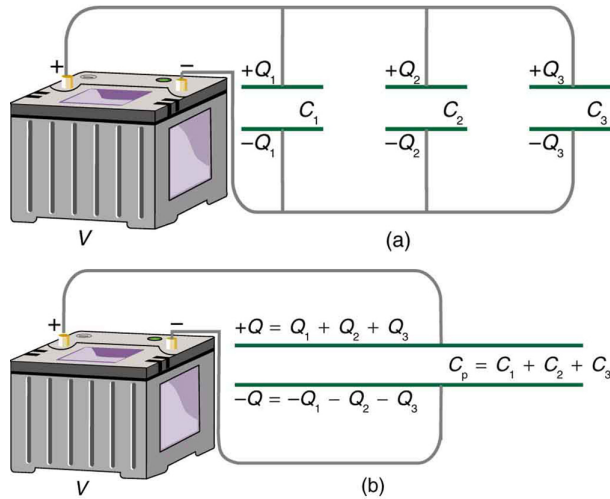
$$C_S = \frac{40 \mu\text{F}}{53} = 0.755 \mu\text{F}.$$

Capacitors in Parallel

[\[link\]](#)(a) shows a parallel connection of three capacitors with a voltage applied. Here the total capacitance is easier to find than in the series case. To find the equivalent total capacitance C_p , we first note that the voltage across each capacitor is V , the same as that of the source, since they are connected directly to it through a conductor. (Conductors are equipotentials, and so the voltage across the capacitors is the same as that across the voltage source.) Thus the capacitors have the same charges on them as they would have if connected individually to the voltage source. The total charge Q is the sum of the individual charges:

Equation:

$$Q = Q_1 + Q_2 + Q_3.$$



(a) Capacitors in parallel. Each is connected directly to the voltage source just as if it were all alone, and so the total capacitance in parallel is just the sum of the individual capacitances. (b) The equivalent capacitor has a larger plate area and can therefore hold more charge than the individual capacitors.

Using the relationship $Q = CV$, we see that the total charge is $Q = C_p V$, and the individual charges are $Q_1 = C_1 V$, $Q_2 = C_2 V$, and $Q_3 = C_3 V$. Entering these into the previous equation gives

Equation:

$$C_p V = C_1 V + C_2 V + C_3 V.$$

Canceling V from the equation, we obtain the equation for the total capacitance in parallel C_p :

Equation:

$$C_p = C_1 + C_2 + C_3 + \dots$$

Total capacitance in parallel is simply the sum of the individual capacitances. (Again the “...” indicates the expression is valid for any number of capacitors connected in parallel.) So, for example, if the capacitors in the example above were connected in parallel, their capacitance would be

Equation:

$$C_p = 1.000 \mu\text{F} + 5.000 \mu\text{F} + 8.000 \mu\text{F} = 14.000 \mu\text{F}.$$

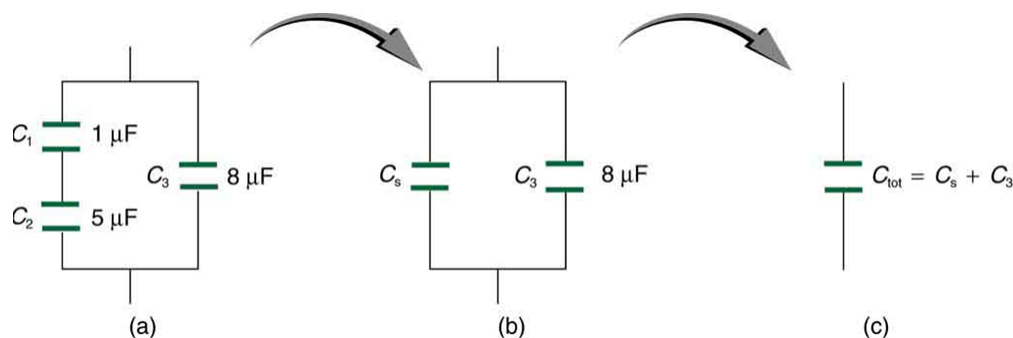
The equivalent capacitor for a parallel connection has an effectively larger plate area and, thus, a larger capacitance, as illustrated in [\[link\]](#)(b).

Note:

Total Capacitance in Parallel, C_p

Total capacitance in parallel $C_p = C_1 + C_2 + C_3 + \dots$

More complicated connections of capacitors can sometimes be combinations of series and parallel. (See [\[link\]](#).) To find the total capacitance of such combinations, we identify series and parallel parts, compute their capacitances, and then find the total.



(a) This circuit contains both series and parallel connections of capacitors. See [\[link\]](#) for the calculation of the overall capacitance of the circuit. (b) C_1 and C_2 are in series; their equivalent capacitance C_s is less than either of them. (c) Note that C_s is in parallel with C_3 . The total capacitance is, thus, the sum of C_s and C_3 .

Example:

A Mixture of Series and Parallel Capacitance

Find the total capacitance of the combination of capacitors shown in [\[link\]](#). Assume the capacitances in [\[link\]](#) are known to three decimal places ($C_1 = 1.000\ \mu\text{F}$, $C_2 = 5.000\ \mu\text{F}$, and $C_3 = 8.000\ \mu\text{F}$), and round your answer to three decimal places.

Strategy

To find the total capacitance, we first identify which capacitors are in series and which are in parallel. Capacitors C_1 and C_2 are in series. Their combination, labeled C_s in the figure, is in parallel with C_3 .

Solution

Since C_1 and C_2 are in series, their total capacitance is given by $\frac{1}{C_s} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$. Entering their values into the equation gives

Equation:

$$\frac{1}{C_s} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{1.000\ \mu\text{F}} + \frac{1}{5.000\ \mu\text{F}} = \frac{1.200}{\mu\text{F}}.$$

Inverting gives

Equation:

$$C_S = 0.833 \mu\text{F}.$$

This equivalent series capacitance is in parallel with the third capacitor; thus, the total is the sum

Equation:

$$\begin{aligned} C_{\text{tot}} &= C_S + C_S \\ &= 0.833 \mu\text{F} + 8.000 \mu\text{F} \\ &= 8.833 \mu\text{F}. \end{aligned}$$

Discussion

This technique of analyzing the combinations of capacitors piece by piece until a total is obtained can be applied to larger combinations of capacitors.

Section Summary

- Total capacitance in series $\frac{1}{C_s} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \dots$
- Total capacitance in parallel $C_p = C_1 + C_2 + C_3 + \dots$
- If a circuit contains a combination of capacitors in series and parallel, identify series and parallel parts, compute their capacitances, and then find the total.

Conceptual Questions

Exercise:

Problem:

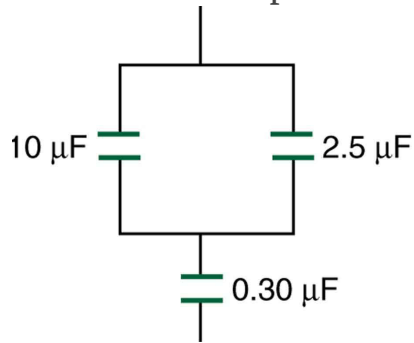
If you wish to store a large amount of energy in a capacitor bank, would you connect capacitors in series or parallel? Explain.

Problems & Exercises

Exercise:

Problem:

Find the total capacitance of the combination of capacitors in [\[link\]](#).



A combination of
series and parallel
connections of
capacitors.

Solution:

$0.293\ \mu\text{F}$

Exercise:

Problem:

Suppose you want a capacitor bank with a total capacitance of $0.750\ \text{F}$ and you possess numerous $1.50\ \text{mF}$ capacitors. What is the smallest number you could hook together to achieve your goal, and how would you connect them?

Exercise:

Problem:

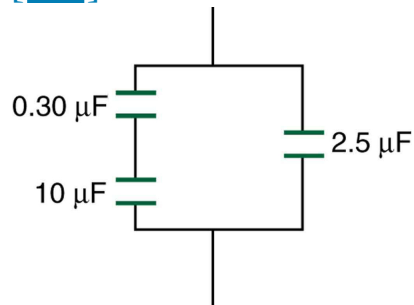
What total capacitances can you make by connecting a $5.00\ \mu\text{F}$ and an $8.00\ \mu\text{F}$ capacitor together?

Solution:

$3.08\ \mu\text{F}$ in series combination, $13.0\ \mu\text{F}$ in parallel combination

Exercise:**Problem:**

Find the total capacitance of the combination of capacitors shown in [\[link\]](#).



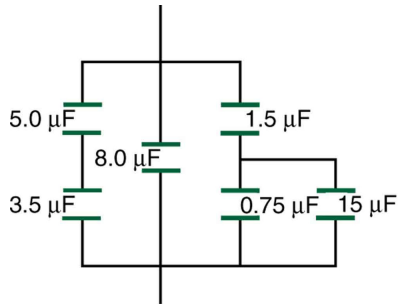
A combination of
series and parallel
connections of
capacitors.

Solution:

$2.79\ \mu\text{F}$

Exercise:**Problem:**

Find the total capacitance of the combination of capacitors shown in [\[link\]](#).



A combination of
series and parallel
connections of
capacitors.

Exercise:

Problem: Unreasonable Results

(a) An $8.00 \mu\text{F}$ capacitor is connected in parallel to another capacitor, producing a total capacitance of $5.00 \mu\text{F}$. What is the capacitance of the second capacitor? (b) What is unreasonable about this result? (c) Which assumptions are unreasonable or inconsistent?

Solution:

(a) $-3.00 \mu\text{F}$

(b) You cannot have a negative value of capacitance.

(c) The assumption that the capacitors were hooked up in parallel, rather than in series, was incorrect. A parallel connection always produces a greater capacitance, while here a smaller capacitance was assumed. This could happen only if the capacitors are connected in series.

Energy Stored in Capacitors

LEARNING OBJECTIVES

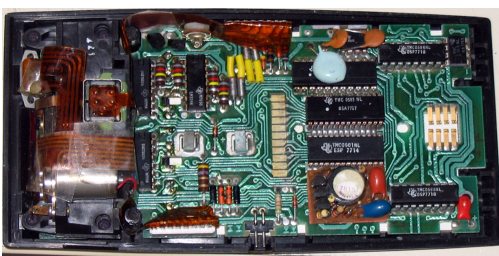
By the end of this section, you will be able to:

- List some uses of capacitors.
- Express in equation form the energy stored in a capacitor.
- Explain the function of a defibrillator.

The information presented in this section supports the following AP® learning objectives and science practices:

- **5.B.2.1** The student is able to calculate the expected behavior of a system using the object model (i.e., by ignoring changes in internal structure) to analyze a situation. Then, when the model fails, the student can justify the use of conservation of energy principles to calculate the change in internal energy due to changes in internal structure because the object is actually a system. **(S.P. 1.4, 2.1)**
- **5.B.3.1** The student is able to describe and make qualitative and/or quantitative predictions about everyday examples of systems with internal potential energy. **(S.P. 2.2, 6.4, 7.2)**
- **5.B.3.2** The student is able to make quantitative calculations of the internal potential energy of a system from a description or diagram of that system. **(S.P. 1.4, 2.2)**
- **5.B.3.3** The student is able to apply mathematical reasoning to create a description of the internal potential energy of a system from a description or diagram of the objects and interactions in that system. **(S.P. 1.4, 2.2)**

Most of us have seen dramatizations in which medical personnel use a **defibrillator** to pass an electric current through a patient's heart to get it to beat normally. (Review [\[link\]](#).) Often realistic in detail, the person applying the shock directs another person to “make it 400 joules this time.” The energy delivered by the defibrillator is stored in a capacitor and can be adjusted to fit the situation. SI units of joules are often employed. Less dramatic is the use of capacitors in microelectronics, such as certain handheld calculators, to supply energy when batteries are charged. (See [\[link\]](#).) Capacitors are also used to supply energy for flash lamps on cameras.



Energy stored in the large capacitor is used to preserve the memory of an electronic calculator when its batteries are charged. (credit: Kucharek, Wikimedia Commons)

Energy stored in a capacitor is electrical potential energy, and it is thus related to the charge Q and voltage V on the capacitor. We must be careful when applying the equation for electrical potential energy $\Delta PE = q\Delta V$ to a capacitor. Remember that ΔPE is the potential energy of a charge q going through a voltage ΔV . But the capacitor starts with zero voltage and gradually comes up to its full voltage as it is charged. The first charge placed on a capacitor experiences a change in voltage $\Delta V = 0$, since the

capacitor has zero voltage when uncharged. The final charge placed on a capacitor experiences $\Delta V = V$, since the capacitor now has its full voltage V on it. The average voltage on the capacitor during the charging process is $V/2$, and so the average voltage experienced by the full charge q is $V/2$.

Thus the energy stored in a capacitor, E_{cap} , is

Equation:

$$E_{\text{cap}} = \frac{QV}{2},$$

where Q is the charge on a capacitor with a voltage V applied. (Note that the energy is not QV , but $QV/2$.) Charge and voltage are related to the capacitance C of a capacitor by $Q = CV$, and so the expression for E_{cap} can be algebraically manipulated into three equivalent expressions:

Equation:

$$E_{\text{cap}} = \frac{QV}{2} = \frac{CV^2}{2} = \frac{Q^2}{2C},$$

where Q is the charge and V the voltage on a capacitor C . The energy is in joules for a charge in coulombs, voltage in volts, and capacitance in farads.

Note:

Energy Stored in Capacitors

The energy stored in a capacitor can be expressed in three ways:

Equation:

$$E_{\text{cap}} = \frac{QV}{2} = \frac{CV^2}{2} = \frac{Q^2}{2C},$$

where Q is the charge, V is the voltage, and C is the capacitance of the capacitor. The energy is in joules for a charge in coulombs, voltage in volts, and capacitance in farads. Energy stored in the capacitor is internal potential energy.

Note:**Making Connections: Point Charges and Capacitors**

Recall that we were able to calculate the stored potential energy of a configuration of point charges, and how the energy changed when the configuration changed in [Applying the Science Practices: Work and Potential Energy in Point Charges](#). Since the charges in a capacitor are, ultimately, all point charges, we can do the same with capacitors. However, we write it down in terms of the macroscopic quantities of (total) charge, voltage, and capacitance; hence Equation (19.76).

For example, consider a parallel plate capacitor with a variable distance between the plates connected to a battery (fixed voltage). When you move the plates closer together, the voltage still doesn't change. However, this increases the capacitance, and hence the internal energy stored in this system (the capacitor) increases. It turns out that the increase in capacitance for a fixed voltage results in an increased charge. The work you did moving the plates closer together ultimately went into moving more electrons from the positive plate to the negative plate.

In a defibrillator, the delivery of a large charge in a short burst to a set of paddles across a person's chest can be a lifesaver. The person's heart attack might have arisen from the onset of fast, irregular beating of the heart—cardiac or ventricular fibrillation. The application of a large shock of electrical energy can terminate the arrhythmia and allow the body's pacemaker to resume normal patterns. Today it is common for ambulances to carry a defibrillator, which also uses an electrocardiogram to analyze the patient's heartbeat pattern. Automated external defibrillators (AED) are found in many public places ([\[link\]](#)). These are designed to be used by lay persons. The device automatically diagnoses the patient's heart condition and then applies the shock with appropriate energy and waveform. CPR is recommended in many cases before use of an AED.



Automated external
defibrillators are found in
many public places.
These portable units
provide verbal
instructions for use in the
important first few
minutes for a person
suffering a cardiac attack.
(credit: Owain Davies,
Wikimedia Commons)

Example:

Capacitance in a Heart Defibrillator

A heart defibrillator delivers 4.00×10^2 J of energy by discharging a capacitor initially at 1.00×10^4 V. What is its capacitance?

Strategy

We are given E_{cap} and V , and we are asked to find the capacitance C . Of the three expressions in the equation for E_{cap} , the most convenient relationship is

Equation:

$$E_{\text{cap}} = \frac{CV^2}{2}.$$

Solution

Solving this expression for C and entering the given values yields

Equation:

$$\begin{aligned} C &= \frac{2E_{\text{cap}}}{V^2} = \frac{2(4.00 \times 10^2 \text{ J})}{(1.00 \times 10^4 \text{ V})^2} = 8.00 \times 10^{-6} \text{ F} \\ &= 8.00 \text{ } \mu\text{F}. \end{aligned}$$

Discussion

This is a fairly large, but manageable, capacitance at $1.00 \times 10^4 \text{ V}$.

Test Prep for AP Courses

Exercise:

Problem:

Consider a parallel plate capacitor, with no dielectric material, attached to a battery with a fixed voltage. What happens when a dielectric is inserted into the capacitor?

- Nothing changes, except now there is a dielectric in the capacitor.
 - The energy in the system decreases, making it very easy to move the dielectric in.
 - You have to do work to move the dielectric, increasing the energy in the system.
 - The reversed polarity destroys the battery.
-

Solution:

(c)

Exercise:

Problem:

Consider a parallel plate capacitor with no dielectric material. It was attached to a battery with a fixed voltage to charge up, but now the battery has been disconnected. What happens to the energy of the system and the dielectric material when a dielectric is inserted into the capacitor?

Exercise:

Problem:

What happens to the energy stored in a circuit as you increase the number of capacitors connected in parallel? Series?

- a. increases, increases
- b. increases, decreases
- c. decreases, increases
- d. decreases, decreases

Solution:

(b)

Exercise:

Problem:

What would the capacitance of a capacitor with the same total internal energy as the car battery in Example 19.1 have to be? Can you explain why we use batteries instead of capacitors for this application?

Exercise:

Problem:

Consider a parallel plate capacitor with metal plates, each of square shape of 1.00 m on a side, separated by 1.00 mm. What is the energy of this capacitor with 3.00×10^3 V applied to it?

- a. 3.98×10^{-2} J
- b. 5.08×10^{14} J
- c. 1.33×10^{-5} J
- d. 1.69×10^{11} J

Solution:

(a)

Exercise:**Problem:**

Consider a parallel plate capacitor with metal plates, each of square shape of 1.00 m on a side, separated by 1.00 mm. What is the internal energy stored in this system if the charge on the capacitor is 30.0 μC ?

Exercise:**Problem:**

Consider a parallel plate capacitor with metal plates, each of square shape of 1.00 m on a side, separated by 1.00 mm. If the plates grow in area while the voltage is held fixed, the capacitance ____ and the stored energy ____.

- a. decreases, decreases
- b. decreases, increases
- c. increases, decreases
- d. increases, increases

Solution:

(d)

Exercise:

Problem:

Consider a parallel plate capacitor with metal plates, each of square shape of 1.00 m on a side, separated by 1.00 mm. What happens to the energy of this system if the area of the plates increases while the charge remains fixed?

Section Summary

- Capacitors are used in a variety of devices, including defibrillators, microelectronics such as calculators, and flash lamps, to supply energy.
- The energy stored in a capacitor can be expressed in three ways:

Equation:

$$E_{\text{cap}} = \frac{QV}{2} = \frac{CV^2}{2} = \frac{Q^2}{2C},$$

where Q is the charge, V is the voltage, and C is the capacitance of the capacitor. The energy is in joules when the charge is in coulombs, voltage is in volts, and capacitance is in farads.

Conceptual Questions

Exercise:

Problem:

How does the energy contained in a charged capacitor change when a dielectric is inserted, assuming the capacitor is isolated and its charge is constant? Does this imply that work was done?

Exercise:

Problem:

What happens to the energy stored in a capacitor connected to a battery when a dielectric is inserted? Was work done in the process?

Problems & Exercises**Exercise:****Problem:**

(a) What is the energy stored in the $10.0\ \mu\text{F}$ capacitor of a heart defibrillator charged to $9.00 \times 10^3\ \text{V}$? (b) Find the amount of stored charge.

Solution:

(a) 405 J

(b) 90.0 mC

Exercise:**Problem:**

In open heart surgery, a much smaller amount of energy will defibrillate the heart. (a) What voltage is applied to the $8.00\ \mu\text{F}$ capacitor of a heart defibrillator that stores 40.0 J of energy? (b) Find the amount of stored charge.

Solution:

(a) 3.16 kV

(b) 25.3 mC

Exercise:

Problem:

A $165\ \mu\text{F}$ capacitor is used in conjunction with a motor. How much energy is stored in it when $119\ \text{V}$ is applied?

Exercise:**Problem:**

Suppose you have a $9.00\ \text{V}$ battery, a $2.00\ \mu\text{F}$ capacitor, and a $7.40\ \mu\text{F}$ capacitor. (a) Find the charge and energy stored if the capacitors are connected to the battery in series. (b) Do the same for a parallel connection.

Solution:

(a) $1.42 \times 10^{-5}\ \text{C}$, $6.38 \times 10^{-5}\ \text{J}$

(b) $8.46 \times 10^{-5}\ \text{C}$, $3.81 \times 10^{-4}\ \text{J}$

Exercise:**Problem:**

A nervous physicist worries that the two metal shelves of his wood frame bookcase might obtain a high voltage if charged by static electricity, perhaps produced by friction. (a) What is the capacitance of the empty shelves if they have area $1.00 \times 10^2\ \text{m}^2$ and are $0.200\ \text{m}$ apart? (b) What is the voltage between them if opposite charges of magnitude $2.00\ \text{nC}$ are placed on them? (c) To show that this voltage poses a small hazard, calculate the energy stored.

Solution:

(a) $4.43 \times 10^{-12}\ \text{F}$

(b) $452\ \text{V}$

(c) $4.52 \times 10^{-7}\ \text{J}$

Exercise:**Problem:**

Show that for a given dielectric material the maximum energy a parallel plate capacitor can store is directly proportional to the volume of dielectric ($\text{Volume} = A \cdot d$). Note that the applied voltage is limited by the dielectric strength.

Exercise:**Problem: Construct Your Own Problem**

Consider a heart defibrillator similar to that discussed in [\[link\]](#). Construct a problem in which you examine the charge stored in the capacitor of a defibrillator as a function of stored energy. Among the things to be considered are the applied voltage and whether it should vary with energy to be delivered, the range of energies involved, and the capacitance of the defibrillator. You may also wish to consider the much smaller energy needed for defibrillation during open-heart surgery as a variation on this problem.

Exercise:**Problem: Unreasonable Results**

(a) On a particular day, it takes $9.60 \times 10^3 \text{ J}$ of electric energy to start a truck's engine. Calculate the capacitance of a capacitor that could store that amount of energy at 12.0 V. (b) What is unreasonable about this result? (c) Which assumptions are responsible?

Solution:

(a) 133 F

(b) Such a capacitor would be too large to carry with a truck. The size of the capacitor would be enormous.

(c) It is unreasonable to assume that a capacitor can store the amount of energy needed.

Glossary

defibrillator

a machine used to provide an electrical shock to a heart attack victim's heart in order to restore the heart's normal rhythmic pattern

Connection for AP® Courses
class="introduction"

Electric
energy in
massive
quantities is
transmitted
from this
hydroelectric facility, the

[Srisailem](#)
[power](#)
[station](#)
[located](#)
[along the](#)
[Krishna](#)
[River in](#)
[India](#), by the
movement
of charge—
that is, by
electric
current.
(credit:
Chintohere,
Wikimedia
Commons)



In our daily lives, we see and experience many examples of electricity which involve electric current, the movement of charge. These include the flicker of numbers on a handheld calculator, nerve impulses carrying signals of vision to the brain, an ultrasound device sending a signal to a computer screen, the brain sending a message for a baby to twitch its toes, an electric train pulling its load over a mountain pass, and a hydroelectric plant sending energy to metropolitan and rural users.

Humankind has indeed harnessed electricity, the basis of technology, to improve the quality of life. While the previous two chapters concentrated on static electricity and the fundamental force underlying its behavior, the next few chapters will be devoted to electric and magnetic phenomena involving electric current. In addition to exploring applications of electricity, we shall gain new insights into its nature – in particular, the fact that all magnetism results from electric current.

This chapter supports learning objectives covered under Big Ideas 1, 4, and 5 of the AP Physics Curriculum Framework. Electric charge is a property of a system (Big Idea 1) that affects its interaction with other charged systems (Enduring Understanding 1.B), whereas electric current is fundamentally the movement of charge through a conductor and is based on the fact that electric charge is conserved within a system (Essential Knowledge 1.B.1). The conservation of charge also leads to the concept of an electric circuit as a closed loop of electrical current. In addition, this chapter discusses examples showing that the current in a circuit is resisted by the elements of the circuit and the strength of the resistance depends on the material of the elements. The macroscopic properties of materials, including resistivity, depend on their molecular and atomic structure (Enduring Understanding 1.E). In addition, resistivity depends on the temperature of the material (Essential Knowledge 1.E.2).

The chapter also describes how the interaction of systems of objects can result in changes in those systems (Big Idea 4). For example, electric properties of a system of charged objects can change in response to the presence of, or changes in, other charged objects or systems (Enduring Understanding 4.E). A simple circuit with a resistor and an energy source is an example of such a system. The current through the resistor in the circuit is equal to the difference of potentials across the resistor divided by its resistance (Essential Knowledge 4.E.4).

The unifying theme of the physics curriculum is that any changes in the systems due to interactions are governed by laws of conservation (Big Idea 5). This chapter applies the idea of energy conservation (Enduring Understanding 5.B) to electric circuits and connects concepts of electric energy and electric power as rates of energy use (Essential Knowledge 5.B.5). While the laws of conservation of energy in electric circuits are fully described by Kirchoff's rules, which are introduced in the next chapter (Essential Knowledge 5.B.9), the specific definition of power (based on Essential Knowledge 5.B.9) is that it is the rate at which energy is transferred from a resistor as the product of the electric potential difference across the resistor and the current through the resistor.

Big Idea 1 Objects and systems have properties such as mass and charge. Systems may have internal structure.

Enduring Understanding 1.B Electric charge is a property of an object or system that affects its interactions with other objects or systems containing charge.

Essential Knowledge 1.B.1 Electric charge is conserved. The net charge of a system is equal to the sum of the charges of all the objects in the system.

Enduring Understanding 1.E Materials have many macroscopic properties that result from the arrangement and interactions of the atoms and molecules that make up the material.

Essential Knowledge 1.E.2 Matter has a property called resistivity.

Big Idea 4 Interactions between systems can result in changes in those systems.

Enduring Understanding 4.E The electric and magnetic properties of a system can change in response to the presence of, or changes in, other objects or systems.

Essential Knowledge 4.E.4 The resistance of a resistor, and the capacitance of a capacitor, can be understood from the basic properties of electric fields and forces, as well as the properties of materials and their geometry.

Big Idea 5: Changes that occur as a result of interactions are constrained by conservation laws.

Enduring Understanding 5.B The energy of a system is conserved.

Essential Knowledge 5.B.5 Energy can be transferred by an external force exerted on an object or system that moves the object or system through a distance; this energy transfer is called work. Energy transfer in mechanical or electrical systems may occur at different rates. Power is defined as the rate of energy transfer into, out of, or within a system. [A piston filled with

gas getting compressed or expanded is treated in Physics 2 as a part of thermodynamics.]

Essential Knowledge 5.B.9 Kirchhoff's loop rule describes conservation of energy in electrical circuits. [The application of Kirchhoff's laws to circuits is introduced in Physics 1 and further developed in Physics 2 in the context of more complex circuits, including those with capacitors.]

Current

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define electric current, ampere, and drift velocity.
- Describe the direction of charge flow in conventional current.
- Use drift velocity to calculate current and vice versa.

The information presented in this section supports the following AP® learning objectives and science practices:

- **1.B.1.1** The student is able to make claims about natural phenomena based on conservation of electric charge. **(S.P. 6.4)**
- **1.B.1.2** The student is able to make predictions, using the conservation of electric charge, about the sign and relative quantity of net charge of objects or systems after various charging processes, including conservation of charge in simple circuits. **(S.P. 6.4, 7.2)**

Electric Current

Electric current is defined to be the rate at which charge flows. A large current, such as that used to start a truck engine, moves a large amount of charge in a small time, whereas a small current, such as that used to operate a hand-held calculator, moves a small amount of charge over a long period of time. In equation form, **electric current** I is defined to be

Equation:

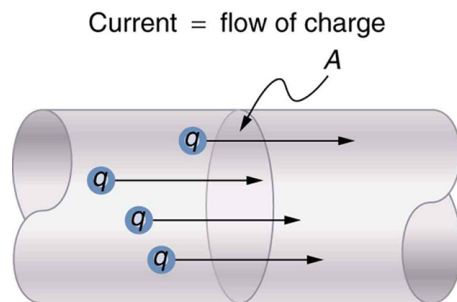
$$I = \frac{\Delta Q}{\Delta t},$$

where ΔQ is the amount of charge passing through a given area in time Δt . (As in previous chapters, initial time is often taken to be zero, in which case $\Delta t = t$.) (See [\[link\]](#).) The SI unit for current is the **ampere** (A), named for the French physicist André-Marie Ampère (1775–1836). Since $I = \Delta Q / \Delta t$, we see that an ampere is one coulomb per second:

Equation:

$$1 \text{ A} = 1 \text{ C/s}$$

Not only are fuses and circuit breakers rated in amperes (or amps), so are many electrical appliances.



The rate of flow of charge is current. An ampere is the flow of one coulomb through an area in one second.

Example:

Calculating Currents: Current in a Truck Battery and a Handheld Calculator

(a) What is the current involved when a truck battery sets in motion 720 C of charge in 4.00 s while starting an engine? (b) How long does it take 1.00 C of charge to flow through a handheld calculator if a 0.300-mA current is flowing?

Strategy

We can use the definition of current in the equation $I = \Delta Q / \Delta t$ to find the current in part (a), since charge and time are given. In part (b), we rearrange the definition of current and use the given values of charge and current to find the time required.

Solution for (a)

Entering the given values for charge and time into the definition of current gives

Equation:

$$\begin{aligned} I &= \frac{\Delta Q}{\Delta t} = \frac{720 \text{ C}}{4.00 \text{ s}} = 180 \text{ C/s} \\ &= 180 \text{ A.} \end{aligned}$$

Discussion for (a)

This large value for current illustrates the fact that a large charge is moved in a small amount of time. The currents in these “starter motors” are fairly large because large frictional forces need to be overcome when setting something in motion.

Solution for (b)

Solving the relationship $I = \Delta Q / \Delta t$ for time Δt , and entering the known values for charge and current gives

Equation:

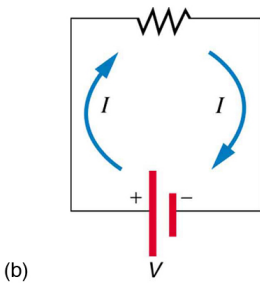
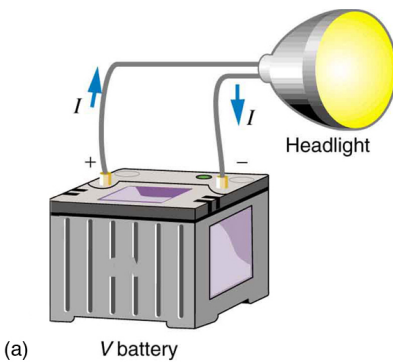
$$\begin{aligned} \Delta t &= \frac{\Delta Q}{I} = \frac{1.00 \text{ C}}{0.300 \times 10^{-3} \text{ C/s}} \\ &= 3.33 \times 10^3 \text{ s.} \end{aligned}$$

Discussion for (b)

This time is slightly less than an hour. The small current used by the handheld calculator takes a much longer time to move a smaller charge than the large current of the truck starter. So why can we operate our calculators

only seconds after turning them on? It's because calculators require very little energy. Such small current and energy demands allow handheld calculators to operate from solar cells or to get many hours of use out of small batteries. Remember, calculators do not have moving parts in the same way that a truck engine has with cylinders and pistons, so the technology requires smaller currents.

[\[link\]](#) shows a simple circuit and the standard schematic representation of a battery, conducting path, and load (a resistor). Schematics are very useful in visualizing the main features of a circuit. A single schematic can represent a wide variety of situations. The schematic in [\[link\]](#) (b), for example, can represent anything from a truck battery connected to a headlight lighting the street in front of the truck to a small battery connected to a penlight lighting a keyhole in a door. Such schematics are useful because the analysis is the same for a wide variety of situations. We need to understand a few schematics to apply the concepts and analysis to many more situations.



(a) A simple

electric circuit. A closed path for current to flow through is supplied by conducting wires connecting a load to the terminals of a battery. (b) In this schematic, the battery is represented by the two parallel red lines, conducting wires are shown as straight lines, and the zigzag represents the load.

The schematic represents a wide variety of similar circuits.

Note that the direction of current in [\[link\]](#) is from positive to negative. *The direction of conventional current is the direction that positive charge would flow.* In a single loop circuit (as shown in [\[link\]](#)), the value for current at all points of the circuit should be the same if there are no losses. This is because current is the flow of charge and charge is conserved, i.e., the charge flowing out from the battery will be the same as the charge flowing into the battery. Depending on the situation, positive charges, negative charges, or both may move. In metal wires, for example, current is carried by electrons—that is, negative charges move. In ionic solutions, such as salt water, both positive and negative charges move. This is also true in nerve cells. A Van de Graaff generator used for nuclear research can produce a current of pure positive charges, such as protons. [\[link\]](#) illustrates the

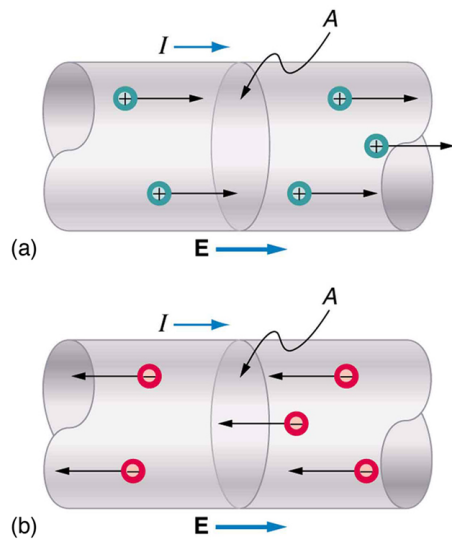
movement of charged particles that compose a current. The fact that conventional current is taken to be in the direction that positive charge would flow can be traced back to American politician and scientist Benjamin Franklin in the 1700s. He named the type of charge associated with electrons negative, long before they were known to carry current in so many situations. Franklin, in fact, was totally unaware of the small-scale structure of electricity.

It is important to realize that there is an electric field in conductors responsible for producing the current, as illustrated in [\[link\]](#). Unlike static electricity, where a conductor in equilibrium cannot have an electric field in it, conductors carrying a current have an electric field and are not in static equilibrium. An electric field is needed to supply energy to move the charges.

Note:**Making Connections: Take-Home Investigation—Electric Current Illustration**

Find a straw and little peas that can move freely in the straw. Place the straw flat on a table and fill the straw with peas. When you pop one pea in at one end, a different pea should pop out the other end. This demonstration is an analogy for an electric current. Identify what compares to the electrons and what compares to the supply of energy. What other analogies can you find for an electric current?

Note that the flow of peas is based on the peas physically bumping into each other; electrons flow due to mutually repulsive electrostatic forces.



Current I is the rate at which charge moves through an area A , such as the cross-section of a wire.

Conventional current is defined to move in the direction of the electric field. (a)

Positive charges move in the direction of the electric field and the same direction as conventional current.

(b) Negative charges move in the direction opposite to the electric field. Conventional current is in the direction opposite to the movement of negative charge. The flow of electrons is

sometimes referred to
as electronic flow.

Example:**Calculating the Number of Electrons that Move through a Calculator**

If the 0.300-mA current through the calculator mentioned in the [\[link\]](#) example is carried by electrons, how many electrons per second pass through it?

Strategy

The current calculated in the previous example was defined for the flow of positive charge. For electrons, the magnitude is the same, but the sign is opposite, $I_{\text{electrons}} = -0.300 \times 10^{-3} \text{ C/s}$. Since each electron (e^-) has a charge of $-1.60 \times 10^{-19} \text{ C}$, we can convert the current in coulombs per second to electrons per second.

Solution

Starting with the definition of current, we have

Equation:

$$I_{\text{electrons}} = \frac{\Delta Q_{\text{electrons}}}{\Delta t} = \frac{-0.300 \times 10^{-3} \text{ C}}{\text{s}}.$$

We divide this by the charge per electron, so that

Equation:

$$\begin{aligned} \frac{e^-}{\text{s}} &= \frac{-0.300 \times 10^{-3} \text{ C}}{\text{s}} \times \frac{1 e^-}{-1.60 \times 10^{-19} \text{ C}} \\ &= 1.88 \times 10^{15} \frac{e^-}{\text{s}}. \end{aligned}$$

Discussion

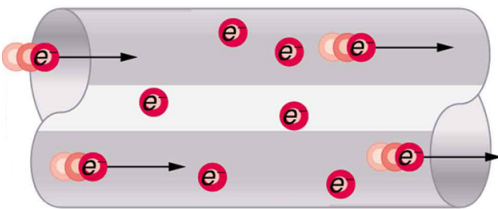
There are so many charged particles moving, even in small currents, that individual charges are not noticed, just as individual water molecules are not noticed in water flow. Even more amazing is that they do not always keep moving forward like soldiers in a parade. Rather they are like a crowd of people with movement in different directions but a general trend to

move forward. There are lots of collisions with atoms in the metal wire and, of course, with other electrons.

Drift Velocity

Electrical signals are known to move very rapidly. Telephone conversations carried by currents in wires cover large distances without noticeable delays. Lights come on as soon as a switch is flicked. Most electrical signals carried by currents travel at speeds on the order of 10^8 m/s, a significant fraction of the speed of light. Interestingly, the individual charges that make up the current move *much* more slowly on average, typically drifting at speeds on the order of 10^{-4} m/s. How do we reconcile these two speeds, and what does it tell us about standard conductors?

The high speed of electrical signals results from the fact that the force between charges acts rapidly at a distance. Thus, when a free charge is forced into a wire, as in [\[link\]](#), the incoming charge pushes other charges ahead of it, which in turn push on charges farther down the line. The density of charge in a system cannot easily be increased, and so the signal is passed on rapidly. The resulting electrical shock wave moves through the system at nearly the speed of light. To be precise, this rapidly moving signal or shock wave is a rapidly propagating change in electric field.

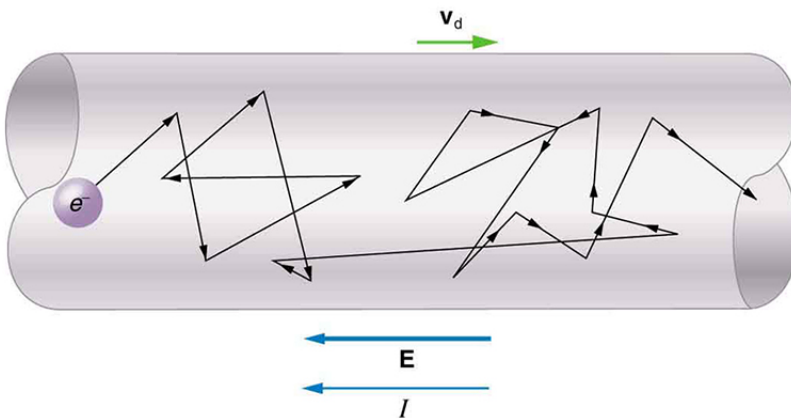


When charged particles are forced into this volume of a conductor, an equal number are quickly forced to leave. The

repulsion between like charges makes it difficult to increase the number of charges in a volume.

Thus, as one charge enters, another leaves almost immediately, carrying the signal rapidly forward.

Good conductors have large numbers of free charges in them. In metals, the free charges are free electrons. [\[link\]](#) shows how free electrons move through an ordinary conductor. The distance that an individual electron can move between collisions with atoms or other electrons is quite small. The electron paths thus appear nearly random, like the motion of atoms in a gas. But there is an electric field in the conductor that causes the electrons to drift in the direction shown (opposite to the field, since they are negative). The **drift velocity** v_d is the average velocity of the free charges. Drift velocity is quite small, since there are so many free charges. If we have an estimate of the density of free electrons in a conductor, we can calculate the drift velocity for a given current. The larger the density, the lower the velocity required for a given current.



Free electrons moving in a conductor make many collisions with other electrons and atoms. The path of one electron is shown. The average velocity of the free charges is called the drift velocity, v_d , and it is in the direction opposite to the electric field for electrons. The collisions normally transfer energy to the conductor, requiring a constant supply of energy to maintain a steady current.

Note:

Conduction of Electricity and Heat

Good electrical conductors are often good heat conductors, too. This is because large numbers of free electrons can carry electrical current and can transport thermal energy.

The free-electron collisions transfer energy to the atoms of the conductor. The electric field does work in moving the electrons through a distance, but that work does not increase the kinetic energy (nor speed, therefore) of the electrons. The work is transferred to the conductor's atoms, possibly increasing temperature. Thus a continuous power input is required to maintain current. An exception, of course, is found in superconductors, for reasons we shall explore in a later chapter. Superconductors can have a steady current without a continual supply of energy—a great energy savings. In contrast, the supply of energy can be useful, such as in a lightbulb filament. The supply of energy is necessary to increase the temperature of the tungsten filament, so that the filament glows.

Note:

Making Connections: Take-Home Investigation—Filament Observations
Find a lightbulb with a filament. Look carefully at the filament and describe its structure. To what points is the filament connected?

We can obtain an expression for the relationship between current and drift velocity by considering the number of free charges in a segment of wire, as illustrated in [\[link\]](#). The number of free charges per unit volume is given the symbol n and depends on the material. The shaded segment has a volume Ax , so that the number of free charges in it is nAx . The charge ΔQ in this segment is thus $qnAx$, where q is the amount of charge on each carrier. (Recall that for electrons, q is -1.60×10^{-19} C.) Current is charge moved per unit time; thus, if all the original charges move out of this segment in time Δt , the current is

Equation:

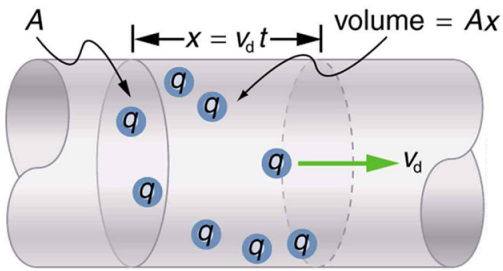
$$I = \frac{\Delta Q}{\Delta t} = \frac{qnAx}{\Delta t}.$$

Note that $x/\Delta t$ is the magnitude of the drift velocity, v_d , since the charges move an average distance x in a time Δt . Rearranging terms gives

Equation:

$$I = nqAv_d,$$

where I is the current through a wire of cross-sectional area A made of a material with a free charge density n . The carriers of the current each have charge q and move with a drift velocity of magnitude v_d .



All the charges in the shaded volume of this wire move out in a time t , having a drift velocity of magnitude $v_d = x/t$. See text for further discussion.

Note that simple drift velocity is not the entire story. The speed of an electron is much greater than its drift velocity. In addition, not all of the electrons in a conductor can move freely, and those that do might move somewhat faster or slower than the drift velocity. So what do we mean by free electrons? Atoms in a metallic conductor are packed in the form of a lattice structure. Some electrons are far enough away from the atomic nuclei that they do not experience the attraction of the nuclei as much as the inner electrons do. These are the free electrons. They are not bound to a single atom but can instead move freely among the atoms in a “sea” of electrons. These free electrons respond by accelerating when an electric field is applied. Of course as they move they collide with the atoms in the lattice and other electrons, generating thermal energy, and the conductor gets warmer. In an insulator, the organization of the atoms and the structure do not allow for such free electrons.

Example: **Calculating Drift Velocity in a Common Wire**

Calculate the drift velocity of electrons in a 12-gauge copper wire (which has a diameter of 2.053 mm) carrying a 20.0-A current, given that there is one free electron per copper atom. (Household wiring often contains 12-gauge copper wire, and the maximum current allowed in such wire is usually 20 A.) The density of copper is $8.80 \times 10^3 \text{ kg/m}^3$.

Strategy

We can calculate the drift velocity using the equation $I = nqAv_d$. The current $I = 20.0 \text{ A}$ is given, and $q = -1.60 \times 10^{-19} \text{ C}$ is the charge of an electron. We can calculate the area of a cross-section of the wire using the formula $A = \pi r^2$, where r is one-half the given diameter, 2.053 mm. We are given the density of copper, $8.80 \times 10^3 \text{ kg/m}^3$, and the periodic table shows that the atomic mass of copper is 63.54 g/mol. We can use these two quantities along with Avogadro's number, $6.02 \times 10^{23} \text{ atoms/mol}$, to determine n , the number of free electrons per cubic meter.

Solution

First, calculate the density of free electrons in copper. There is one free electron per copper atom. Therefore, is the same as the number of copper atoms per m^3 . We can now find n as follows:

Equation:

$$\begin{aligned} n &= \frac{1 e^-}{\text{atom}} \times \frac{6.02 \times 10^{23} \text{ atoms}}{\text{mol}} \times \frac{1 \text{ mol}}{63.54 \text{ g}} \times \frac{1000 \text{ g}}{\text{kg}} \times \frac{8.80 \times 10^3 \text{ kg}}{1 \text{ m}^3} \\ &= 8.342 \times 10^{28} e^-/\text{m}^3. \end{aligned}$$

The cross-sectional area of the wire is

Equation:

$$\begin{aligned} A &= \pi r^2 \\ &= \pi \left(\frac{2.053 \times 10^{-3} \text{ m}}{2} \right)^2 \\ &= 3.310 \times 10^{-6} \text{ m}^2. \end{aligned}$$

Rearranging $I = nqAv_d$ to isolate drift velocity gives

Equation:

$$\begin{aligned}
 v_d &= \frac{I}{nqA} \\
 &= \frac{20.0 \text{ A}}{(8.342 \times 10^{28} / \text{m}^3)(-1.60 \times 10^{-19} \text{ C})(3.310 \times 10^{-6} \text{ m}^2)} \\
 &= -4.53 \times 10^{-4} \text{ m/s.}
 \end{aligned}$$

Discussion

The minus sign indicates that the negative charges are moving in the direction opposite to conventional current. The small value for drift velocity (on the order of 10^{-4} m/s) confirms that the signal moves on the order of 10^{12} times faster (about 10^8 m/s) than the charges that carry it.

Test Prep for AP Courses

Exercise:

Problem:

Which of the following can be explained on the basis of conservation of charge in a closed circuit consisting of a battery, resistor, and metal wires?

- The number of electrons leaving the battery will be equal to the number of electrons entering the battery.
- The number of electrons leaving the battery will be less than the number of electrons entering the battery.
- The number of protons leaving the battery will be equal to the number of protons entering the battery.
- The number of protons leaving the battery will be less than the number of protons entering the battery.

Solution:

(a)

Exercise:

Problem:

When a battery is connected to a bulb, there is 2.5 A of current in the circuit. What amount of charge will flow through the circuit in a time of 0.5 s?

- a. 0.5 C
- b. 1 C
- c. 1.25 C
- d. 1.5 C

Exercise:**Problem:**

If 0.625×10^{20} electrons flow through a circuit each second, what is the current in the circuit?

Solution:

10 A

Exercise:**Problem:**

Two students calculate the charge flowing through a circuit. The first student concludes that 300 C of charge flows in 1 minute. The second student concludes that 3.125×10^{19} electrons flow per second. If the current measured in the circuit is 5 A, which of the two students (if any) have performed the calculations correctly?

Section Summary

- Electric current I is the rate at which charge flows, given by
Equation:

$$I = \frac{\Delta Q}{\Delta t},$$

- where ΔQ is the amount of charge passing through an area in time Δt .
- The direction of conventional current is taken as the direction in which positive charge moves.
 - The SI unit for current is the ampere (A), where $1 \text{ A} = 1 \text{ C/s}$.
 - Current is the flow of free charges, such as electrons and ions.
 - Drift velocity v_d is the average speed at which these charges move.
 - Current I is proportional to drift velocity v_d , as expressed in the relationship $I = nqAv_d$. Here, I is the current through a wire of cross-sectional area A . The wire's material has a free-charge density n , and each carrier has charge q and a drift velocity v_d .
 - Electrical signals travel at speeds about 10^{12} times greater than the drift velocity of free electrons.

Conceptual Questions

Exercise:

Problem:

Can a wire carry a current and still be neutral—that is, have a total charge of zero? Explain.

Exercise:

Problem:

Car batteries are rated in ampere-hours (A · h). To what physical quantity do ampere-hours correspond (voltage, charge, . . .), and what relationship do ampere-hours have to energy content?

Exercise:

Problem:

If two different wires having identical cross-sectional areas carry the same current, will the drift velocity be higher or lower in the better conductor? Explain in terms of the equation $v_d = \frac{I}{nqA}$, by considering how the density of charge carriers n relates to whether or not a material is a good conductor.

Exercise:**Problem:**

Why are two conducting paths from a voltage source to an electrical device needed to operate the device?

Exercise:**Problem:**

In cars, one battery terminal is connected to the metal body. How does this allow a single wire to supply current to electrical devices rather than two wires?

Exercise:**Problem:**

Why isn't a bird sitting on a high-voltage power line electrocuted? Contrast this with the situation in which a large bird hits two wires simultaneously with its wings.

Problems & Exercises**Exercise:****Problem:**

What is the current in milliamperes produced by the solar cells of a pocket calculator through which 4.00 C of charge passes in 4.00 h?

Solution:

0.278 mA

Exercise:**Problem:**

A total of 600 C of charge passes through a flashlight in 0.500 h. What is the average current?

Exercise:**Problem:**

What is the current when a typical static charge of $0.250\ \mu\text{C}$ moves from your finger to a metal doorknob in $1.00\ \mu\text{s}$?

Solution:

0.250 A

Exercise:**Problem:**

Find the current when 2.00 nC jumps between your comb and hair over a $0.500\ \mu\text{s}$ time interval.

Exercise:**Problem:**

A large lightning bolt had a 20,000-A current and moved 30.0 C of charge. What was its duration?

Solution:

1.50ms

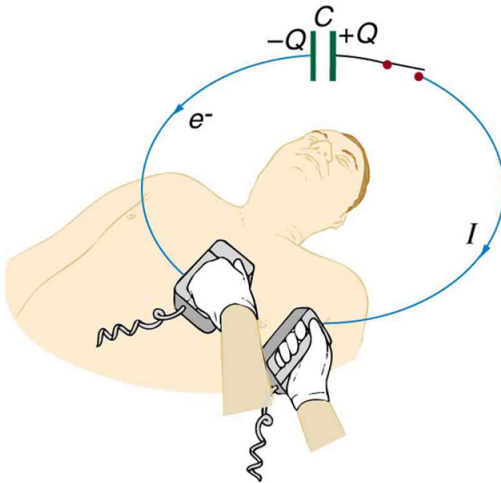
Exercise:

Problem:

The 200-A current through a spark plug moves 0.300 mC of charge. How long does the spark last?

Exercise:**Problem:**

(a) A defibrillator sends a 6.00-A current through the chest of a patient by applying a 10,000-V potential as in the figure below. What is the resistance of the path? (b) The defibrillator paddles make contact with the patient through a conducting gel that greatly reduces the path resistance. Discuss the difficulties that would ensue if a larger voltage were used to produce the same current through the patient, but with the path having perhaps 50 times the resistance. (Hint: The current must be about the same, so a higher voltage would imply greater power. Use this equation for power: $P = I^2 R$.)



The capacitor in a defibrillation unit drives a current through the heart of a patient.

Solution:

(a) $1.67\text{k}\Omega$

(b) If a 50 times larger resistance existed, keeping the current about the same, the power would be increased by a factor of about 50 (based on the equation $P = I^2 R$), causing much more energy to be transferred to the skin, which could cause serious burns. The gel used reduces the resistance, and therefore reduces the power transferred to the skin.

Exercise:

Problem:

During open-heart surgery, a defibrillator can be used to bring a patient out of cardiac arrest. The resistance of the path is $500\ \Omega$ and a 10.0-mA current is needed. What voltage should be applied?

Exercise:

Problem:

(a) A defibrillator passes 12.0 A of current through the torso of a person for 0.0100 s . How much charge moves? (b) How many electrons pass through the wires connected to the patient? (See figure two problems earlier.)

Solution:

(a) 0.120 C

(b) 7.50×10^{17} electrons

Exercise:

Problem:

A clock battery wears out after moving $10,000\text{ C}$ of charge through the clock at a rate of 0.500 mA . (a) How long did the clock run? (b) How many electrons per second flowed?

Exercise:

Problem:

The batteries of a submerged non-nuclear submarine supply 1000 A at full speed ahead. How long does it take to move Avogadro's number (6.02×10^{23}) of electrons at this rate?

Solution:

96.3 s

Exercise:**Problem:**

Electron guns are used in X-ray tubes. The electrons are accelerated through a relatively large voltage and directed onto a metal target, producing X-rays. (a) How many electrons per second strike the target if the current is 0.500 mA? (b) What charge strikes the target in 0.750 s?

Exercise:**Problem:**

A large cyclotron directs a beam of He^{++} nuclei onto a target with a beam current of 0.250 mA. (a) How many He^{++} nuclei per second is this? (b) How long does it take for 1.00 C to strike the target? (c) How long before 1.00 mol of He^{++} nuclei strike the target?

Solution:

(a) $7.81 \times 10^{14} \text{ He}^{++} \text{ nuclei/s}$

(b) $4.00 \times 10^3 \text{ s}$

(c) $7.71 \times 10^8 \text{ s}$

Exercise:

Problem:

Repeat the above example on [\[link\]](#), but for a wire made of silver and given there is one free electron per silver atom.

Exercise:**Problem:**

Using the results of the above example on [\[link\]](#), find the drift velocity in a copper wire of twice the diameter and carrying 20.0 A.

Solution:

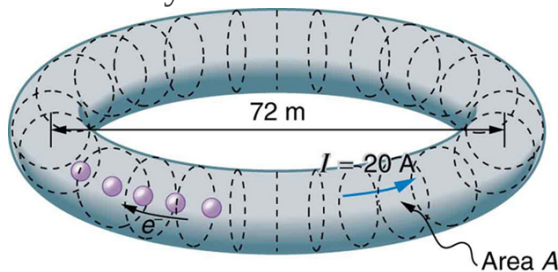
$$-1.13 \times 10^{-4} \text{ m/s}$$

Exercise:**Problem:**

A 14-gauge copper wire has a diameter of 1.628 mm. What magnitude current flows when the drift velocity is 1.00 mm/s? (See above example on [\[link\]](#) for useful information.)

Exercise:**Problem:**

SPEAR, a storage ring about 72.0 m in diameter at the Stanford Linear Accelerator (closed in 2009), has a 20.0-A circulating beam of electrons that are moving at nearly the speed of light. (See [\[link\]](#).) How many electrons are in the beam?



Electrons circulating in the storage ring called SPEAR

constitute a 20.0-A current.
Because they travel close to
the speed of light, each
electron completes many
orbits in each second.

Solution:

9.42×10^{13} electrons

Glossary

electric current

the rate at which charge flows, $I = \Delta Q / \Delta t$

ampere

(amp) the SI unit for current; $1 \text{ A} = 1 \text{ C/s}$

drift velocity

the average velocity at which free charges flow in response to an
electric field

Ohm's Law: Resistance and Simple Circuits

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain the origin of Ohm's law.
- Calculate voltages, currents, and resistances with Ohm's law.
- Explain the difference between ohmic and non-ohmic materials.
- Describe a simple circuit.

The information presented in this section supports the following AP[®] learning objectives and science practices:

- **4.E.4.1** The student is able to make predictions about the properties of resistors and/or capacitors when placed in a simple circuit based on the geometry of the circuit element and supported by scientific theories and mathematical relationships. **(S.P. 2.2, 6.4)**

What drives current? We can think of various devices—such as batteries, generators, wall outlets, and so on—which are necessary to maintain a current. All such devices create a potential difference and are loosely referred to as voltage sources. When a voltage source is connected to a conductor, it applies a potential difference V that creates an electric field. The electric field in turn exerts force on charges, causing current.

Ohm's Law

The current that flows through most substances is directly proportional to the voltage V applied to it. The German physicist Georg Simon Ohm (1787–1854) was the first to demonstrate experimentally that the current in a metal wire is *directly proportional to the voltage applied*:

Equation:

$$I \propto V.$$

This important relationship is known as **Ohm's law**. It can be viewed as a cause-and-effect relationship, with voltage the cause and current the effect. This is an empirical law like that for friction—an experimentally observed phenomenon. Such a linear relationship doesn't always occur.

Resistance and Simple Circuits

If voltage drives current, what impedes it? The electric property that impedes current (crudely similar to friction and air resistance) is called **resistance** R . Collisions of moving charges with atoms and molecules in a substance transfer energy to the substance and limit current. Resistance is defined as inversely proportional to current, or

Equation:

$$I \propto \frac{1}{R}.$$

Thus, for example, current is cut in half if resistance doubles. Combining the relationships of current to voltage and current to resistance gives

Equation:

$$I = \frac{V}{R}.$$

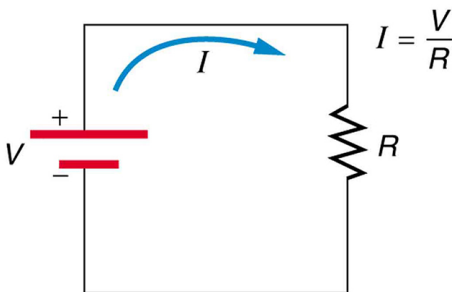
This relationship is also called Ohm's law. Ohm's law in this form really defines resistance for certain materials. Ohm's law (like Hooke's law) is not universally valid. The many substances for which Ohm's law holds are

called **ohmic**. These include good conductors like copper and aluminum, and some poor conductors under certain circumstances. Ohmic materials have a resistance R that is independent of voltage V and current I . An object that has simple resistance is called a *resistor*, even if its resistance is small. The unit for resistance is an **ohm** and is given the symbol Ω (upper case Greek omega). Rearranging $I = V/R$ gives $R = V/I$, and so the units of resistance are 1 ohm = 1 volt per ampere:

Equation:

$$1 \Omega = 1 \frac{V}{A}.$$

[\[link\]](#) shows the schematic for a simple circuit. A **simple circuit** has a single voltage source and a single resistor. The wires connecting the voltage source to the resistor can be assumed to have negligible resistance, or their resistance can be included in R .



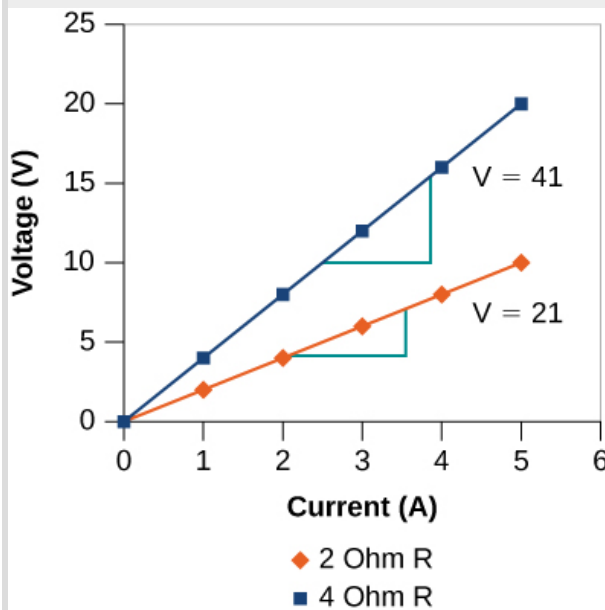
A simple electric circuit in which a closed path for current to flow is supplied by conductors (usually metal wires) connecting a load to the terminals of a battery, represented by the red parallel lines. The zigzag symbol

represents the single resistor and includes any resistance in the connections to the voltage source.

Note:

Making Connections: Real World Connections

Ohm's law ($V = IR$) is a fundamental relationship that could be presented by a linear function with the slope of the line being the resistance. The resistance represents the voltage that needs to be applied to the resistor to create a current of 1 A through the circuit. The graph (in the figure below) shows this representation for two simple circuits with resistors that have different resistances and thus different slopes.



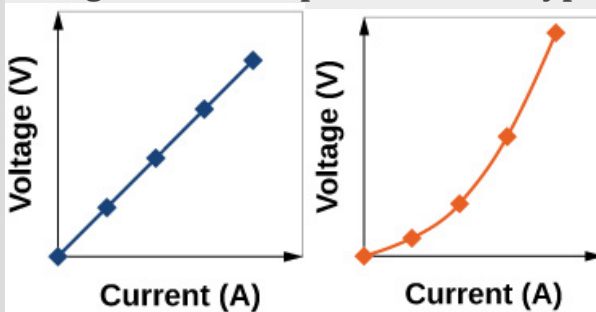
The figure illustrates the relationship between current and voltage for two different resistors. The slope of the graph represents the resistance value,

which is 2Ω and 4Ω for the two lines shown.

Note:

Making Connections: Real World Connections

The materials which follow Ohm's law by having a linear relationship between voltage and current are known as ohmic materials. On the other hand, some materials exhibit a nonlinear voltage-current relationship and hence are known as non-ohmic materials. The figure below shows current voltage relationships for the two types of materials.



The relationship between voltage and current for ohmic and non-ohmic materials are shown.

Clearly the resistance of an ohmic material (shown in (a)) remains constant and can be calculated by finding the slope of the graph but that is not true for a non-ohmic material (shown in (b)).

Example:

Calculating Resistance: An Automobile Headlight

What is the resistance of an automobile headlight through which 2.50 A flows when 12.0 V is applied to it?

Strategy

We can rearrange Ohm's law as stated by $I = V/R$ and use it to find the resistance.

Solution

Rearranging $I = V/R$ and substituting known values gives

Equation:

$$R = \frac{V}{I} = \frac{12.0 \text{ V}}{2.50 \text{ A}} = 4.80 \Omega.$$

Discussion

This is a relatively small resistance, but it is larger than the cold resistance of the headlight. As we shall see in [Resistance and Resistivity](#), resistance usually increases with temperature, and so the bulb has a lower resistance when it is first switched on and will draw considerably more current during its brief warm-up period.

Resistances range over many orders of magnitude. Some ceramic insulators, such as those used to support power lines, have resistances of $10^{12} \Omega$ or more. A dry person may have a hand-to-foot resistance of $10^5 \Omega$, whereas the resistance of the human heart is about $10^3 \Omega$. A meter-long piece of large-diameter copper wire may have a resistance of $10^{-5} \Omega$, and superconductors have no resistance at all (they are non-ohmic). Resistance is related to the shape of an object and the material of which it is composed, as will be seen in [Resistance and Resistivity](#).

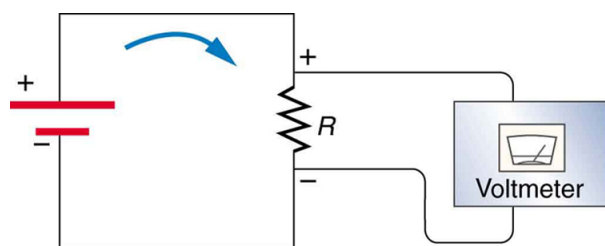
Additional insight is gained by solving $I = V/R$ for V , yielding

Equation:

$$V = IR.$$

This expression for V can be interpreted as the *voltage drop across a resistor produced by the current I* . The phrase *IR drop* is often used for this

voltage. For instance, the headlight in [\[link\]](#) has an IR drop of 12.0 V. If voltage is measured at various points in a circuit, it will be seen to increase at the voltage source and decrease at the resistor. Voltage is similar to fluid pressure. The voltage source is like a pump, creating a pressure difference, causing current—the flow of charge. The resistor is like a pipe that reduces pressure and limits flow because of its resistance. Conservation of energy has important consequences here. The voltage source supplies energy (causing an electric field and a current), and the resistor converts it to another form (such as thermal energy). In a simple circuit (one with a single simple resistor), the voltage supplied by the source equals the voltage drop across the resistor, since $PE = q\Delta V$, and the same q flows through each. Thus the energy supplied by the voltage source and the energy converted by the resistor are equal. (See [\[link\]](#).)



$$V = IR = 18 \text{ V}$$

The voltage drop across a resistor in a simple circuit equals the voltage output of the battery.

Note:**Making Connections: Conservation of Energy**

In a simple electrical circuit, the sole resistor converts energy supplied by the source into another form. Conservation of energy is evidenced here by the fact that all of the energy supplied by the source is converted to another

form by the resistor alone. We will find that conservation of energy has other important applications in circuits and is a powerful tool in circuit analysis.

Note:

PhET Explorations: Ohm's Law

See how the equation form of Ohm's law relates to a simple circuit. Adjust the voltage and resistance, and see the current change according to Ohm's law. The sizes of the symbols in the equation change to match the circuit diagram.

[Ohm's Law](#)

Test Prep for AP Courses

Exercise:

Problem:

If the voltage across a fixed resistance is doubled, what happens to the current?

- a. It doubles.
- b. It halves.
- c. It stays the same.
- d. The current cannot be determined.

Solution:

(a)

Exercise:

Problem:

The table below gives the voltages and currents recorded across a resistor.

Voltage (V)	2.50	5.00	7.50	10.00	12.50
Current (A)	0.69	1.38	2.09	2.76	3.49

- Plot the graph and comment on the shape.
- Calculate the value of the resistor.

Exercise:

Problem:

What is the resistance of a bulb if the current in it is 1.25 A when a 4 V voltage supply is connected to it? If the voltage supply is increased to 7 V, what will be the current in the bulb?

Solution:

3.2 Ω , 2.19 A

Section Summary

- A simple circuit *is* one in which there is a single voltage source and a single resistance.

- One statement of Ohm's law gives the relationship between current I , voltage V , and resistance R in a simple circuit to be $I = \frac{V}{R}$.
- Resistance has units of ohms (Ω), related to volts and amperes by $1 \Omega = 1 \text{ V/A}$.
- There is a voltage or IR drop across a resistor, caused by the current flowing through it, given by $V = IR$.

Conceptual Questions

Exercise:

Problem:

The IR drop across a resistor means that there is a change in potential or voltage across the resistor. Is there any change in current as it passes through a resistor? Explain.

Exercise:

Problem:

How is the IR drop in a resistor similar to the pressure drop in a fluid flowing through a pipe?

Problems & Exercises

Exercise:

Problem:

What current flows through the bulb of a 3.00-V flashlight when its hot resistance is 3.60Ω ?

Solution:

0.833 A

Exercise:

Problem:

Calculate the effective resistance of a pocket calculator that has a 1.35-V battery and through which 0.200 mA flows.

Exercise:**Problem:**

What is the effective resistance of a car's starter motor when 150 A flows through it as the car battery applies 11.0 V to the motor?

Solution:

$$7.33 \times 10^{-2} \, \Omega$$

Exercise:**Problem:**

How many volts are supplied to operate an indicator light on a DVD player that has a resistance of $140 \, \Omega$, given that 25.0 mA passes through it?

Exercise:**Problem:**

(a) Find the voltage drop in an extension cord having a $0.0600\text{-}\Omega$ resistance and through which 5.00 A is flowing. (b) A cheaper cord utilizes thinner wire and has a resistance of $0.300 \, \Omega$. What is the voltage drop in it when 5.00 A flows? (c) Why is the voltage to whatever appliance is being used reduced by this amount? What is the effect on the appliance?

Solution:

(a) 0.300 V

(b) 1.50 V

(c) The voltage supplied to whatever appliance is being used is reduced because the total voltage drop from the wall to the final output of the appliance is fixed. Thus, if the voltage drop across the extension cord is large, the voltage drop across the appliance is significantly decreased, so the power output by the appliance can be significantly decreased, reducing the ability of the appliance to work properly.

Exercise:

Problem:

A power transmission line is hung from metal towers with glass insulators having a resistance of $1.00 \times 10^9 \Omega$. What current flows through the insulator if the voltage is 200 kV? (Some high-voltage lines are DC.)

Glossary

Ohm's law

an empirical relation stating that the current I is proportional to the potential difference V , $\propto V$; it is often written as $I = V/R$, where R is the resistance

resistance

the electric property that impedes current; for ohmic materials, it is the ratio of voltage to current, $R = V/I$

ohm

the unit of resistance, given by $1\Omega = 1 \text{ V/A}$

ohmic

a type of a material for which Ohm's law is valid

simple circuit

a circuit with a single voltage source and a single resistor

Resistance and Resistivity

LEARNING OBJECTIVES

By the end of this section, you will be able to:

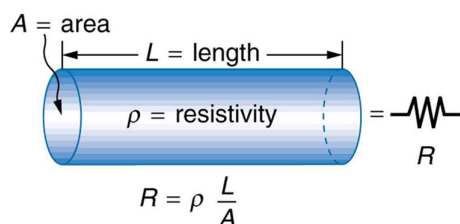
- Explain the concept of resistivity.
- Use resistivity to calculate the resistance of specified configurations of material.
- Use the thermal coefficient of resistivity to calculate the change of resistance with temperature.

The information presented in this section supports the following AP® learning objectives and science practices:

- **1.E.2.1** The student is able to choose and justify the selection of data needed to determine resistivity for a given material. **(S.P. 4.1)**
- **4.E.4.2** The student is able to design a plan for the collection of data to determine the effect of changing the geometry and/or materials on the resistance or capacitance of a circuit element and relate results to the basic properties of resistors and capacitors. **(S.P. 4.1, 4.2)**
- **4.E.4.3** The student is able to analyze data to determine the effect of changing the geometry and/or materials on the resistance or capacitance of a circuit element and relate results to the basic properties of resistors and capacitors. **(S.P. 5.1)**

Material and Shape Dependence of Resistance

The resistance of an object depends on its shape and the material of which it is composed. The cylindrical resistor in [\[link\]](#) is easy to analyze, and, by so doing, we can gain insight into the resistance of more complicated shapes. As you might expect, the cylinder's electric resistance R is directly proportional to its length L , similar to the resistance of a pipe to fluid flow. The longer the cylinder, the more collisions charges will make with its atoms. The greater the diameter of the cylinder, the more current it can carry (again similar to the flow of fluid through a pipe). In fact, R is inversely proportional to the cylinder's cross-sectional area A .



A uniform cylinder of length L and cross-sectional area A . Its resistance to the flow of current is similar to the resistance posed by a pipe to fluid flow.

The longer the cylinder, the greater its resistance. The larger its cross-sectional area A , the smaller its resistance.

For a given shape, the resistance depends on the material of which the object is composed. Different materials offer different resistance to the flow of charge. We define the **resistivity** ρ of a substance so that the **resistance**

R of an object is directly proportional to ρ . Resistivity ρ is an *intrinsic* property of a material, independent of its shape or size. The resistance R of a uniform cylinder of length L , of cross-sectional area A , and made of a material with resistivity ρ , is

Equation:

$$R = \frac{\rho L}{A}.$$

[\[link\]](#) gives representative values of ρ . The materials listed in the table are separated into categories of conductors, semiconductors, and insulators, based on broad groupings of resistivities. Conductors have the smallest resistivities, and insulators have the largest; semiconductors have intermediate resistivities. Conductors have varying but large free charge densities, whereas most charges in insulators are bound to atoms and are not free to move. Semiconductors are intermediate, having far fewer free charges than conductors, but having properties that make the number of free charges depend strongly on the type and amount of impurities in the semiconductor. These unique properties of semiconductors are put to use in modern electronics, as will be explored in later chapters.

Material	Resistivity ρ ($\Omega \cdot \text{m}$)
Conductors	
Silver	1.59×10^{-8}

Material	Resistivity ρ ($\Omega \cdot \text{m}$)
Copper	1.72×10^{-8}
Gold	2.44×10^{-8}
Aluminum	2.65×10^{-8}
Tungsten	5.6×10^{-8}
Iron	9.71×10^{-8}
Platinum	10.6×10^{-8}
Steel	20×10^{-8}
Lead	22×10^{-8}
Manganin (Cu, Mn, Ni alloy)	44×10^{-8}

Material	Resistivity ρ ($\Omega \cdot \text{m}$)
Constantan (Cu, Ni alloy)	49×10^{-8}
Mercury	96×10^{-8}
Nichrome (Ni, Fe, Cr alloy)	100×10^{-8}
<i>Semiconductors</i> [footnote] Values depend strongly on amounts and types of impurities	
Carbon (pure)	3.5×10^5
Carbon	$(3.5 - 60) \times 10^5$
Germanium (pure)	600×10^{-3}
Germanium	$(1 - 600) \times 10^{-3}$
Silicon (pure)	2300

Material	Resistivity ρ ($\Omega \cdot \text{m}$)
Silicon	0.1–2300
<i>Insulators</i>	
Amber	5×10^{14}
Glass	$10^9 - 10^{14}$
Lucite	$>10^{13}$
Mica	$10^{11} - 10^{15}$
Quartz (fused)	75×10^{16}
Rubber (hard)	$10^{13} - 10^{16}$
Sulfur	10^{15}

Material	Resistivity ρ ($\Omega \cdot \text{m}$)
Teflon	$>10^{13}$
Wood	$10^8 - 10^{14}$

Resistivities ρ of Various materials at 20°C

Example:

Calculating Resistor Diameter: A Headlight Filament

A car headlight filament is made of tungsten and has a cold resistance of 0.350Ω . If the filament is a cylinder 4.00 cm long (it may be coiled to save space), what is its diameter?

Strategy

We can rearrange the equation $R = \frac{\rho L}{A}$ to find the cross-sectional area A of the filament from the given information. Then its diameter can be found by assuming it has a circular cross-section.

Solution

The cross-sectional area, found by rearranging the expression for the resistance of a cylinder given in $R = \frac{\rho L}{A}$, is

Equation:

$$A = \frac{\rho L}{R}.$$

Substituting the given values, and taking ρ from [\[link\]](#), yields

Equation:

$$\begin{aligned} A &= \frac{(5.6 \times 10^{-8} \Omega \cdot \text{m})(4.00 \times 10^{-2} \text{ m})}{0.350 \Omega} \\ &= 6.40 \times 10^{-9} \text{ m}^2. \end{aligned}$$

The area of a circle is related to its diameter D by

Equation:

$$A = \frac{\pi D^2}{4}.$$

Solving for the diameter D , and substituting the value found for A , gives

Equation:

$$\begin{aligned} D &= 2\left(\frac{A}{\pi}\right)^{\frac{1}{2}} = 2\left(\frac{6.40 \times 10^{-9} \text{ m}^2}{3.14}\right)^{\frac{1}{2}} \\ &= 9.0 \times 10^{-5} \text{ m}. \end{aligned}$$

Discussion

The diameter is just under a tenth of a millimeter. It is quoted to only two digits, because ρ is known to only two digits.

Temperature Variation of Resistance

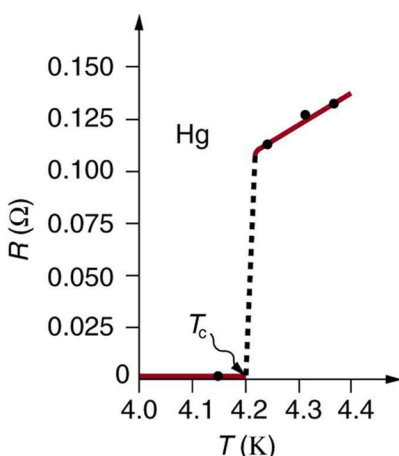
The resistivity of all materials depends on temperature. Some even become superconductors (zero resistivity) at very low temperatures. (See [\[link\]](#).) Conversely, the resistivity of conductors increases with increasing temperature. Since the atoms vibrate more rapidly and over larger distances at higher temperatures, the electrons moving through a metal make more collisions, effectively making the resistivity higher. Over relatively small temperature changes (about 100°C or less), resistivity ρ varies with temperature change ΔT as expressed in the following equation

Equation:

$$\rho = \rho_0(1 + \alpha\Delta T),$$

where ρ_0 is the original resistivity and α is the **temperature coefficient of resistivity**. (See the values of α in [\[link\]](#) below.) For larger temperature changes, α may vary or a nonlinear equation may be needed to find ρ . Note that α is positive for metals, meaning their resistivity increases with

temperature. Some alloys have been developed specifically to have a small temperature dependence. Manganin (which is made of copper, manganese and nickel), for example, has α close to zero (to three digits on the scale in [\[link\]](#)), and so its resistivity varies only slightly with temperature. This is useful for making a temperature-independent resistance standard, for example.



The resistance of a sample of mercury is zero at very low temperatures—it is a superconductor up to about 4.2 K. Above that critical temperature, its resistance makes a sudden jump and then increases nearly linearly with temperature.

Material	Coefficient $\alpha(1/^{\circ}\text{C})$ [footnote] Values at 20°C.
<i>Conductors</i>	
Silver	3.8×10^{-3}
Copper	3.9×10^{-3}
Gold	3.4×10^{-3}
Aluminum	3.9×10^{-3}
Tungsten	4.5×10^{-3}
Iron	5.0×10^{-3}
Platinum	3.93×10^{-3}
Lead	4.3×10^{-3}
Manganin (Cu, Mn, Ni alloy)	0.000×10^{-3}

Material	Coefficient $\alpha(1/^{\circ}\text{C})$ [footnote] Values at 20°C.
Constantan (Cu, Ni alloy)	0.002×10^{-3}
Mercury	0.89×10^{-3}
Nichrome (Ni, Fe, Cr alloy)	0.4×10^{-3}
<i>Semiconductors</i>	
Carbon (pure)	-0.5×10^{-3}
Germanium (pure)	-50×10^{-3}
Silicon (pure)	-70×10^{-3}

Temperature Coefficients of Resistivity α

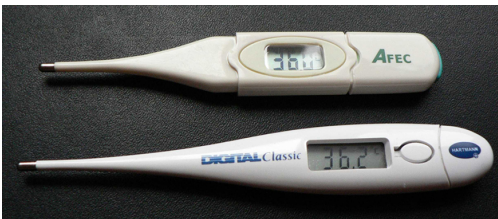
Note also that α is negative for the semiconductors listed in [\[link\]](#), meaning that their resistivity decreases with increasing temperature. They become better conductors at higher temperature, because increased thermal agitation increases the number of free charges available to carry current. This property of decreasing ρ with temperature is also related to the type and amount of impurities present in the semiconductors.

The resistance of an object also depends on temperature, since R_0 is directly proportional to ρ . For a cylinder we know $R = \rho L/A$, and so, if L and A do not change greatly with temperature, R will have the same temperature dependence as ρ . (Examination of the coefficients of linear expansion shows them to be about two orders of magnitude less than typical temperature coefficients of resistivity, and so the effect of temperature on L and A is about two orders of magnitude less than on ρ .) Thus,

Equation:

$$R = R_0(1 + \alpha\Delta T)$$

is the temperature dependence of the resistance of an object, where R_0 is the original resistance and R is the resistance after a temperature change ΔT . Numerous thermometers are based on the effect of temperature on resistance. (See [\[link\]](#).) One of the most common is the thermistor, a semiconductor crystal with a strong temperature dependence, the resistance of which is measured to obtain its temperature. The device is small, so that it quickly comes into thermal equilibrium with the part of a person it touches.



These familiar
thermometers are based
on the automated
measurement of a
thermistor's temperature-
dependent resistance.
(credit: Biol, Wikimedia
Commons)

Example:**Calculating Resistance: Hot-Filament Resistance**

Although caution must be used in applying $\rho = \rho_0(1 + \alpha\Delta T)$ and $R = R_0(1 + \alpha\Delta T)$ for temperature changes greater than 100°C, for tungsten the equations work reasonably well for very large temperature changes. What, then, is the resistance of the tungsten filament in the previous example if its temperature is increased from room temperature (20°C) to a typical operating temperature of 2850°C?

Strategy

This is a straightforward application of $R = R_0(1 + \alpha\Delta T)$, since the original resistance of the filament was given to be $R_0 = 0.350 \, \Omega$, and the temperature change is $\Delta T = 2830^\circ\text{C}$.

Solution

The hot resistance R is obtained by entering known values into the above equation:

Equation:

$$\begin{aligned} R &= R_0(1 + \alpha\Delta T) \\ &= (0.350 \, \Omega)[1 + (4.5 \times 10^{-3}/^\circ\text{C})(2830^\circ\text{C})] \\ &= 4.8 \, \Omega. \end{aligned}$$

Discussion

This value is consistent with the headlight resistance example in [Ohm's Law: Resistance and Simple Circuits](#).

Note:**PhET Explorations: Resistance in a Wire**

Learn about the physics of resistance in a wire. Change its resistivity, length, and area to see how they affect the wire's resistance. The sizes of the symbols in the equation change along with the diagram of a wire.

Resistance in a Wire

Note:

Applying the Science Practices: Examining Resistance

Using the PhET Simulation “Resistance in a Wire”, design an experiment to determine how different variables – resistivity, length, and area – affect the resistance of a resistor. For each variable, you should record your results in a table and then create a graph to determine the relationship.

Test Prep for AP Courses

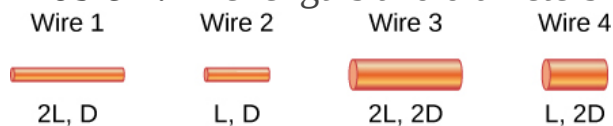
Exercise:

Problem: Which of the following affect the resistivity of a wire?

- a. length
- b. area of cross section
- c. material
- d. all of the above

Exercise:

Problem: The lengths and diameters of four wires are given as shown.



If the four wires are made from the same material, which of the following is true? Select *two* answers.

- a. Resistance of Wire 3 > Resistance of Wire 2
 - b. Resistance of Wire 1 > Resistance of Wire 2
 - c. Resistance of Wire 1 < Resistance of Wire 4
 - d. Resistance of Wire 4 < Resistance of Wire 3
-

Solution:

(b), (d)

Exercise:

Problem:

Suppose the resistance of a wire is $R \, \Omega$. What will be the resistance of another wire of the same material having the same length but double the diameter?

- a. $R/2$
- b. $2R$
- c. $R/4$
- d. $4R$

Exercise:

Problem:

The resistances of two wires having the same lengths and cross section areas are $3 \, \Omega$ and $11 \, \Omega$. If the resistivity of the $3 \, \Omega$ wire is $2.65 \times 10^{-8} \, \Omega \cdot \text{m}$, find the resistivity of the $1 \, \Omega$ wire.

Solution:

$$9.72 \times 10^{-8} \, \Omega \cdot \text{m}$$

Exercise:

Problem:

The lengths and diameters of three wires are given below. If they all have the same resistance, find the ratio of their resistivities.

<i>Wire</i>	<i>Length</i>	<i>Diameter</i>
Wire 1	2 m	1 cm
Wire 2	1 m	0.5 cm
Wire 3	1 m	1 cm

Exercise:

Problem:

Suppose the resistance of a wire is $2\ \Omega$. If the wire is stretched to three times its length, what will be its resistance? Assume that the volume does not change.

Solution:

$18\ \Omega$

Section Summary

- The resistance R of a cylinder of length L and cross-sectional area A is $R = \frac{\rho L}{A}$, where ρ is the resistivity of the material.
- Values of ρ in [\[link\]](#) show that materials fall into three groups—*conductors, semiconductors, and insulators*.
- Temperature affects resistivity; for relatively small temperature changes ΔT , resistivity is $\rho = \rho_0(1 + \alpha\Delta T)$, where ρ_0 is the original resistivity and α is the temperature coefficient of resistivity.
- [\[link\]](#) gives values for α , the temperature coefficient of resistivity.
- The resistance R of an object also varies with temperature: $R = R_0(1 + \alpha\Delta T)$, where R_0 is the original resistance, and R is the resistance after the temperature change.

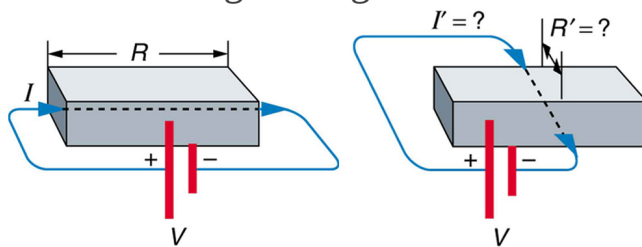
Conceptual Questions

Exercise:**Problem:**

In which of the three semiconducting materials listed in [\[link\]](#) do impurities supply free charges? (Hint: Examine the range of resistivity for each and determine whether the pure semiconductor has the higher or lower conductivity.)

Exercise:**Problem:**

Does the resistance of an object depend on the path current takes through it? Consider, for example, a rectangular bar—is its resistance the same along its length as across its width? (See [\[link\]](#).)



Does current taking two different paths through the same object encounter different resistance?

Exercise:**Problem:**

If aluminum and copper wires of the same length have the same resistance, which has the larger diameter? Why?

Exercise:

Problem:

Explain why $R = R_0(1 + \alpha\Delta T)$ for the temperature variation of the resistance R of an object is not as accurate as $\rho = \rho_0(1 + \alpha\Delta T)$, which gives the temperature variation of resistivity ρ .

Problems & Exercises**Exercise:****Problem:**

What is the resistance of a 20.0-m-long piece of 12-gauge copper wire having a 2.053-mm diameter?

Solution:

0.104 Ω

Exercise:**Problem:**

The diameter of 0-gauge copper wire is 8.252 mm. Find the resistance of a 1.00-km length of such wire used for power transmission.

Exercise:**Problem:**

If the 0.100-mm diameter tungsten filament in a light bulb is to have a resistance of 0.200 Ω at 20.0°C, how long should it be?

Solution:

2.8×10^{-2} m

Exercise:

Problem:

Find the ratio of the diameter of aluminum to copper wire, if they have the same resistance per unit length (as they might in household wiring).

Exercise:**Problem:**

What current flows through a 2.54-cm-diameter rod of pure silicon that is 20.0 cm long, when 1.00×10^3 V is applied to it? (Such a rod may be used to make nuclear-particle detectors, for example.)

Solution:

$$1.10 \times 10^{-3} \text{ A}$$

Exercise:**Problem:**

(a) To what temperature must you raise a copper wire, originally at 20.0°C , to double its resistance, neglecting any changes in dimensions? (b) Does this happen in household wiring under ordinary circumstances?

Exercise:**Problem:**

A resistor made of Nichrome wire is used in an application where its resistance cannot change more than 1.00% from its value at 20.0°C . Over what temperature range can it be used?

Solution:

$$-5^\circ\text{C to } 45^\circ\text{C}$$

Exercise:

Problem:

Of what material is a resistor made if its resistance is 40.0% greater at 100°C than at 20.0°C?

Exercise:**Problem:**

An electronic device designed to operate at any temperature in the range from -10.0°C to 55.0°C contains pure carbon resistors. By what factor does their resistance increase over this range?

Solution:

1.03

Exercise:**Problem:**

(a) Of what material is a wire made, if it is 25.0 m long with a 0.100 mm diameter and has a resistance of $77.7\ \Omega$ at 20.0°C ? (b) What is its resistance at 150°C ?

Exercise:**Problem:**

Assuming a constant temperature coefficient of resistivity, what is the maximum percent decrease in the resistance of a constantan wire starting at 20.0°C ?

Solution:

0.06%

Exercise:

Problem:

A wire is drawn through a die, stretching it to four times its original length. By what factor does its resistance increase?

Exercise:**Problem:**

A copper wire has a resistance of $0.500\ \Omega$ at 20.0°C , and an iron wire has a resistance of $0.525\ \Omega$ at the same temperature. At what temperature are their resistances equal?

Solution:

-17°C

Exercise:**Problem:**

(a) Digital medical thermometers determine temperature by measuring the resistance of a semiconductor device called a thermistor (which has $\alpha = -0.0600/^\circ\text{C}$) when it is at the same temperature as the patient. What is a patient's temperature if the thermistor's resistance at that temperature is 82.0% of its value at 37.0°C (normal body temperature)? (b) The negative value for α may not be maintained for very low temperatures. Discuss why and whether this is the case here. (Hint: Resistance can't become negative.)

Exercise:**Problem: Integrated Concepts**

(a) Redo [\[link\]](#) taking into account the thermal expansion of the tungsten filament. You may assume a thermal expansion coefficient of $12 \times 10^{-6}/^\circ\text{C}$. (b) By what percentage does your answer differ from that in the example?

Solution:

(a) $4.7\ \Omega$ (total)

(b) 3.0% decrease

Exercise:

Problem: Unreasonable Results

(a) To what temperature must you raise a resistor made of constantan to double its resistance, assuming a constant temperature coefficient of resistivity? (b) To cut it in half? (c) What is unreasonable about these results? (d) Which assumptions are unreasonable, or which premises are inconsistent?

Glossary

resistivity

an intrinsic property of a material, independent of its shape or size, directly proportional to the resistance, denoted by ρ

temperature coefficient of resistivity

an empirical quantity, denoted by α , which describes the change in resistance or resistivity of a material with temperature

Electric Power and Energy

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Calculate the power dissipated by a resistor and the power supplied by a power supply.
- Calculate the cost of electricity under various circumstances.

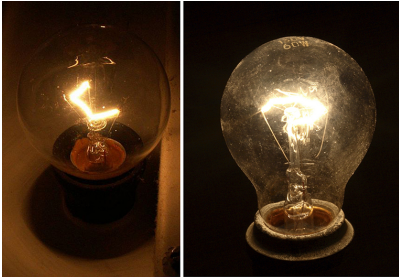
The information presented in this section supports the following AP® learning objectives and science practices:

- **5.B.9.8** The student is able to translate between graphical and symbolic representations of experimental data describing relationships among power, current, and potential difference across a resistor. **(S.P. 1.5)**

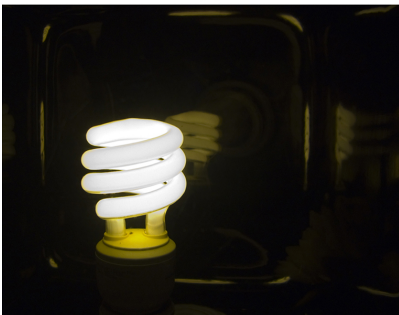
Power in Electric Circuits

Power is associated by many people with electricity. Knowing that power is the rate of energy use or energy conversion, what is the expression for **electric power**? Power transmission lines might come to mind. We also think of lightbulbs in terms of their power ratings in watts. Let us compare a 25-W bulb with a 60-W bulb. (See [\[link\]](#)(a).) Since both operate on the same voltage, the 60-W bulb must draw more current to have a greater power rating. Thus the 60-W bulb's resistance must be lower than that of a 25-W bulb. If we increase voltage, we also increase power. For example, when a 25-W bulb that is designed to operate on 120 V is connected to 240

V, it briefly glows very brightly and then burns out. Precisely how are voltage, current, and resistance related to electric power?



(a)



(b)

(a) Which of these lightbulbs, the 25-W bulb (upper left) or the 60-W bulb (upper right), has the higher resistance? Which draws more current? Which uses the most energy? Can you tell from the color that the 25-W filament is cooler? Is the brighter bulb a different color and if so why?

(credits:
Dickbauch,
Wikimedia
Commons; Greg
Westfall, Flickr) (b)
This compact
fluorescent light
(CFL) puts out the
same intensity of
light as the 60-W
bulb, but at 1/4 to
1/10 the input
power. (credit:
dbgg1979, Flickr)

Electric energy depends on both the voltage involved and the charge moved. This is expressed most simply as $PE = qV$, where q is the charge moved and V is the voltage (or more precisely, the potential difference the charge moves through). Power is the rate at which energy is moved, and so electric power is

Equation:

$$P = \frac{PE}{t} = \frac{qV}{t}.$$

Recognizing that current is $I = q/t$ (note that $\Delta t = t$ here), the expression for power becomes

Equation:

$$P = IV.$$

Electric power (P) is simply the product of current times voltage. Power has familiar units of watts. Since the SI unit for potential energy (PE) is the joule, power has units of joules per second, or watts. Thus, $1 \text{ A} \cdot \text{V} = 1 \text{ W}$.

For example, cars often have one or more auxiliary power outlets with which you can charge a cell phone or other electronic devices. These outlets may be rated at 20 A, so that the circuit can deliver a maximum power $P = IV = (20 \text{ A})(12 \text{ V}) = 240 \text{ W}$. In some applications, electric power may be expressed as volt-amperes or even kilovolt-amperes ($1 \text{ kA} \cdot \text{V} = 1 \text{ kW}$).

To see the relationship of power to resistance, we combine Ohm's law with $P = IV$. Substituting $I = V/R$ gives $P = (V/R)V = V^2/R$. Similarly, substituting $V = IR$ gives $P = I(IR) = I^2R$. Three expressions for electric power are listed together here for convenience:

Equation:

$$P = IV$$

Equation:

$$P = \frac{V^2}{R}$$

Equation:

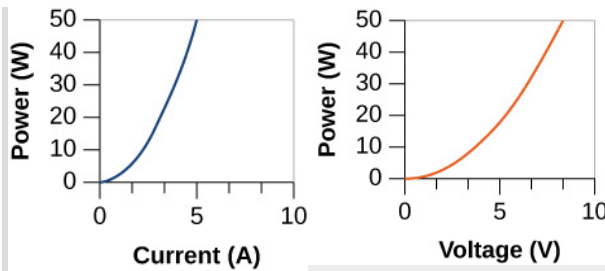
$$P = I^2R.$$

Note that the first equation is always valid, whereas the other two can be used only for resistors. In a simple circuit, with one voltage source and a single resistor, the power supplied by the voltage source and that dissipated by the resistor are identical. (In more complicated circuits, P can be the power dissipated by a single device and not the total power in the circuit.)

Note:

Making Connections: Using Graphs to Calculate Resistance

As $p \propto I^2$ and $p \propto V^2$, the graph for power versus current or voltage is quadratic. An example is shown in the figure below.



The figure shows (a) power versus current and (b) power versus voltage relationships for simple resistor circuits.

Using equations (20.29) and (20.30), we can calculate the resistance in each case. In graph (a), the power is 50 W when current is 5 A; hence, the resistance can be calculated as $R = P/I^2 = 50/5^2 = 2 \Omega$. Similarly, the resistance value can be calculated in graph (b) as

$$R = V^2/P = 10^2/50 = 2 \Omega$$

Different insights can be gained from the three different expressions for electric power. For example, $P = V^2/R$ implies that the lower the resistance connected to a given voltage source, the greater the power delivered. Furthermore, since voltage is squared in $P = V^2/R$, the effect of applying a higher voltage is perhaps greater than expected. Thus, when the voltage is doubled to a 25-W bulb, its power nearly quadruples to about 100 W, burning it out. If the bulb's resistance remained constant, its power would be exactly 100 W, but at the higher temperature its resistance is higher, too.

Example:

Calculating Power Dissipation and Current: Hot and Cold Power

(a) Consider the examples given in [Ohm's Law: Resistance and Simple Circuits](#) and [Resistance and Resistivity](#). Then find the power dissipated by

the car headlight in these examples, both when it is hot and when it is cold.
(b) What current does it draw when cold?

Strategy for (a)

For the hot headlight, we know voltage and current, so we can use $P = IV$ to find the power. For the cold headlight, we know the voltage and resistance, so we can use $P = V^2/R$ to find the power.

Solution for (a)

Entering the known values of current and voltage for the hot headlight, we obtain

Equation:

$$P = IV = (2.50 \text{ A})(12.0 \text{ V}) = 30.0 \text{ W}.$$

The cold resistance was 0.350Ω , and so the power it uses when first switched on is

Equation:

$$P = \frac{V^2}{R} = \frac{(12.0 \text{ V})^2}{0.350 \Omega} = 411 \text{ W}.$$

Discussion for (a)

The 30 W dissipated by the hot headlight is typical. But the 411 W when cold is surprisingly higher. The initial power quickly decreases as the bulb's temperature increases and its resistance increases.

Strategy and Solution for (b)

The current when the bulb is cold can be found several different ways. We rearrange one of the power equations, $P = I^2 R$, and enter known values, obtaining

Equation:

$$I = \sqrt{\frac{P}{R}} = \sqrt{\frac{411 \text{ W}}{0.350 \Omega}} = 34.3 \text{ A}.$$

Discussion for (b)

The cold current is remarkably higher than the steady-state value of 2.50 A, but the current will quickly decline to that value as the bulb's temperature increases. Most fuses and circuit breakers (used to limit the

current in a circuit) are designed to tolerate very high currents briefly as a device comes on. In some cases, such as with electric motors, the current remains high for several seconds, necessitating special “slow blow” fuses.

The Cost of Electricity

The more electric appliances you use and the longer they are left on, the higher your electric bill. This familiar fact is based on the relationship between energy and power. You pay for the energy used. Since $P = E/t$, we see that

Equation:

$$E = Pt$$

is the energy used by a device using power P for a time interval t . For example, the more lightbulbs burning, the greater P used; the longer they are on, the greater t is. The energy unit on electric bills is the kilowatt-hour ($\text{kW} \cdot \text{h}$), consistent with the relationship $E = Pt$. It is easy to estimate the cost of operating electric appliances if you have some idea of their power consumption rate in watts or kilowatts, the time they are on in hours, and the cost per kilowatt-hour for your electric utility. Kilowatt-hours, like all other specialized energy units such as food calories, can be converted to joules. You can prove to yourself that $1 \text{ kW} \cdot \text{h} = 3.6 \times 10^6 \text{ J}$.

The electrical energy (E) used can be reduced either by reducing the time of use or by reducing the power consumption of that appliance or fixture. This will not only reduce the cost, but it will also result in a reduced impact on the environment. Improvements to lighting are some of the fastest ways to reduce the electrical energy used in a home or business. About 20% of a home's use of energy goes to lighting, while the number for commercial establishments is closer to 40%. Fluorescent lights are about four times more efficient than incandescent lights—this is true for both the long tubes and the compact fluorescent lights (CFL). (See [\[link\]](#)(b).) Thus, a 60-W incandescent bulb can be replaced by a 15-W CFL, which has the same brightness and color. CFLs have a bent tube inside a globe or a spiral-

shaped tube, all connected to a standard screw-in base that fits standard incandescent light sockets. (Original problems with color, flicker, shape, and high initial investment for CFLs have been addressed in recent years.) The heat transfer from these CFLs is less, and they last up to 10 times longer. The significance of an investment in such bulbs is addressed in the next example. New white LED lights (which are clusters of small LED bulbs) are even more efficient (twice that of CFLs) and last 5 times longer than CFLs. However, their cost is still high.

Note:**Making Connections: Energy, Power, and Time**

The relationship $E = Pt$ is one that you will find useful in many different contexts. The energy your body uses in exercise is related to the power level and duration of your activity, for example. The amount of heating by a power source is related to the power level and time it is applied. Even the radiation dose of an X-ray image is related to the power and time of exposure.

Example:**Calculating the Cost Effectiveness of Compact Fluorescent Lights (CFL)**

If the cost of electricity in your area is 12 cents per kWh, what is the total cost (capital plus operation) of using a 60-W incandescent bulb for 1000 hours (the lifetime of that bulb) if the bulb cost 25 cents? (b) If we replace this bulb with a compact fluorescent light that provides the same light output, but at one-quarter the wattage, and which costs \$1.50 but lasts 10 times longer (10,000 hours), what will that total cost be?

Strategy

To find the operating cost, we first find the energy used in kilowatt-hours and then multiply by the cost per kilowatt-hour.

Solution for (a)

The energy used in kilowatt-hours is found by entering the power and time into the expression for energy:

Equation:

$$E = Pt = (60 \text{ W})(1000 \text{ h}) = 60,000 \text{ W} \cdot \text{h}.$$

In kilowatt-hours, this is

Equation:

$$E = 60.0 \text{ kW} \cdot \text{h}.$$

Now the electricity cost is

Equation:

$$\text{cost} = (60.0 \text{ kW} \cdot \text{h})(\$0.12/\text{kW} \cdot \text{h}) = \$7.20.$$

The total cost will be \$7.20 for 1000 hours (about one-half year at 5 hours per day).

Solution for (b)

Since the CFL uses only 15 W and not 60 W, the electricity cost will be $\$7.20/4 = \1.80 . The CFL will last 10 times longer than the incandescent, so that the investment cost will be 1/10 of the bulb cost for that time period of use, or $0.1(\$1.50) = \0.15 . Therefore, the total cost will be \$1.95 for 1000 hours.

Discussion

Therefore, it is much cheaper to use the CFLs, even though the initial investment is higher. The increased cost of labor that a business must include for replacing the incandescent bulbs more often has not been figured in here.

Note:**Making Connections: Take-Home Experiment—Electrical Energy Use Inventory**

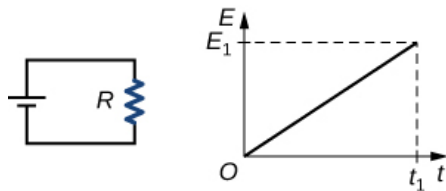
1) Make a list of the power ratings on a range of appliances in your home or room. Explain why something like a toaster has a higher rating than a digital clock. Estimate the energy consumed by these appliances in an average day (by estimating their time of use). Some appliances might only state the operating current. If the household voltage is 120 V, then use

$P = IV$. 2) Check out the total wattage used in the rest rooms of your school's floor or building. (You might need to assume the long fluorescent lights in use are rated at 32 W.) Suppose that the building was closed all weekend and that these lights were left on from 6 p.m. Friday until 8 a.m. Monday. What would this oversight cost? How about for an entire year of weekends?

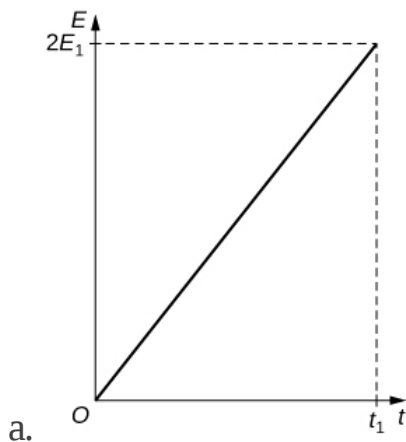
Test Prep for AP Courses

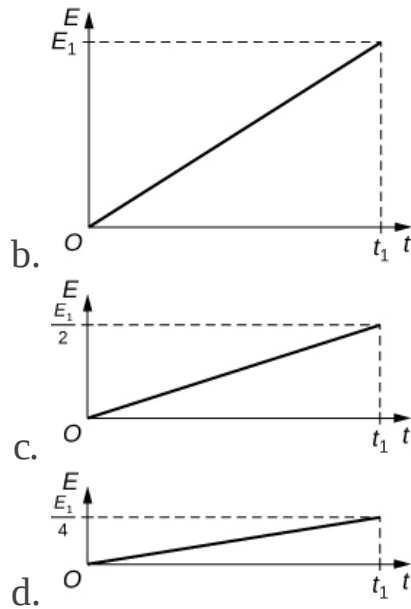
Exercise:

Problem:



The circuit shown contains a resistor R connected to a voltage supply. The graph shows the total energy E dissipated by the resistance as a function of time. Which of the following shows the corresponding graph for double resistance, i.e., if R is replaced by $2R$?





Exercise:

Problem:

What will be the ratio of the resistance of a 120 W, 220 V lamp to that of a 100 W, 110 V lamp?

Solution:

10:3 or 3.33

Section Summary

- Electric power P is the rate (in watts) that energy is supplied by a source or dissipated by a device.
- Three expressions for electrical power are

Equation:

$$P = IV,$$

Equation:

$$P = \frac{V^2}{R},$$

and

Equation:

$$P = I^2 R.$$

- The energy used by a device with a power P over a time t is $E = Pt$.

Conceptual Questions

Exercise:

Problem:

Why do incandescent lightbulbs grow dim late in their lives, particularly just before their filaments break?

Exercise:

Problem:

The power dissipated in a resistor is given by $P = V^2/R$, which means power decreases if resistance increases. Yet this power is also given by $P = I^2 R$, which means power increases if resistance increases. Explain why there is no contradiction here.

Problem Exercises

Exercise:

Problem:

What is the power of a 1.00×10^2 MV lightning bolt having a current of 2.00×10^4 A?

Solution:

$$2.00 \times 10^{12} \text{ W}$$

Exercise:

Problem:

What power is supplied to the starter motor of a large truck that draws 250 A of current from a 24.0-V battery hookup?

Exercise:

Problem:

A charge of 4.00 C of charge passes through a pocket calculator's solar cells in 4.00 h. What is the power output, given the calculator's voltage output is 3.00 V? (See [\[link\]](#).)



The strip of solar cells just above the keys of this calculator convert light to electricity to supply its energy needs. (credit: Evan-Amos, Wikimedia Commons)

Exercise:**Problem:**

How many watts does a flashlight that has 6.00×10^2 C pass through it in 0.500 h use if its voltage is 3.00 V?

Exercise:**Problem:**

Find the power dissipated in each of these extension cords: (a) an extension cord having a $0.0600\text{ }\Omega$ resistance and through which 5.00 A is flowing; (b) a cheaper cord utilizing thinner wire and with a resistance of $0.300\text{ }\Omega$.

Solution:

(a) 1.50 W

(b) 7.50 W

Exercise:**Problem:**

Verify that the units of a volt-ampere are watts, as implied by the equation $P = IV$.

Exercise:**Problem:**

Show that the units $1\text{ V}^2/\Omega = 1\text{ W}$, as implied by the equation $P = V^2/R$.

Solution:

$$\frac{V^2}{\Omega} = \frac{V^2}{V/A} = AV = \left(\frac{C}{s}\right)\left(\frac{J}{C}\right) = \frac{J}{s} = 1\text{ W}$$

Exercise:

Problem:

Show that the units $1 \text{ A}^2 \cdot \Omega = 1 \text{ W}$, as implied by the equation $P = I^2 R$.

Exercise:**Problem:**

Verify the energy unit equivalence that $1 \text{ kW} \cdot \text{h} = 3.60 \times 10^6 \text{ J}$.

Solution:

$$1 \text{ kW} \cdot \text{h} = \left(\frac{1 \times 10^3 \text{ J}}{1 \text{ s}} \right) (1 \text{ h}) \left(\frac{3600 \text{ s}}{1 \text{ h}} \right) = 3.60 \times 10^6 \text{ J}$$

Exercise:**Problem:**

Electrons in an X-ray tube are accelerated through $1.00 \times 10^2 \text{ kV}$ and directed toward a target to produce X-rays. Calculate the power of the electron beam in this tube if it has a current of 15.0 mA .

Exercise:**Problem:**

An electric water heater consumes 5.00 kW for 2.00 h per day. What is the cost of running it for one year if electricity costs $12.0 \text{ cents/kW} \cdot \text{h}$? See [\[link\]](#).



On-demand electric
hot water heater. Heat

is supplied to water
only when needed.
(credit: aviddavid,
Flickr)

Solution:

\$438/y

Exercise:

Problem:

With a 1200-W toaster, how much electrical energy is needed to make a slice of toast (cooking time = 1 minute)? At 9.0 cents/kW · h, how much does this cost?

Exercise:

Problem:

What would be the maximum cost of a CFL such that the total cost (investment plus operating) would be the same for both CFL and incandescent 60-W bulbs? Assume the cost of the incandescent bulb is 25 cents and that electricity costs 10 cents/kWh. Calculate the cost for 1000 hours, as in the cost effectiveness of CFL example.

Solution:

\$6.25

Exercise:

Problem:

Some makes of older cars have 6.00-V electrical systems. (a) What is the hot resistance of a 30.0-W headlight in such a car? (b) What current flows through it?

Exercise:**Problem:**

Alkaline batteries have the advantage of putting out constant voltage until very nearly the end of their life. How long will an alkaline battery rated at $1.00 \text{ A} \cdot \text{h}$ and 1.58 V keep a 1.00-W flashlight bulb burning?

Solution:

1.58 h

Exercise:**Problem:**

A cauterizer, used to stop bleeding in surgery, puts out 2.00 mA at 15.0 kV . (a) What is its power output? (b) What is the resistance of the path?

Exercise:**Problem:**

The average television is said to be on 6 hours per day. Estimate the yearly cost of electricity to operate 100 million TVs, assuming their power consumption averages 150 W and the cost of electricity averages $12.0 \text{ cents/kW} \cdot \text{h}$.

Solution:

$\$3.94 \text{ billion/year}$

Exercise:**Problem:**

An old lightbulb draws only 50.0 W , rather than its original 60.0 W , due to evaporative thinning of its filament. By what factor is its diameter reduced, assuming uniform thinning along its length? Neglect any effects caused by temperature differences.

Exercise:**Problem:**

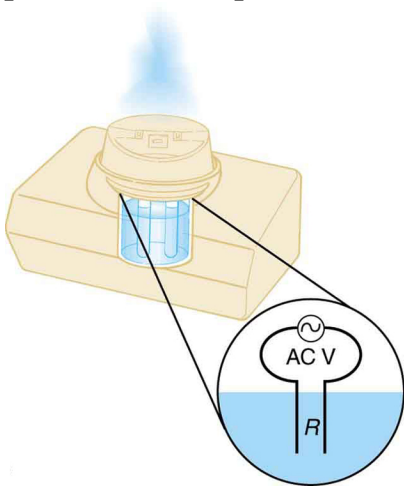
00-gauge copper wire has a diameter of 9.266 mm. Calculate the power loss in a kilometer of such wire when it carries 1.00×10^2 A.

Solution:

25.5 W

Exercise:**Problem: Integrated Concepts**

Cold vaporizers pass a current through water, evaporating it with only a small increase in temperature. One such home device is rated at 3.50 A and utilizes 120 V AC with 95.0% efficiency. (a) What is the vaporization rate in grams per minute? (b) How much water must you put into the vaporizer for 8.00 h of overnight operation? (See [link](#).)



This cold vaporizer
passes current
directly through
water, vaporizing it
directly with
relatively little

temperature
increase.

Exercise:

Problem: Integrated Concepts

(a) What energy is dissipated by a lightning bolt having a 20,000-A current, a voltage of 1.00×10^2 MV, and a length of 1.00 ms? (b) What mass of tree sap could be raised from 18.0°C to its boiling point and then evaporated by this energy, assuming sap has the same thermal characteristics as water?

Solution:

(a) 2.00×10^9 J

(b) 769 kg

Exercise:

Problem: Integrated Concepts

What current must be produced by a 12.0-V battery-operated bottle warmer in order to heat 75.0 g of glass, 250 g of baby formula, and 3.00×10^2 g of aluminum from 20.0°C to 90.0°C in 5.00 min?

Exercise:

Problem: Integrated Concepts

How much time is needed for a surgical cauterizer to raise the temperature of 1.00 g of tissue from 37.0°C to 100°C and then boil away 0.500 g of water, if it puts out 2.00 mA at 15.0 kV? Ignore heat transfer to the surroundings.

Solution:

45.0 s

Exercise:

Problem: Integrated Concepts

Hydroelectric generators (see [\[link\]](#)) at Hoover Dam produce a maximum current of $8.00 \times 10^3 \text{ A}$ at 250 kV. (a) What is the power output? (b) The water that powers the generators enters and leaves the system at low speed (thus its kinetic energy does not change) but loses 160 m in altitude. How many cubic meters per second are needed, assuming 85.0% efficiency?



Hydroelectric generators
at the Hoover dam.
(credit: Jon Sullivan)

Exercise:

Problem: Integrated Concepts

(a) Assuming 95.0% efficiency for the conversion of electrical power by the motor, what current must the 12.0-V batteries of a 750-kg electric car be able to supply: (a) To accelerate from rest to 25.0 m/s in 1.00 min? (b) To climb a 2.00×10^2 -m-high hill in 2.00 min at a constant 25.0-m/s speed while exerting $5.00 \times 10^2 \text{ N}$ of force to overcome air resistance and friction? (c) To travel at a constant 25.0-

m/s speed, exerting a 5.00×10^2 N force to overcome air resistance and friction? See [\[link\]](#).



This REVAi, an electric car, gets recharged on a street in London. (credit: Frank Hebbert)

Solution:

- (a) 343 A
- (b) 2.17×10^3 A
- (c) 1.10×10^3 A

Exercise:

Problem: Integrated Concepts

A light-rail commuter train draws 630 A of 650-V DC electricity when accelerating. (a) What is its power consumption rate in kilowatts? (b) How long does it take to reach 20.0 m/s starting from rest if its loaded mass is 5.30×10^4 kg, assuming 95.0% efficiency and constant power? (c) Find its average acceleration. (d) Discuss how the acceleration you found for the light-rail train compares to what might be typical for an automobile.

Exercise:**Problem: Integrated Concepts**

(a) An aluminum power transmission line has a resistance of $0.0580\ \Omega/\text{km}$. What is its mass per kilometer? (b) What is the mass per kilometer of a copper line having the same resistance? A lower resistance would shorten the heating time. Discuss the practical limits to speeding the heating by lowering the resistance.

Solution:

(a) $1.23 \times 10^3\ \text{kg}$

(b) $2.64 \times 10^3\ \text{kg}$

Exercise:**Problem: Integrated Concepts**

(a) An immersion heater utilizing 120 V can raise the temperature of a $1.00 \times 10^2\text{-g}$ aluminum cup containing 350 g of water from 20.0°C to 95.0°C in 2.00 min. Find its resistance, assuming it is constant during the process. (b) A lower resistance would shorten the heating time. Discuss the practical limits to speeding the heating by lowering the resistance.

Exercise:**Problem: Integrated Concepts**

(a) What is the cost of heating a hot tub containing 1500 kg of water from 10.0°C to 40.0°C , assuming 75.0% efficiency to account for heat transfer to the surroundings? The cost of electricity is 9 cents/kW · h. (b) What current was used by the 220-V AC electric heater, if this took 4.00 h?

Exercise:

Problem: Unreasonable Results

(a) What current is needed to transmit 1.00×10^2 MW of power at 480 V? (b) What power is dissipated by the transmission lines if they have a $1.00\text{-}\Omega$ resistance? (c) What is unreasonable about this result? (d) Which assumptions are unreasonable, or which premises are inconsistent?

Solution:

(a) 2.08×10^5 A

(b) 4.33×10^4 MW

(c) The transmission lines dissipate more power than they are supposed to transmit.

(d) A voltage of 480 V is unreasonably low for a transmission voltage. Long-distance transmission lines are kept at much higher voltages (often hundreds of kilovolts) to reduce power losses.

Exercise:**Problem: Unreasonable Results**

(a) What current is needed to transmit 1.00×10^2 MW of power at 10.0 kV? (b) Find the resistance of 1.00 km of wire that would cause a 0.0100% power loss. (c) What is the diameter of a 1.00-km-long copper wire having this resistance? (d) What is unreasonable about these results? (e) Which assumptions are unreasonable, or which premises are inconsistent?

Exercise:**Problem: Construct Your Own Problem**

Consider an electric immersion heater used to heat a cup of water to make tea. Construct a problem in which you calculate the needed resistance of the heater so that it increases the temperature of the water and cup in a reasonable amount of time. Also calculate the cost of the electrical energy used in your process. Among the things to be considered are the voltage used, the masses and heat capacities involved, heat losses, and the time over which the heating takes place. Your instructor may wish for you to consider a thermal safety switch (perhaps bimetallic) that will halt the process before damaging temperatures are reached in the immersion unit.

Glossary

electric power

the rate at which electrical energy is supplied by a source or dissipated by a device; it is the product of current times voltage

Alternating Current versus Direct Current

LEARNING OBJECTIVES

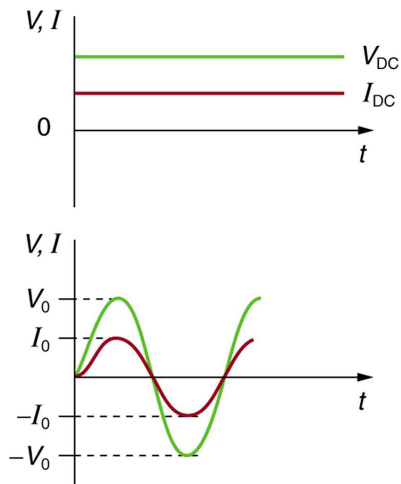
By the end of this section, you will be able to:

- Explain the differences and similarities between AC and DC current.
- Calculate *rms* voltage, current, and average power.
- Explain why AC current is used for power transmission.

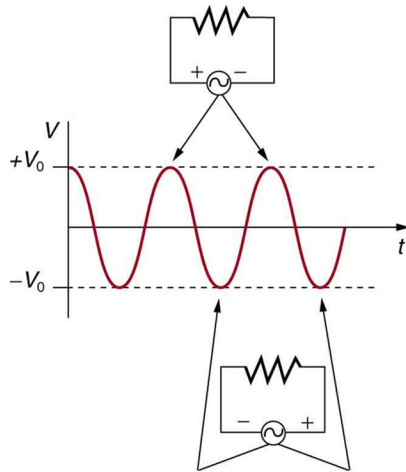
Alternating Current

Most of the examples dealt with so far, and particularly those utilizing batteries, have constant voltage sources. Once the current is established, it is thus also a constant. **Direct current** (DC) is the flow of electric charge in only one direction. It is the steady state of a constant-voltage circuit. Most well-known applications, however, use a time-varying voltage source.

Alternating current (AC) is the flow of electric charge that periodically reverses direction. If the source varies periodically, particularly sinusoidally, the circuit is known as an alternating current circuit. Examples include the commercial and residential power that serves so many of our needs. [\[link\]](#) shows graphs of voltage and current versus time for typical DC and AC power. The AC voltages and frequencies commonly used in homes and businesses vary around the world.



(a) DC voltage and current are constant in time, once the current is established. (b) A graph of voltage and current versus time for 60-Hz AC power. The voltage and current are sinusoidal and are in phase for a simple resistance circuit. The frequencies and peak voltages of AC sources differ greatly.



The potential difference V between the terminals of an AC voltage source fluctuates as shown. The mathematical expression for V is given by

$$V = V_0 \sin 2\pi ft.$$

[\[link\]](#) shows a schematic of a simple circuit with an AC voltage source. The voltage between the terminals fluctuates as shown, with the **AC voltage** given by

Equation:

$$V = V_0 \sin 2\pi ft,$$

where V is the voltage at time t , V_0 is the peak voltage, and f is the frequency in hertz. For this simple resistance circuit, $I = V/R$, and so the **AC current** is

Equation:

$$I = I_0 \sin 2\pi ft,$$

where I is the current at time t , and $I_0 = V_0/R$ is the peak current. For this example, the voltage and current are said to be in phase, as seen in [\[link\]](#)(b).

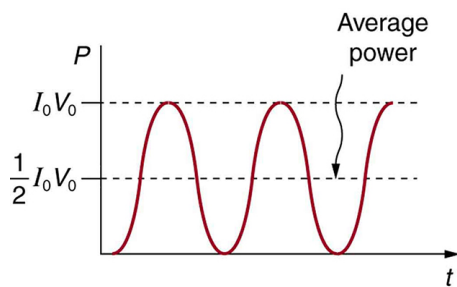
Current in the resistor alternates back and forth just like the driving voltage, since $I = V/R$. If the resistor is a fluorescent light bulb, for example, it brightens and dims 120 times per second as the current repeatedly goes through zero. A 120-Hz flicker is too rapid for your eyes to detect, but if you wave your hand back and forth between your face and a fluorescent light, you will see a stroboscopic effect evidencing AC. The fact that the light output fluctuates means that the power is fluctuating. The power supplied is $P = IV$. Using the expressions for I and V above, we see that the time dependence of power is $P = I_0 V_0 \sin^2 2\pi ft$, as shown in [\[link\]](#).

Note:

Making Connections: Take-Home Experiment—AC/DC Lights

Wave your hand back and forth between your face and a fluorescent light bulb. Do you observe the same thing with the headlights on your car?

Explain what you observe. *Warning: Do not look directly at very bright light.*



AC power as a function of time. Since the voltage and current are in phase here, their

product is non-negative and fluctuates between zero and $I_0 V_0$

. Average power is $(1/2)I_0 V_0$.

We are most often concerned with average power rather than its fluctuations—that 60-W light bulb in your desk lamp has an average power consumption of 60 W, for example. As illustrated in [\[link\]](#), the average power P_{ave} is

Equation:

$$P_{\text{ave}} = \frac{1}{2} I_0 V_0.$$

This is evident from the graph, since the areas above and below the $(1/2)I_0 V_0$ line are equal, but it can also be proven using trigonometric identities. Similarly, we define an average or **rms current** I_{rms} and average or **rms voltage** V_{rms} to be, respectively,

Equation:

$$I_{\text{rms}} = \frac{I_0}{\sqrt{2}}$$

and

Equation:

$$V_{\text{rms}} = \frac{V_0}{\sqrt{2}}.$$

where rms stands for root mean square, a particular kind of average. In general, to obtain a root mean square, the particular quantity is squared, its

mean (or average) is found, and the square root is taken. This is useful for AC, since the average value is zero. Now,

Equation:

$$P_{\text{ave}} = I_{\text{rms}} V_{\text{rms}},$$

which gives

Equation:

$$P_{\text{ave}} = \frac{I_0}{\sqrt{2}} \cdot \frac{V_0}{\sqrt{2}} = \frac{1}{2} I_0 V_0,$$

as stated above. It is standard practice to quote I_{rms} , V_{rms} , and P_{ave} rather than the peak values. For example, most household electricity is 120 V AC, which means that V_{rms} is 120 V. The common 10-A circuit breaker will interrupt a sustained I_{rms} greater than 10 A. Your 1.0-kW microwave oven consumes $P_{\text{ave}} = 1.0$ kW, and so on. You can think of these rms and average values as the equivalent DC values for a simple resistive circuit.

To summarize, when dealing with AC, Ohm's law and the equations for power are completely analogous to those for DC, but rms and average values are used for AC. Thus, for AC, Ohm's law is written

Equation:

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{R}.$$

The various expressions for AC power P_{ave} are

Equation:

$$P_{\text{ave}} = I_{\text{rms}} V_{\text{rms}},$$

Equation:

$$P_{\text{ave}} = \frac{V_{\text{rms}}^2}{R},$$

and

Equation:

$$P_{\text{ave}} = I_{\text{rms}}^2 R.$$

Example:

Peak Voltage and Power for AC

(a) What is the value of the peak voltage for 120-V AC power? (b) What is the peak power consumption rate of a 60.0-W AC light bulb?

Strategy

We are told that V_{rms} is 120 V and P_{ave} is 60.0 W. We can use $V_{\text{rms}} = \frac{V_0}{\sqrt{2}}$ to find the peak voltage, and we can manipulate the definition of power to find the peak power from the given average power.

Solution for (a)

Solving the equation $V_{\text{rms}} = \frac{V_0}{\sqrt{2}}$ for the peak voltage V_0 and substituting the known value for V_{rms} gives

Equation:

$$V_0 = \sqrt{2}V_{\text{rms}} = 1.414(120 \text{ V}) = 170 \text{ V}.$$

Discussion for (a)

This means that the AC voltage swings from 170 V to -170 V and back 60 times every second. An equivalent DC voltage is a constant 120 V.

Solution for (b)

Peak power is peak current times peak voltage. Thus,

Equation:

$$P_0 = I_0 V_0 = 2 \left(\frac{1}{2} I_0 V_0 \right) = 2P_{\text{ave}}.$$

We know the average power is 60.0 W, and so

Equation:

$$P_0 = 2(60.0 \text{ W}) = 120 \text{ W}.$$

Discussion

So the power swings from zero to 120 W one hundred twenty times per second (twice each cycle), and the power averages 60 W.

Why Use AC for Power Distribution?

Most large power-distribution systems are AC. Moreover, the power is transmitted at much higher voltages than the 120-V AC (240 V in most parts of the world) we use in homes and on the job. Economies of scale make it cheaper to build a few very large electric power-generation plants than to build numerous small ones. This necessitates sending power long distances, and it is obviously important that energy losses en route be minimized. High voltages can be transmitted with much smaller power losses than low voltages, as we shall see. (See [\[link\]](#).) For safety reasons, the voltage at the user is reduced to familiar values. The crucial factor is that it is much easier to increase and decrease AC voltages than DC, so AC is used in most large power distribution systems.



Power is distributed over large distances at high voltage to reduce power loss in the transmission lines. The

voltages generated at the power plant are stepped up by passive devices called transformers (see [Transformers](#)) to 330,000 volts (or more in some places worldwide). At the point of use, the transformers reduce the voltage transmitted for safe residential and commercial use.
(Credit: GeorgHH, Wikimedia Commons)

Example:**Power Losses Are Less for High-Voltage Transmission**

(a) What current is needed to transmit 100 MW of power at 200 kV? (b) What is the power dissipated by the transmission lines if they have a resistance of $1.00\ \Omega$? (c) What percentage of the power is lost in the transmission lines?

Strategy

We are given $P_{\text{ave}} = 100\ \text{MW}$, $V_{\text{rms}} = 200\ \text{kV}$, and the resistance of the lines is $R = 1.00\ \Omega$. Using these givens, we can find the current (from $P = IV$) and then the power dissipated in the lines ($P = I^2 R$), and we take the ratio to the total power transmitted.

Solution

To find the current, we rearrange the relationship $P_{\text{ave}} = I_{\text{rms}} V_{\text{rms}}$ and substitute known values. This gives

Equation:

$$I_{\text{rms}} = \frac{P_{\text{ave}}}{V_{\text{rms}}} = \frac{100 \times 10^6 \text{ W}}{200 \times 10^3 \text{ V}} = 500 \text{ A}.$$

Solution

Knowing the current and given the resistance of the lines, the power dissipated in them is found from $P_{\text{ave}} = I_{\text{rms}}^2 R$. Substituting the known values gives

Equation:

$$P_{\text{ave}} = I_{\text{rms}}^2 R = (500 \text{ A})^2 (1.00 \Omega) = 250 \text{ kW}.$$

Solution

The percent loss is the ratio of this lost power to the total or input power, multiplied by 100:

Equation:

$$\% \text{ loss} = \frac{250 \text{ kW}}{100 \text{ MW}} \times 100 = 0.250 \ \%.$$

Discussion

One-fourth of a percent is an acceptable loss. Note that if 100 MW of power had been transmitted at 25 kV, then a current of 4000 A would have been needed. This would result in a power loss in the lines of 16.0 MW, or 16.0% rather than 0.250%. The lower the voltage, the more current is needed, and the greater the power loss in the fixed-resistance transmission lines. Of course, lower-resistance lines can be built, but this requires larger and more expensive wires. If superconducting lines could be economically produced, there would be no loss in the transmission lines at all. But, as we shall see in a later chapter, there is a limit to current in superconductors, too. In short, high voltages are more economical for transmitting power, and AC voltage is much easier to raise and lower, so that AC is used in most large-scale power distribution systems.

It is widely recognized that high voltages pose greater hazards than low voltages. But, in fact, some high voltages, such as those associated with common static electricity, can be harmless. So it is not voltage alone that

determines a hazard. It is not so widely recognized that AC shocks are often more harmful than similar DC shocks. Thomas Edison thought that AC shocks were more harmful and set up a DC power-distribution system in New York City in the late 1800s. There were bitter fights, in particular between Edison and George Westinghouse and Nikola Tesla, who were advocating the use of AC in early power-distribution systems. AC has prevailed largely due to transformers and lower power losses with high-voltage transmission.

Note:

PhET Explorations: Generator

Generate electricity with a bar magnet! Discover the physics behind the phenomena by exploring magnets and how you can use them to make a bulb light.

[Generato](#)

[r](#)

Section Summary

- Direct current (DC) is the flow of electric current in only one direction. It refers to systems where the source voltage is constant.
- The voltage source of an alternating current (AC) system puts out $V = V_0 \sin 2\pi ft$, where V is the voltage at time t , V_0 is the peak voltage, and f is the frequency in hertz.
- In a simple circuit, $I = V/R$ and AC current is $I = I_0 \sin 2\pi ft$, where I is the current at time t , and $I_0 = V_0/R$ is the peak current.
- The average AC power is $P_{\text{ave}} = \frac{1}{2} I_0 V_0$.
- Average (rms) current I_{rms} and average (rms) voltage V_{rms} are $I_{\text{rms}} = \frac{I_0}{\sqrt{2}}$ and $V_{\text{rms}} = \frac{V_0}{\sqrt{2}}$, where rms stands for root mean square.

- Thus, $P_{\text{ave}} = I_{\text{rms}} V_{\text{rms}}$.
- Ohm's law for AC is $I_{\text{rms}} = \frac{V_{\text{rms}}}{R}$.
- Expressions for the average power of an AC circuit are $P_{\text{ave}} = I_{\text{rms}} V_{\text{rms}}$, $P_{\text{ave}} = \frac{V_{\text{rms}}^2}{R}$, and $P_{\text{ave}} = I_{\text{rms}}^2 R$, analogous to the expressions for DC circuits.

Conceptual Questions

Exercise:

Problem:

Give an example of a use of AC power other than in the household. Similarly, give an example of a use of DC power other than that supplied by batteries.

Exercise:

Problem:

Why do voltage, current, and power go through zero 120 times per second for 60-Hz AC electricity?

Exercise:

Problem:

You are riding in a train, gazing into the distance through its window. As close objects streak by, you notice that the nearby fluorescent lights make *dashed* streaks. Explain.

Problem Exercises

Exercise:

Problem:

(a) What is the hot resistance of a 25-W light bulb that runs on 120-V AC? (b) If the bulb's operating temperature is 2700°C , what is its resistance at 2600°C ?

Exercise:**Problem:**

Certain heavy industrial equipment uses AC power that has a peak voltage of 679 V. What is the rms voltage?

Solution:

480 V

Exercise:**Problem:**

A certain circuit breaker trips when the rms current is 15.0 A. What is the corresponding peak current?

Exercise:**Problem:**

Military aircraft use 400-Hz AC power, because it is possible to design lighter-weight equipment at this higher frequency. What is the time for one complete cycle of this power?

Solution:

2.50 ms

Exercise:

Problem:

A North American tourist takes his 25.0-W, 120-V AC razor to Europe, finds a special adapter, and plugs it into 240 V AC. Assuming constant resistance, what power does the razor consume as it is ruined?

Exercise:**Problem:**

In this problem, you will verify statements made at the end of the power losses for [\[link\]](#). (a) What current is needed to transmit 100 MW of power at a voltage of 25.0 kV? (b) Find the power loss in a 1.00 - Ω transmission line. (c) What percent loss does this represent?

Solution:

- (a) 4.00 kA
- (b) 16.0 MW
- (c) 16.0%

Exercise:**Problem:**

A small office-building air conditioner operates on 408-V AC and consumes 50.0 kW. (a) What is its effective resistance? (b) What is the cost of running the air conditioner during a hot summer month when it is on 8.00 h per day for 30 days and electricity costs 9.00 cents/kW · h?

Exercise:**Problem:**

What is the peak power consumption of a 120-V AC microwave oven that draws 10.0 A?

Solution:

2.40 kW

Exercise:

Problem:

What is the peak current through a 500-W room heater that operates on 120-V AC power?

Exercise:

Problem:

Two different electrical devices have the same power consumption, but one is meant to be operated on 120-V AC and the other on 240-V AC. (a) What is the ratio of their resistances? (b) What is the ratio of their currents? (c) Assuming its resistance is unaffected, by what factor will the power increase if a 120-V AC device is connected to 240-V AC?

Solution:

(a) 4.0

(b) 0.50

(c) 4.0

Exercise:

Problem:

Nichrome wire is used in some radiative heaters. (a) Find the resistance needed if the average power output is to be 1.00 kW utilizing 120-V AC. (b) What length of Nichrome wire, having a cross-sectional area of 5.00mm^2 , is needed if the operating temperature is 500°C ? (c) What power will it draw when first switched on?

Exercise:

Problem:

Find the time after $t = 0$ when the instantaneous voltage of 60-Hz AC first reaches the following values: (a) $V_0/2$ (b) V_0 (c) 0.

Solution:

(a) 1.39 ms

(b) 4.17 ms

(c) 8.33 ms

Exercise:**Problem:**

(a) At what two times in the first period following $t = 0$ does the instantaneous voltage in 60-Hz AC equal V_{rms} ? (b) $-V_{\text{rms}}$?

Glossary

direct current

(DC) the flow of electric charge in only one direction

alternating current

(AC) the flow of electric charge that periodically reverses direction

AC voltage

voltage that fluctuates sinusoidally with time, expressed as $V = V_0 \sin 2\pi ft$, where V is the voltage at time t , V_0 is the peak voltage, and f is the frequency in hertz

AC current

current that fluctuates sinusoidally with time, expressed as $I = I_0 \sin 2\pi ft$, where I is the current at time t , I_0 is the peak current, and f is the frequency in hertz

rms current

the root mean square of the current, $I_{\text{rms}} = I_0/\sqrt{2}$, where I_0 is the peak current, in an AC system

rms voltage

the root mean square of the voltage, $V_{\text{rms}} = V_0/\sqrt{2}$, where V_0 is the peak voltage, in an AC system

LEARNING OBJECTIVES

By the end of this section, you will be able to:

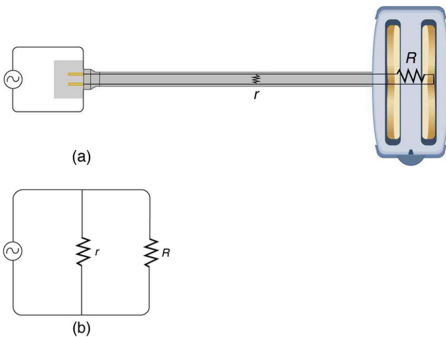
- Define thermal hazard, shock hazard, and short circuit.
- Explain what effects various levels of current have on the human body.

There are two known hazards of electricity—thermal and shock. A **thermal hazard** is one where excessive electric power causes undesired thermal effects, such as starting a fire in the wall of a house. A **shock hazard** occurs when electric current passes through a person. Shocks range in severity from painful, but otherwise harmless, to heart-stopping lethality. This section considers these hazards and the various factors affecting them in a quantitative manner. [Electrical Safety: Systems and Devices](#) will consider systems and devices for preventing electrical hazards.

Thermal Hazards

Electric power causes undesired heating effects whenever electric energy is converted to thermal energy at a rate faster than it can be safely dissipated. A classic example of this is the **short circuit**, a low-resistance path between terminals of a voltage source. An example of a short circuit is shown in [\[link\]](#). Insulation on wires leading to an appliance has worn through, allowing the two wires to come into contact. Such an undesired contact with a high voltage is called a *short*. Since the resistance of the short, r , is very small, the power dissipated in the short, $P = V^2/r$, is very large. For

example, if V is 120 V and r is $0.100\ \Omega$, then the power is 144 kW, *much* greater than that used by a typical household appliance. Thermal energy delivered at this rate will very quickly raise the temperature of surrounding materials, melting or perhaps igniting them.

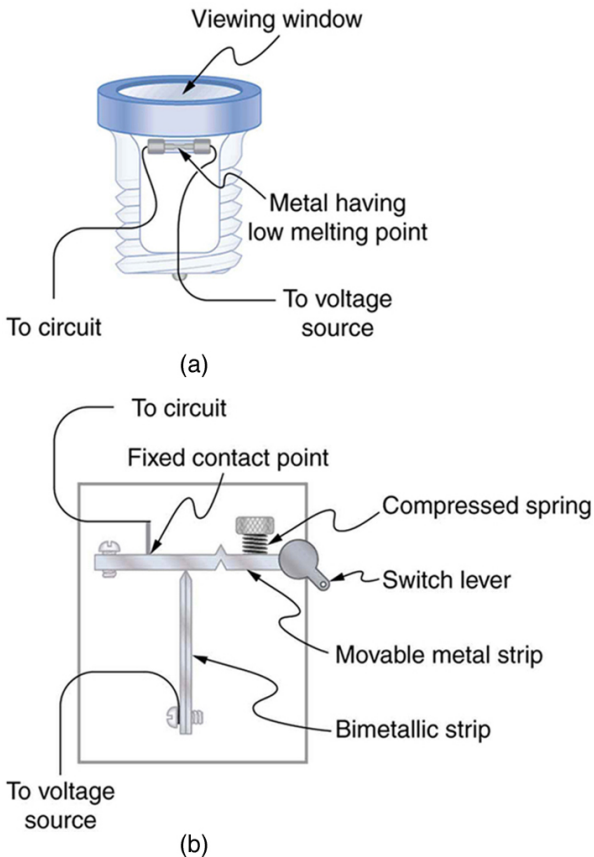


A short circuit is an undesired low-resistance path across a voltage source. (a) Worn insulation on the wires of a toaster allow them to come into contact with a low resistance r . Since $P = V^2/r$, thermal power is created so rapidly that the cord melts or burns. (b) A schematic of the short circuit.

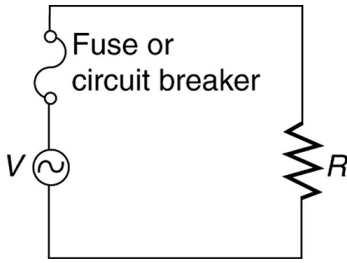
One particularly insidious aspect of a short circuit is that its resistance may actually be decreased due to the increase in temperature. This can happen if the short creates ionization. These charged atoms and molecules are free to move and, thus, lower the resistance r . Since $P = V^2/r$, the power

dissipated in the short rises, possibly causing more ionization, more power, and so on. High voltages, such as the 480-V AC used in some industrial applications, lend themselves to this hazard, because higher voltages create higher initial power production in a short.

Another serious, but less dramatic, thermal hazard occurs when wires supplying power to a user are overloaded with too great a current. As discussed in the previous section, the power dissipated in the supply wires is $P = I^2 R_w$, where R_w is the resistance of the wires and I the current flowing through them. If either I or R_w is too large, the wires overheat. For example, a worn appliance cord (with some of its braided wires broken) may have $R_w = 2.00 \, \Omega$ rather than the $0.100 \, \Omega$ it should be. If 10.0 A of current passes through the cord, then $P = I^2 R_w = 200 \, \text{W}$ is dissipated in the cord—much more than is safe. Similarly, if a wire with a $0.100 \, \Omega$ resistance is meant to carry a few amps, but is instead carrying 100 A, it will severely overheat. The power dissipated in the wire will in that case be $P = 1000 \, \text{W}$. Fuses and circuit breakers are used to limit excessive currents. (See [\[link\]](#) and [\[link\]](#).) Each device opens the circuit automatically when a sustained current exceeds safe limits.



(a) A fuse has a metal strip with a low melting point that, when overheated by an excessive current, permanently breaks the connection of a circuit to a voltage source. (b) A circuit breaker is an automatic but restorable electric switch. The one shown here has a bimetallic strip that bends to the right and into the notch if overheated. The spring then forces the metal strip downward, breaking the electrical connection at the points.



Schematic of a circuit with a fuse or circuit breaker in it.

Fuses and circuit breakers act like automatic switches that open when sustained current exceeds desired limits.

Fuses and circuit breakers for typical household voltages and currents are relatively simple to produce, but those for large voltages and currents experience special problems. For example, when a circuit breaker tries to interrupt the flow of high-voltage electricity, a spark can jump across its points that ionizes the air in the gap and allows the current to continue. Large circuit breakers found in power-distribution systems employ insulating gas and even use jets of gas to blow out such sparks. Here AC is safer than DC, since AC current goes through zero 120 times per second, giving a quick opportunity to extinguish these arcs.

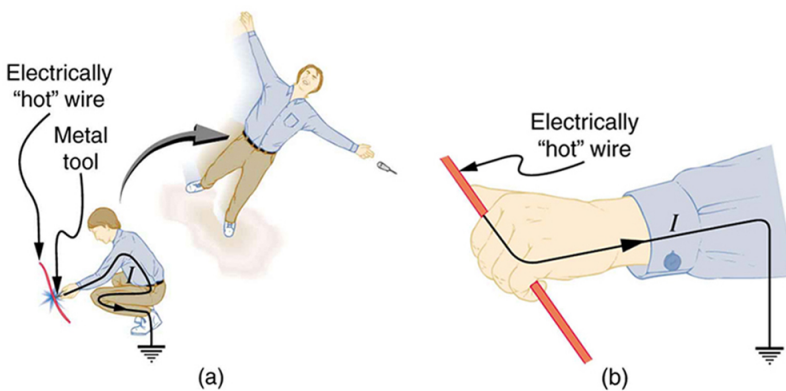
Shock Hazards

Electrical currents through people produce tremendously varied effects. An electrical current can be used to block back pain. The possibility of using

electrical current to stimulate muscle action in paralyzed limbs, perhaps allowing paraplegics to walk, is under study. TV dramatizations in which electrical shocks are used to bring a heart attack victim out of ventricular fibrillation (a massively irregular, often fatal, beating of the heart) are more than common. Yet most electrical shock fatalities occur because a current put the heart into fibrillation. A pacemaker uses electrical shocks to stimulate the heart to beat properly. Some fatal shocks do not produce burns, but warts can be safely burned off with electric current (though freezing using liquid nitrogen is now more common). Of course, there are consistent explanations for these disparate effects. The major factors upon which the effects of electrical shock depend are

1. The amount of current I
2. The path taken by the current
3. The duration of the shock
4. The frequency f of the current ($f = 0$ for DC)

[\[link\]](#) gives the effects of electrical shocks as a function of current for a typical accidental shock. The effects are for a shock that passes through the trunk of the body, has a duration of 1 s, and is caused by 60-Hz power.



An electric current can cause muscular contractions with varying effects. (a) The victim is “thrown” backward by involuntary muscle contractions that extend the legs and torso. (b) The victim can't let go of the wire

that is stimulating all the muscles in the hand. Those that close the fingers are stronger than those that open them.

Current (mA)	Effect
1	Threshold of sensation
5	Maximum harmless current
10–20	Onset of sustained muscular contraction; cannot let go for duration of shock; contraction of chest muscles may stop breathing during shock
50	Onset of pain
100–300+	Ventricular fibrillation possible; often fatal
300	Onset of burns depending on concentration of current
6000 (6 A)	Onset of sustained ventricular contraction and respiratory paralysis; both cease when shock ends; heartbeat may return to normal; used to defibrillate the heart

Effects of Electrical Shock as a Function of Current^{[footnote](#)}

For an average male shocked through trunk of body for 1 s by 60-Hz AC. Values for females are 60–80% of those listed.

Our bodies are relatively good conductors due to the water in our bodies. Given that larger currents will flow through sections with lower resistance (to be further discussed in the next chapter), electric currents preferentially flow through paths in the human body that have a minimum resistance in a direct path to Earth. The Earth is a natural electron sink. Wearing insulating shoes, a requirement in many professions, prohibits a pathway for electrons by providing a large resistance in that path. Whenever working with high-power tools (drills), or in risky situations, ensure that you do not provide a pathway for current flow (especially through the heart).

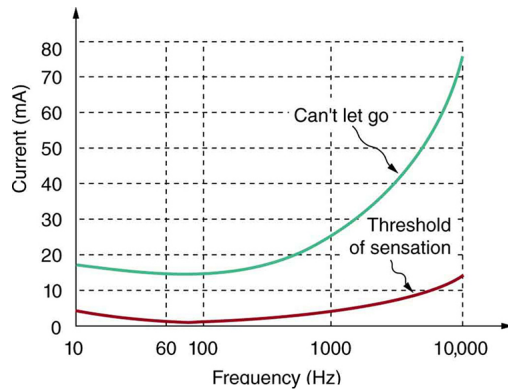
Very small currents pass harmlessly and unfelt through the body. This happens to you regularly without your knowledge. The threshold of sensation is only 1 mA and, although unpleasant, shocks are apparently harmless for currents less than 5 mA. A great number of safety rules take the 5-mA value for the maximum allowed shock. At 10 to 20 mA and above, the current can stimulate sustained muscular contractions much as regular nerve impulses do. People sometimes say they were knocked across the room by a shock, but what really happened was that certain muscles contracted, propelling them in a manner not of their own choosing. (See [\[link\]](#)(a).) More frightening, and potentially more dangerous, is the “can't let go” effect illustrated in [\[link\]](#)(b). The muscles that close the fingers are stronger than those that open them, so the hand closes involuntarily on the wire shocking it. This can prolong the shock indefinitely. It can also be a danger to a person trying to rescue the victim, because the rescuer's hand may close about the victim's wrist. Usually the best way to help the victim is to give the fist a hard knock/blow/jar with an insulator or to throw an insulator at the fist. Modern electric fences, used in animal enclosures, are now pulsed on and off to allow people who touch them to get free, rendering them less lethal than in the past.

Greater currents may affect the heart. Its electrical patterns can be disrupted, so that it beats irregularly and ineffectively in a condition called “ventricular fibrillation.” This condition often lingers after the shock and is fatal due to a lack of blood circulation. The threshold for ventricular fibrillation is between 100 and 300 mA. At about 300 mA and above, the shock can cause burns, depending on the concentration of current—the more concentrated, the greater the likelihood of burns.

Very large currents cause the heart and diaphragm to contract for the duration of the shock. Both the heart and breathing stop. Interestingly, both often return to normal following the shock. The electrical patterns on the heart are completely erased in a manner that the heart can start afresh with normal beating, as opposed to the permanent disruption caused by smaller currents that can put the heart into ventricular fibrillation. The latter is something like scribbling on a blackboard, whereas the former completely erases it. TV dramatizations of electric shock used to bring a heart attack victim out of ventricular fibrillation also show large paddles. These are used to spread out current passed through the victim to reduce the likelihood of burns.

Current is the major factor determining shock severity (given that other conditions such as path, duration, and frequency are fixed, such as in the table and preceding discussion). A larger voltage is more hazardous, but since $I = V/R$, the severity of the shock depends on the combination of voltage and resistance. For example, a person with dry skin has a resistance of about 200 k Ω . If he comes into contact with 120-V AC, a current $I = (120 \text{ V})/(200 \text{ k}\Omega) = 0.6 \text{ mA}$ passes harmlessly through him. The same person soaking wet may have a resistance of 10.0 k Ω and the same 120 V will produce a current of 12 mA—above the “can't let go” threshold and potentially dangerous.

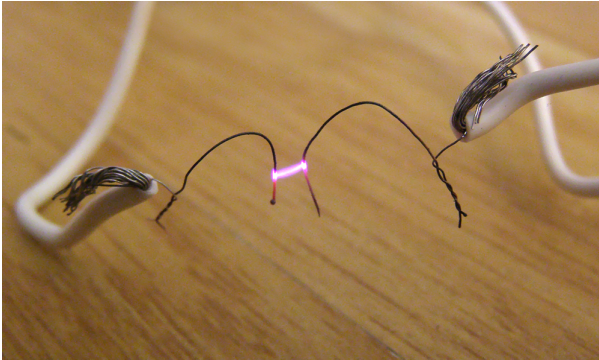
Most of the body's resistance is in its dry skin. When wet, salts go into ion form, lowering the resistance significantly. The interior of the body has a much lower resistance than dry skin because of all the ionic solutions and fluids it contains. If skin resistance is bypassed, such as by an intravenous infusion, a catheter, or exposed pacemaker leads, a person is rendered **microshock sensitive**. In this condition, currents about 1/1000 those listed in [\[link\]](#) produce similar effects. During open-heart surgery, currents as small as 20 μA can be used to still the heart. Stringent electrical safety requirements in hospitals, particularly in surgery and intensive care, are related to the doubly disadvantaged microshock-sensitive patient. The break in the skin has reduced his resistance, and so the same voltage causes a greater current, and a much smaller current has a greater effect.



Graph of average values for the threshold of sensation and the “can't let go” current as a function of frequency. The lower the value, the more sensitive the body is at that frequency.

Factors other than current that affect the severity of a shock are its path, duration, and AC frequency. Path has obvious consequences. For example, the heart is unaffected by an electric shock through the brain, such as may be used to treat manic depression. And it is a general truth that the longer the duration of a shock, the greater its effects. [\[link\]](#) presents a graph that illustrates the effects of frequency on a shock. The curves show the minimum current for two different effects, as a function of frequency. The lower the current needed, the more sensitive the body is at that frequency. Ironically, the body is most sensitive to frequencies near the 50- or 60-Hz frequencies in common use. The body is slightly less sensitive for DC ($f = 0$), mildly confirming Edison's claims that AC presents a greater hazard. At higher and higher frequencies, the body becomes progressively less sensitive to any effects that involve nerves. This is related to the maximum rates at which nerves can fire or be stimulated. At very high frequencies, electrical current travels only on the surface of a person. Thus a wart can be burned off with very high frequency current without causing the heart to stop. (Do not try this at home with 60-Hz AC!) Some of the

spectacular demonstrations of electricity, in which high-voltage arcs are passed through the air and over people's bodies, employ high frequencies and low currents. (See [\[link\]](#).) Electrical safety devices and techniques are discussed in detail in [Electrical Safety: Systems and Devices](#).



Is this electric arc dangerous?
The answer depends on the AC
frequency and the power
involved. (credit: Khimich
Alex, Wikimedia Commons)

Section Summary

- The two types of electric hazards are thermal (excessive power) and shock (current through a person).
- Shock severity is determined by current, path, duration, and AC frequency.
- [\[link\]](#) lists shock hazards as a function of current.
- [\[link\]](#) graphs the threshold current for two hazards as a function of frequency.

Conceptual Questions

Exercise:

Problem:

Using an ohmmeter, a student measures the resistance between various points on his body. He finds that the resistance between two points on the same finger is about the same as the resistance between two points on opposite hands—both are several hundred thousand ohms.

Furthermore, the resistance decreases when more skin is brought into contact with the probes of the ohmmeter. Finally, there is a dramatic drop in resistance (to a few thousand ohms) when the skin is wet.

Explain these observations and their implications regarding skin and internal resistance of the human body.

Exercise:

Problem: What are the two major hazards of electricity?

Exercise:

Problem: Why isn't a short circuit a shock hazard?

Exercise:**Problem:**

What determines the severity of a shock? Can you say that a certain voltage is hazardous without further information?

Exercise:**Problem:**

An electrified needle is used to burn off warts, with the circuit being completed by having the patient sit on a large butt plate. Why is this plate large?

Exercise:

Problem:

Some surgery is performed with high-voltage electricity passing from a metal scalpel through the tissue being cut. Considering the nature of electric fields at the surface of conductors, why would you expect most of the current to flow from the sharp edge of the scalpel? Do you think high- or low-frequency AC is used?

Exercise:**Problem:**

Some devices often used in bathrooms, such as hairdryers, often have safety messages saying “Do not use when the bathtub or basin is full of water.” Why is this so?

Exercise:**Problem:**

We are often advised to not flick electric switches with wet hands, dry your hand first. We are also advised to never throw water on an electric fire. Why is this so?

Exercise:**Problem:**

Before working on a power transmission line, linemen will touch the line with the back of the hand as a final check that the voltage is zero. Why the back of the hand?

Exercise:**Problem:**

Why is the resistance of wet skin so much smaller than dry, and why do blood and other bodily fluids have low resistances?

Exercise:

Problem:

Could a person on intravenous infusion (an IV) be microshock sensitive?

Exercise:**Problem:**

In view of the small currents that cause shock hazards and the larger currents that circuit breakers and fuses interrupt, how do they play a role in preventing shock hazards?

Problem Exercises**Exercise:****Problem:**

(a) How much power is dissipated in a short circuit of 240-V AC through a resistance of $0.250\ \Omega$? (b) What current flows?

Solution:

(a) 230 kW

(b) 960 A

Exercise:**Problem:**

What voltage is involved in a 1.44-kW short circuit through a $0.100\text{-}\Omega$ resistance?

Exercise:

Problem:

Find the current through a person and identify the likely effect on her if she touches a 120-V AC source: (a) if she is standing on a rubber mat and offers a total resistance of $300\text{ k}\Omega$; (b) if she is standing barefoot on wet grass and has a resistance of only $4000\text{ k}\Omega$.

Solution:

(a) 0.400 mA , no effect

(b) 26.7 mA , muscular contraction for duration of the shock (can't let go)

Exercise:**Problem:**

While taking a bath, a person touches the metal case of a radio. The path through the person to the drainpipe and ground has a resistance of $4000\text{ }\Omega$. What is the smallest voltage on the case of the radio that could cause ventricular fibrillation?

Exercise:**Problem:**

Foolishly trying to fish a burning piece of bread from a toaster with a metal butter knife, a man comes into contact with 120-V AC. He does not even feel it since, luckily, he is wearing rubber-soled shoes. What is the minimum resistance of the path the current follows through the person?

Solution:

$1.20 \times 10^5\text{ }\Omega$

Exercise:

Problem:

(a) During surgery, a current as small as $20.0\ \mu\text{A}$ applied directly to the heart may cause ventricular fibrillation. If the resistance of the exposed heart is $300\ \Omega$, what is the smallest voltage that poses this danger? (b) Does your answer imply that special electrical safety precautions are needed?

Exercise:**Problem:**

(a) What is the resistance of a 220-V AC short circuit that generates a peak power of 96.8 kW? (b) What would the average power be if the voltage was 120 V AC?

Solution:

(a) $1.00\ \Omega$

(b) 14.4 kW

Exercise:**Problem:**

A heart defibrillator passes 10.0 A through a patient's torso for 5.00 ms in an attempt to restore normal beating. (a) How much charge passed? (b) What voltage was applied if 500 J of energy was dissipated? (c) What was the path's resistance? (d) Find the temperature increase caused in the 8.00 kg of affected tissue.

Exercise:**Problem: Integrated Concepts**

A short circuit in a 120-V appliance cord has a $0.500\text{-}\Omega$ resistance. Calculate the temperature rise of the 2.00 g of surrounding materials, assuming their specific heat capacity is $0.200\ \text{cal/g}\cdot^\circ\text{C}$ and that it

takes 0.0500 s for a circuit breaker to interrupt the current. Is this likely to be damaging?

Solution:

Temperature increases 860° C. It is very likely to be damaging.

Exercise:

Problem: Construct Your Own Problem

Consider a person working in an environment where electric currents might pass through her body. Construct a problem in which you calculate the resistance of insulation needed to protect the person from harm. Among the things to be considered are the voltage to which the person might be exposed, likely body resistance (dry, wet, ...), and acceptable currents (safe but sensed, safe and unfelt, ...).

Glossary

thermal hazard

a hazard in which electric current causes undesired thermal effects

shock hazard

when electric current passes through a person

short circuit

also known as a “short,” a low-resistance path between terminals of a voltage source

microshock sensitive

a condition in which a person's skin resistance is bypassed, possibly by a medical procedure, rendering the person vulnerable to electrical shock at currents about 1/1000 the normally required level

Nerve Conduction–Electrocardiograms

LEARNING OBJECTIVES

By the end of this section, you will be able to:

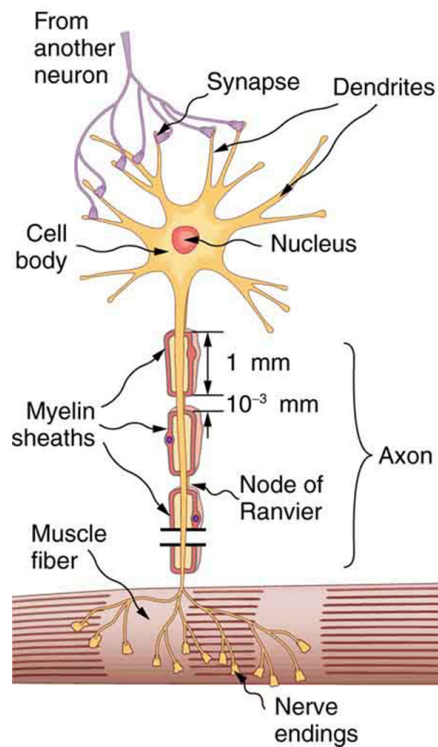
- Explain the process by which electric signals are transmitted along a neuron.
- Explain the effects myelin sheaths have on signal propagation.
- Explain what the features of an ECG signal indicate.

Nerve Conduction

Electric currents in the vastly complex system of billions of nerves in our body allow us to sense the world, control parts of our body, and think. These are representative of the three major functions of nerves. First, nerves carry messages from our sensory organs and others to the central nervous system, consisting of the brain and spinal cord. Second, nerves carry messages from the central nervous system to muscles and other organs. Third, nerves transmit and process signals within the central nervous system. The sheer number of nerve cells and the incredibly greater number of connections between them makes this system the subtle wonder that it is. **Nerve conduction** is a general term for electrical signals carried by nerve cells. It is one aspect of **bioelectricity**, or electrical effects in and created by biological systems.

Nerve cells, properly called *neurons*, look different from other cells—they have tendrils, some of them many centimeters long, connecting them with other cells. (See [\[link\]](#).) Signals arrive at the cell body across *synapses* or

through *dendrites*, stimulating the neuron to generate its own signal, sent along its long *axon* to other nerve or muscle cells. Signals may arrive from many other locations and be transmitted to yet others, conditioning the synapses by use, giving the system its complexity and its ability to learn.

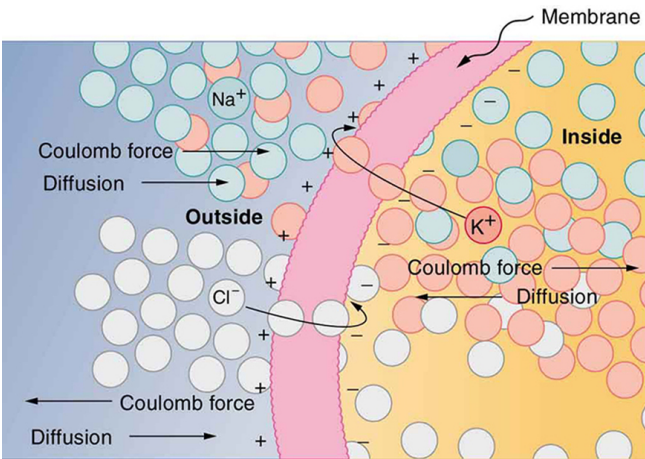


A neuron with its dendrites and long axon. Signals in the form of electric currents reach the cell body through dendrites and across synapses, stimulating the neuron to generate its own signal sent down the axon. The number of interconnections can

be far greater than
shown here.

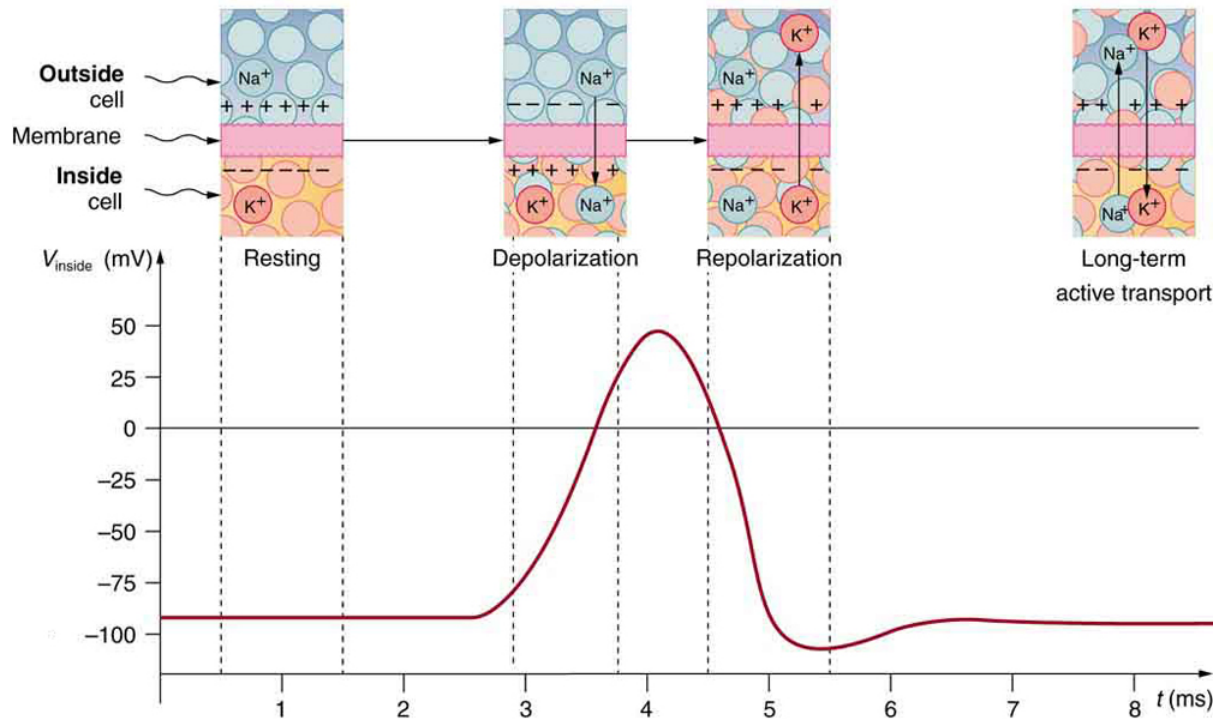
The method by which these electric currents are generated and transmitted is more complex than the simple movement of free charges in a conductor, but it can be understood with principles already discussed in this text. The most important of these are the Coulomb force and diffusion.

[\[link\]](#) illustrates how a voltage (potential difference) is created across the cell membrane of a neuron in its resting state. This thin membrane separates electrically neutral fluids having differing concentrations of ions, the most important varieties being Na^+ , K^+ , and Cl^- (these are sodium, potassium, and chlorine ions with single plus or minus charges as indicated). As discussed in [Molecular Transport Phenomena: Diffusion, Osmosis, and Related Processes](#), free ions will diffuse from a region of high concentration to one of low concentration. But the cell membrane is **semipermeable**, meaning that some ions may cross it while others cannot. In its resting state, the cell membrane is permeable to K^+ and Cl^- , and impermeable to Na^+ . Diffusion of K^+ and Cl^- thus creates the layers of positive and negative charge on the outside and inside of the membrane. The Coulomb force prevents the ions from diffusing across in their entirety. Once the charge layer has built up, the repulsion of like charges prevents more from moving across, and the attraction of unlike charges prevents more from leaving either side. The result is two layers of charge right on the membrane, with diffusion being balanced by the Coulomb force. A tiny fraction of the charges move across and the fluids remain neutral (other ions are present), while a separation of charge and a voltage have been created across the membrane.



The semipermeable membrane of a cell has different concentrations of ions inside and out. Diffusion moves the K^+ and Cl^- ions in the direction shown, until the Coulomb force halts further transfer. This results in a layer of positive charge on the outside, a layer of negative charge on the inside, and thus a voltage across the cell membrane.

The membrane is normally impermeable to Na^+ .

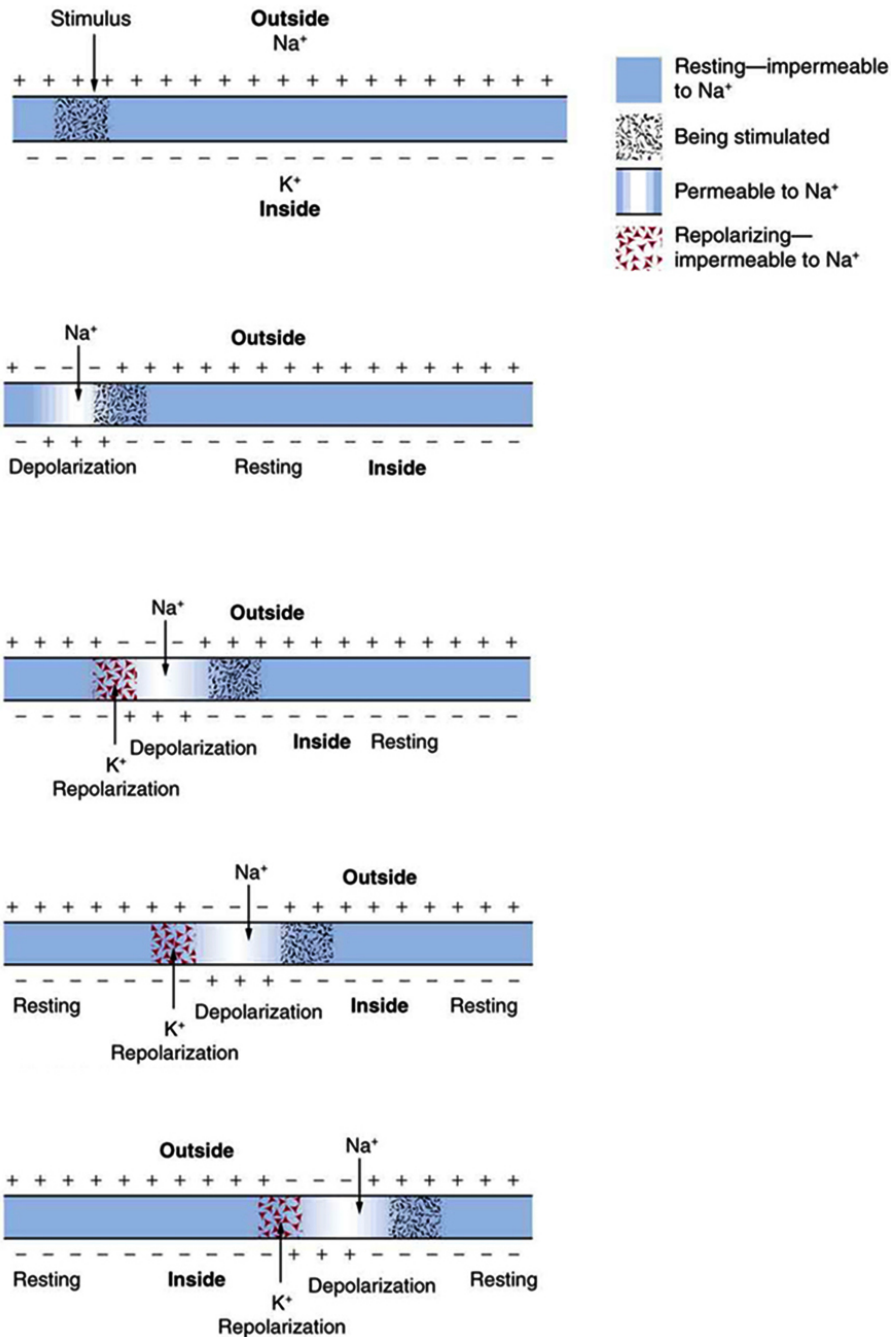


An action potential is the pulse of voltage inside a nerve cell graphed here. It is caused by movements of ions across the cell membrane as shown. Depolarization occurs when a stimulus makes the membrane permeable to Na^+ ions. Repolarization follows as the membrane again becomes impermeable to Na^+ , and K^+ moves from high to low concentration. In the long term, active transport slowly maintains the concentration differences, but the cell may fire hundreds of times in rapid succession without seriously depleting them.

The separation of charge creates a potential difference of 70 to 90 mV across the cell membrane. While this is a small voltage, the resulting electric field ($E = V/d$) across the only 8-nm-thick membrane is immense (on the order of 11 MV/m!) and has fundamental effects on its structure and permeability. Now, if the exterior of a neuron is taken to be at 0 V, then the interior has a *resting potential* of about -90 mV. Such voltages are created across the membranes of almost all types of animal cells but are largest in nerve and muscle cells. In fact, fully 25% of the energy used by cells goes toward creating and maintaining these potentials.

Electric currents along the cell membrane are created by any stimulus that changes the membrane's permeability. The membrane thus temporarily becomes permeable to Na^+ , which then rushes in, driven both by diffusion and the Coulomb force. This inrush of Na^+ first neutralizes the inside membrane, or *depolarizes* it, and then makes it slightly positive. The depolarization causes the membrane to again become impermeable to Na^+ , and the movement of K^+ quickly returns the cell to its resting potential, or *repolarizes* it. This sequence of events results in a voltage pulse, called the *action potential*. (See [\[link\]](#).) Only small fractions of the ions move, so that the cell can fire many hundreds of times without depleting the excess concentrations of Na^+ and K^+ . Eventually, the cell must replenish these ions to maintain the concentration differences that create bioelectricity. This sodium-potassium pump is an example of *active transport*, wherein cell energy is used to move ions across membranes against diffusion gradients and the Coulomb force.

The action potential is a voltage pulse at one location on a cell membrane. How does it get transmitted along the cell membrane, and in particular down an axon, as a nerve impulse? The answer is that the changing voltage and electric fields affect the permeability of the adjacent cell membrane, so that the same process takes place there. The adjacent membrane depolarizes, affecting the membrane further down, and so on, as illustrated in [\[link\]](#). Thus the action potential stimulated at one location triggers a *nerve impulse* that moves slowly (about 1 m/s) along the cell membrane.



A nerve impulse is the propagation of an action potential along a cell membrane. A stimulus causes an action potential at one location, which changes the permeability of the adjacent membrane, causing an action potential there. This in turn affects the membrane further down, so that

the action potential moves slowly (in electrical terms) along the cell membrane. Although the impulse is due to Na^+ and K^+ going across the membrane, it is equivalent to a wave of charge moving along the outside and inside of the membrane.

Some axons, like that in [\[link\]](#), are sheathed with *myelin*, consisting of fat-containing cells. [\[link\]](#) shows an enlarged view of an axon having myelin sheaths characteristically separated by unmyelinated gaps (called nodes of Ranvier). This arrangement gives the axon a number of interesting properties. Since myelin is an insulator, it prevents signals from jumping between adjacent nerves (cross talk). Additionally, the myelinated regions transmit electrical signals at a very high speed, as an ordinary conductor or resistor would. There is no action potential in the myelinated regions, so that no cell energy is used in them. There is an IR signal loss in the myelin, but the signal is regenerated in the gaps, where the voltage pulse triggers the action potential at full voltage. So a myelinated axon transmits a nerve impulse faster, with less energy consumption, and is better protected from cross talk than an unmyelinated one. Not all axons are myelinated, so that cross talk and slow signal transmission are a characteristic of the normal operation of these axons, another variable in the nervous system.

The degeneration or destruction of the myelin sheaths that surround the nerve fibers impairs signal transmission and can lead to numerous neurological effects. One of the most prominent of these diseases comes from the body's own immune system attacking the myelin in the central nervous system—multiple sclerosis. MS symptoms include fatigue, vision problems, weakness of arms and legs, loss of balance, and tingling or numbness in one's extremities (neuropathy). It is more apt to strike younger adults, especially females. Causes might come from infection, environmental or geographic affects, or genetics. At the moment there is no known cure for MS.

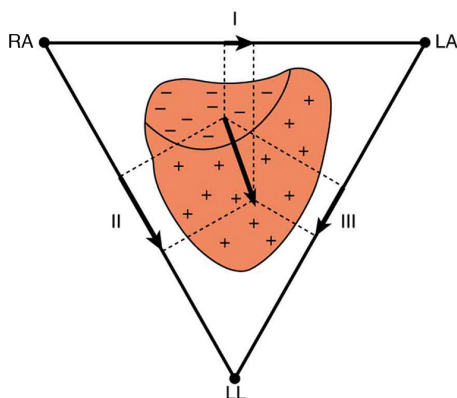




An electric eel flexes its muscles to create a voltage that stuns prey.
(credit: chrisbb, Flickr)

Electrocardiograms

Just as nerve impulses are transmitted by depolarization and repolarization of adjacent membrane, the depolarization that causes muscle contraction can also stimulate adjacent muscle cells to depolarize (fire) and contract. Thus, a depolarization wave can be sent across the heart, coordinating its rhythmic contractions and enabling it to perform its vital function of propelling blood through the circulatory system. [\[link\]](#) is a simplified graphic of a depolarization wave spreading across the heart from the *sinoarterial (SA) node*, the heart's natural pacemaker.



The outer surface of the heart changes from positive to negative during depolarization.

This wave of depolarization is spreading from the top of the heart and is represented by a vector pointing in the direction of the wave.

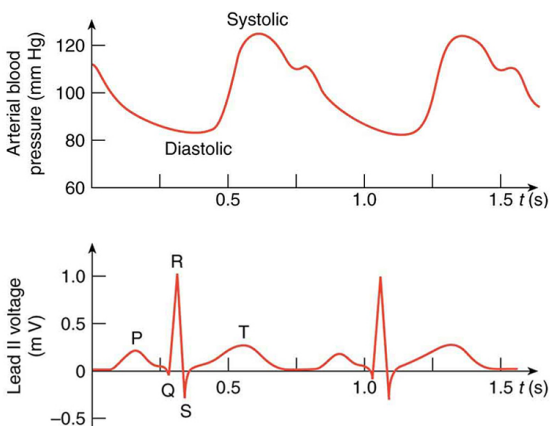
This vector is a voltage (potential difference) vector.

Three electrodes, labeled RA, LA, and LL, are placed on the patient. Each pair (called leads I, II, and III) measures a component of the depolarization vector and is graphed in an ECG.

An **electrocardiogram (ECG)** is a record of the voltages created by the wave of depolarization and subsequent repolarization in the heart. Voltages between pairs of electrodes placed on the chest are vector components of the voltage wave on the heart. Standard ECGs have 12 or more electrodes, but only three are shown in [\[link\]](#) for clarity. Decades ago, three-electrode ECGs were performed by placing electrodes on the left and right arms and the left leg. The voltage between the right arm and the left leg is called the *lead II potential* and is the most often graphed. We shall examine the lead II potential as an indicator of heart-muscle function and see that it is coordinated with arterial blood pressure as well.

Heart function and its four-chamber action are explored in [Viscosity and Laminar Flow; Poiseuille's Law](#). Basically, the right and left atria receive blood from the body and lungs, respectively, and pump the blood into the ventricles. The right and left ventricles, in turn, pump blood through the lungs and the rest of the body, respectively. Depolarization of the heart muscle causes it to contract. After contraction it is repolarized to ready it for the next beat. The ECG measures components of depolarization and repolarization of the heart muscle and can yield significant information on the functioning and malfunctioning of the heart.

[\[link\]](#) shows an ECG of the lead II potential and a graph of the corresponding arterial blood pressure. The major features are labeled P, Q, R, S, and T. The *P wave* is generated by the depolarization and contraction of the atria as they pump blood into the ventricles. The *QRS complex* is created by the depolarization of the ventricles as they pump blood to the lungs and body. Since the shape of the heart and the path of the depolarization wave are not simple, the QRS complex has this typical shape and time span. The lead II QRS signal also masks the repolarization of the atria, which occur at the same time. Finally, the *T wave* is generated by the repolarization of the ventricles and is followed by the next P wave in the next heartbeat. Arterial blood pressure varies with each part of the heartbeat, with systolic (maximum) pressure occurring closely after the QRS complex, which signals contraction of the ventricles.



A lead II ECG with

corresponding arterial blood pressure. The QRS complex is created by the depolarization and contraction of the ventricles and is followed shortly by the maximum or systolic blood pressure. See text for further description.

Taken together, the 12 leads of a state-of-the-art ECG can yield a wealth of information about the heart. For example, regions of damaged heart tissue, called infarcts, reflect electrical waves and are apparent in one or more lead potentials. Subtle changes due to slight or gradual damage to the heart are most readily detected by comparing a recent ECG to an older one. This is particularly the case since individual heart shape, size, and orientation can cause variations in ECGs from one individual to another. ECG technology has advanced to the point where a portable ECG monitor with a liquid crystal instant display and a printer can be carried to patients' homes or used in emergency vehicles. See [\[link\]](#).



This NASA scientist and NEEMO 5 aquanaut's heart rate and other vital signs

are being recorded by
a portable device
while living in an
underwater habitat.
(credit: NASA, Life
Sciences Data Archive
at Johnson Space
Center, Houston,
Texas)

Note:

PhET Explorations: Neuron

[Neuro](#)
[n](#)

Stimulate a neuron and monitor what happens. Pause, rewind, and move forward in time in order to observe the ions as they move across the neuron membrane.

Section Summary

- Electric potentials in neurons and other cells are created by ionic concentration differences across semipermeable membranes.
- Stimuli change the permeability and create action potentials that propagate along neurons.
- Myelin sheaths speed this process and reduce the needed energy input.
- This process in the heart can be measured with an electrocardiogram (ECG).

Conceptual Questions

Exercise:

Problem:

Note that in [\[link\]](#), both the concentration gradient and the Coulomb force tend to move Na^+ ions into the cell. What prevents this?

Exercise:

Problem:

Define depolarization, repolarization, and the action potential.

Exercise:

Problem:

Explain the properties of myelinated nerves in terms of the insulating properties of myelin.

Problems & Exercises

Exercise:

Problem: Integrated Concepts

Use the ECG in [\[link\]](#) to determine the heart rate in beats per minute assuming a constant time between beats.

Solution:

80 beats/minute

Exercise:

Problem: Integrated Concepts

(a) Referring to [\[link\]](#), find the time systolic pressure lags behind the middle of the QRS complex. (b) Discuss the reasons for the time lag.

Glossary

nerve conduction

the transport of electrical signals by nerve cells

bioelectricity

electrical effects in and created by biological systems

semipermeable

property of a membrane that allows only certain types of ions to cross it

electrocardiogram (ECG)

usually abbreviated ECG, a record of voltages created by depolarization and repolarization, especially in the heart

Connection for AP® Courses
class="introduction"

Electric
circuits in
a
computer
allow
large
amounts
of data to
be
quickly
and
accuratel
y
analyzed..
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Airman
1st Class
Mike
Meares,
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Force)



Electric circuits are commonplace in our everyday lives. Some circuits are simple, such as those in flashlights while others are extremely complex, such as those used in supercomputers. This chapter takes the topic of electric circuits a step beyond simple circuits by addressing both changes that result from interactions between systems (Big Idea 4) and constraints on such changes due to laws of conservation (Big Idea 5). When the circuit is purely resistive, everything in this chapter applies to both DC and AC. However, matters become more complex when capacitance is involved. We do consider what happens when capacitors are connected to DC voltage sources, but the interaction of capacitors (and other nonresistive devices) with AC sources is left for a later chapter. In addition, a number of important DC instruments, such as meters that measure voltage and current, are covered in this chapter.

Information and examples presented in the chapter examine cause-effect relationships inherent in interactions involving electrical systems. The electrical properties of an electric circuit can change due to other systems (Enduring Understanding 4.E). More specifically, values of currents and potential differences in electric circuits depend on arrangements of individual circuit components (Essential Knowledge 4.E.5). In this chapter

several series and parallel combinations of resistors are discussed and their effects on currents and potential differences are analyzed.

In electric circuits the total energy (Enduring Understanding 5.B) and the total electric charge (Enduring Understanding 5.C) are conserved. Kirchhoff's rules describe both, energy conservation (Essential Knowledge 5.B.9) and charge conservation (Essential Knowledge 5.C.3). Energy conservation is discussed in terms of the loop rule which specifies that the potential around any closed circuit path must be zero. Charge conservation is applied as conservation of current by equating the sum of all currents entering a junction to the sum of all currents leaving the junction (also known as the junction rule). Kirchhoff's rules are used to calculate currents and potential differences in circuits that combine resistors in series and parallel, and resistors and capacitors.

The concepts in this chapter support:

Big Idea 4 Interactions between systems can result in changes in those systems.

Enduring Understanding 4.E The electric and magnetic properties of a system can change in response to the presence of, or changes in, other objects or systems.

Essential Knowledge 4.E.5 The values of currents and electric potential differences in an electric circuit are determined by the properties and arrangement of the individual circuit elements such as sources of emf, resistors, and capacitors.

Big Idea 5 Changes that occur as a result of interactions are constrained by conservation laws.

Enduring Understanding 5.B The energy of a system is conserved.

Essential Knowledge 5.B.9 Kirchhoff's loop rule describes conservation of energy in electrical circuits.

Enduring Understanding 5.C The electric charge of a system is conserved.

Essential Knowledge 5.C.3 Kirchhoff's junction rule describes the conservation of electric charge in electrical circuits. Since charge is conserved, current must be conserved at each junction in the circuit. Examples should include circuits that combine resistors in series and parallel.

Resistors in Series and Parallel

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Draw a circuit with resistors in parallel and in series.
- Use Ohm's law to calculate the voltage drop across a resistor when current passes through it.
- Contrast the way total resistance is calculated for resistors in series and in parallel.
- Explain why total resistance of a parallel circuit is less than the smallest resistance of any of the resistors in that circuit.
- Calculate total resistance of a circuit that contains a mixture of resistors connected in series and in parallel.

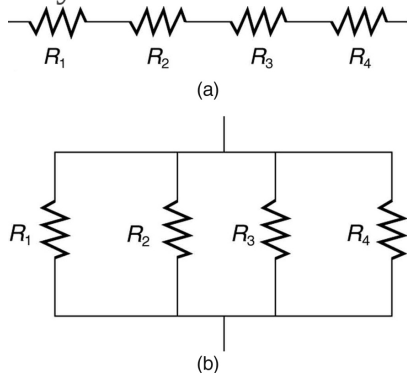
The information presented in this section supports the following AP® learning objectives and science practices:

- **4.E.5.1** The student is able to make and justify a quantitative prediction of the effect of a change in values or arrangements of one or two circuit elements on the currents and potential differences in a circuit containing a small number of sources of emf, resistors, capacitors, and switches in series and/or parallel. **(S.P. 2.2, 6.4)**
- **4.E.5.2** The student is able to make and justify a qualitative prediction of the effect of a change in values or arrangements of one or two circuit elements on currents and potential differences in a circuit containing a small number of sources of emf, resistors, capacitors, and switches in series and/or parallel. **(S.P. 6.1, 6.4)**
- **4.E.5.3** The student is able to plan data collection strategies and perform data analysis to examine the values of currents

and potential differences in an electric circuit that is modified by changing or rearranging circuit elements, including sources of emf, resistors, and capacitors. (S.P. 2.2, 4.2, 5.1)

- **5.B.9.3** The student is able to apply conservation of energy (Kirchhoff's loop rule) in calculations involving the total electric potential difference for complete circuit loops with only a single battery and resistors in series and/or in, at most, one parallel branch. (S.P. 2.2, 6.4, 7.2)

Most circuits have more than one component, called a **resistor** that limits the flow of charge in the circuit. A measure of this limit on charge flow is called **resistance**. The simplest combinations of resistors are the series and parallel connections illustrated in [\[link\]](#). The total resistance of a combination of resistors depends on both their individual values and how they are connected.

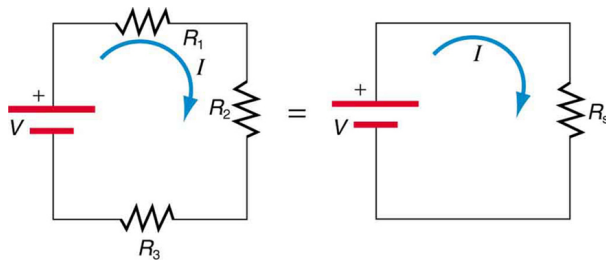


(a) A series connection of resistors. (b) A parallel connection of resistors.

Resistors in Series

When are resistors in **series**? Resistors are in series whenever the flow of charge, called the **current**, must flow through devices sequentially. For example, if current flows through a person holding a screwdriver and into the Earth, then R_1 in [\[link\]](#)(a) could be the resistance of the screwdriver's shaft, R_2 the resistance of its handle, R_3 the person's body resistance, and R_4 the resistance of her shoes.

[\[link\]](#) shows resistors in series connected to a **voltage** source. It seems reasonable that the total resistance is the sum of the individual resistances, considering that the current has to pass through each resistor in sequence. (This fact would be an advantage to a person wishing to avoid an electrical shock, who could reduce the current by wearing high-resistance rubber-soled shoes. It could be a disadvantage if one of the resistances were a faulty high-resistance cord to an appliance that would reduce the operating current.)



Three resistors connected in series to a battery (left) and the equivalent single or series resistance (right).

To verify that resistances in series do indeed add, let us consider the loss of electrical power, called a **voltage drop**, in each resistor in [\[link\]](#).

According to **Ohm's law**, the voltage drop, V , across a resistor when a current flows through it is calculated using the equation $V = IR$, where I equals the current in amps (A) and R is the resistance in ohms (Ω). Another way to think of this is that V is the voltage necessary to make a current I flow through a resistance R .

So the voltage drop across R_1 is $V_1 = IR_1$, that across R_2 is $V_2 = IR_2$, and that across R_3 is $V_3 = IR_3$. The sum of these voltages equals the voltage output of the source; that is,

Equation:

$$V = V_1 + V_2 + V_3.$$

This equation is based on the conservation of energy and conservation of charge. Electrical potential energy can be described by the equation $PE = qV$, where q is the electric charge and V is the voltage. Thus the energy supplied by the source is qV , while that dissipated by the resistors is

Equation:

$$qV_1 + qV_2 + qV_3.$$

Note:

Connections: Conservation Laws

The derivations of the expressions for series and parallel resistance are based on the laws of conservation of energy and conservation of charge, which state that total charge and total energy are constant in any process. These two laws are directly involved in all electrical phenomena and will be invoked repeatedly to explain both specific effects and the general behavior of electricity.

These energies must be equal, because there is no other source and no other destination for energy in the circuit. Thus, $qV = qV_1 + qV_2 + qV_3$. The charge q cancels, yielding $V = V_1 + V_2 + V_3$, as stated. (Note that the

same amount of charge passes through the battery and each resistor in a given amount of time, since there is no capacitance to store charge, there is no place for charge to leak, and charge is conserved.)

Now substituting the values for the individual voltages gives

Equation:

$$V = IR_1 + IR_2 + IR_3 = I(R_1 + R_2 + R_3).$$

Note that for the equivalent single series resistance R_s , we have

Equation:

$$V = IR_s.$$

This implies that the total or equivalent series resistance R_s of three resistors is $R_s = R_1 + R_2 + R_3$.

This logic is valid in general for any number of resistors in series; thus, the total resistance R_s of a series connection is

Equation:

$$R_s = R_1 + R_2 + R_3 + \dots,$$

as proposed. Since all of the current must pass through each resistor, it experiences the resistance of each, and resistances in series simply add up.

Example:

Calculating Resistance, Current, Voltage Drop, and Power

Dissipation: Analysis of a Series Circuit

Suppose the voltage output of the battery in [\[link\]](#) is 12.0 V, and the resistances are $R_1 = 1.00 \, \Omega$, $R_2 = 6.00 \, \Omega$, and $R_3 = 13.0 \, \Omega$. (a) What is the total resistance? (b) Find the current. (c) Calculate the voltage drop in each resistor, and show these add to equal the voltage output of the source. (d) Calculate the power dissipated by each resistor. (e) Find the power

output of the source, and show that it equals the total power dissipated by the resistors.

Strategy and Solution for (a)

The total resistance is simply the sum of the individual resistances, as given by this equation:

Equation:

$$\begin{aligned}R_s &= R_1 + R_2 + R_3 \\&= 1.00\ \Omega + 6.00\ \Omega + 13.0\ \Omega \\&= 20.0\ \Omega.\end{aligned}$$

Strategy and Solution for (b)

The current is found using Ohm's law, $V = IR$. Entering the value of the applied voltage and the total resistance yields the current for the circuit:

Equation:

$$I = \frac{V}{R_s} = \frac{12.0\ \text{V}}{20.0\ \Omega} = 0.600\ \text{A}.$$

Strategy and Solution for (c)

The voltage—or IR drop—in a resistor is given by Ohm's law. Entering the current and the value of the first resistance yields

Equation:

$$V_1 = IR_1 = (0.600\ \text{A})(1.0\ \Omega) = 0.600\ \text{V}.$$

Similarly,

Equation:

$$V_2 = IR_2 = (0.600\ \text{A})(6.0\ \Omega) = 3.60\ \text{V}$$

and

Equation:

$$V_3 = IR_3 = (0.600\ \text{A})(13.0\ \Omega) = 7.80\ \text{V}.$$

Discussion for (c)

The three IR drops add to 12.0 V, as predicted:

Equation:

$$V_1 + V_2 + V_3 = (0.600 + 3.60 + 7.80) \text{ V} = 12.0 \text{ V}.$$

Strategy and Solution for (d)

The easiest way to calculate power in watts (W) dissipated by a resistor in a DC circuit is to use **Joule's law**, $P = IV$, where P is electric power. In this case, each resistor has the same full current flowing through it. By substituting Ohm's law $V = IR$ into Joule's law, we get the power dissipated by the first resistor as

Equation:

$$P_1 = I^2 R_1 = (0.600 \text{ A})^2 (1.00 \Omega) = 0.360 \text{ W}.$$

Similarly,

Equation:

$$P_2 = I^2 R_2 = (0.600 \text{ A})^2 (6.00 \Omega) = 2.16 \text{ W}$$

and

Equation:

$$P_3 = I^2 R_3 = (0.600 \text{ A})^2 (13.0 \Omega) = 4.68 \text{ W}.$$

Discussion for (d)

Power can also be calculated using either $P = IV$ or $P = \frac{V^2}{R}$, where V is the voltage drop across the resistor (not the full voltage of the source). The same values will be obtained.

Strategy and Solution for (e)

The easiest way to calculate power output of the source is to use $P = IV$, where V is the source voltage. This gives

Equation:

$$P = (0.600 \text{ A})(12.0 \text{ V}) = 7.20 \text{ W}.$$

Discussion for (e)

Note, coincidentally, that the total power dissipated by the resistors is also 7.20 W, the same as the power put out by the source. That is,

Equation:

$$P_1 + P_2 + P_3 = (0.360 + 2.16 + 4.68) \text{ W} = 7.20 \text{ W}.$$

Power is energy per unit time (watts), and so conservation of energy requires the power output of the source to be equal to the total power dissipated by the resistors.

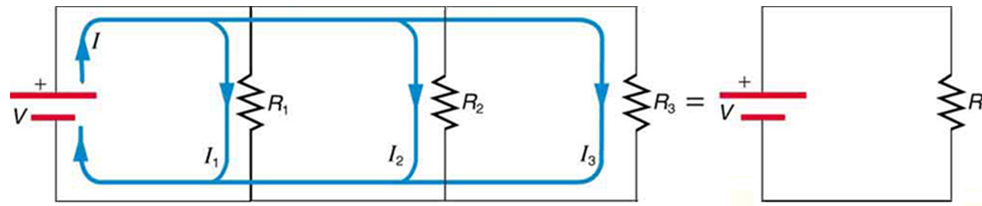
Note:**Major Features of Resistors in Series**

1. Series resistances add: $R_s = R_1 + R_2 + R_3 + \dots$
2. The same current flows through each resistor in series.
3. Individual resistors in series do not get the total source voltage, but divide it.

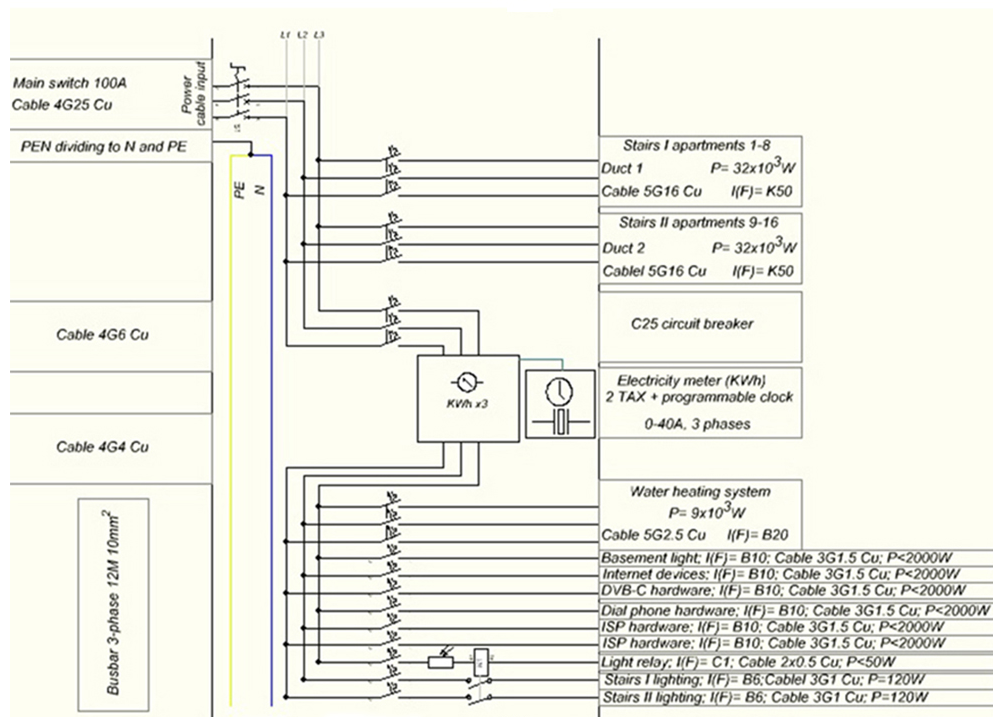
Resistors in Parallel

[\[link\]](#) shows resistors in **parallel**, wired to a voltage source. Resistors are in parallel when each resistor is connected directly to the voltage source by connecting wires having negligible resistance. Each resistor thus has the full voltage of the source applied to it.

Each resistor draws the same current it would if it alone were connected to the voltage source (provided the voltage source is not overloaded). For example, an automobile's headlights, radio, and so on, are wired in parallel, so that they utilize the full voltage of the source and can operate completely independently. The same is true in your house, or any building. (See [\[link\]](#) (b).)



(a)



(b)

(a) Three resistors connected in parallel to a battery and the equivalent single or parallel resistance. (b) Electrical power setup in a house. (credit: Dmitry G, Wikimedia Commons)

To find an expression for the equivalent parallel resistance R_p , let us consider the currents that flow and how they are related to resistance. Since each resistor in the circuit has the full voltage, the currents flowing through the individual resistors are $I_1 = \frac{V}{R_1}$, $I_2 = \frac{V}{R_2}$, and $I_3 = \frac{V}{R_3}$. Conservation of charge implies that the total current I produced by the source is the sum of these currents:

Equation:

$$I = I_1 + I_2 + I_3.$$

Substituting the expressions for the individual currents gives

Equation:

$$I = \frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3} = V \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right).$$

Note that Ohm's law for the equivalent single resistance gives

Equation:

$$I = \frac{V}{R_p} = V \left(\frac{1}{R_p} \right).$$

The terms inside the parentheses in the last two equations must be equal. Generalizing to any number of resistors, the total resistance R_p of a parallel connection is related to the individual resistances by

Equation:

$$\frac{1}{R_p} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \dots$$

This relationship results in a total resistance R_p that is less than the smallest of the individual resistances. (This is seen in the next example.) When resistors are connected in parallel, more current flows from the source than would flow for any of them individually, and so the total resistance is lower.

Example:

Calculating Resistance, Current, Power Dissipation, and Power Output: Analysis of a Parallel Circuit

Let the voltage output of the battery and resistances in the parallel connection in [\[link\]](#) be the same as the previously considered series

connection: $V = 12.0 \text{ V}$, $R_1 = 1.00 \text{ } \Omega$, $R_2 = 6.00 \text{ } \Omega$, and $R_3 = 13.0 \text{ } \Omega$.

(a) What is the total resistance? (b) Find the total current. (c) Calculate the currents in each resistor, and show these add to equal the total current output of the source. (d) Calculate the power dissipated by each resistor. (e) Find the power output of the source, and show that it equals the total power dissipated by the resistors.

Strategy and Solution for (a)

The total resistance for a parallel combination of resistors is found using the equation below. Entering known values gives

Equation:

$$\frac{1}{R_p} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} = \frac{1}{1.00 \text{ } \Omega} + \frac{1}{6.00 \text{ } \Omega} + \frac{1}{13.0 \text{ } \Omega}.$$

Thus,

Equation:

$$\frac{1}{R_p} = \frac{1.00}{\Omega} + \frac{0.1667}{\Omega} + \frac{0.07692}{\Omega} = \frac{1.2436}{\Omega}.$$

(Note that in these calculations, each intermediate answer is shown with an extra digit.)

We must invert this to find the total resistance R_p . This yields

Equation:

$$R_p = \frac{1}{1.2436} \Omega = 0.8041 \text{ } \Omega.$$

The total resistance with the correct number of significant digits is $R_p = 0.804 \text{ } \Omega$.

Discussion for (a)

R_p is, as predicted, less than the smallest individual resistance.

Strategy and Solution for (b)

The total current can be found from Ohm's law, substituting R_p for the total resistance. This gives

Equation:

$$I = \frac{V}{R_p} = \frac{12.0 \text{ V}}{0.8041 \Omega} = 14.92 \text{ A.}$$

Discussion for (b)

Current I for each device is much larger than for the same devices connected in series (see the previous example). A circuit with parallel connections has a smaller total resistance than the resistors connected in series.

Strategy and Solution for (c)

The individual currents are easily calculated from Ohm's law, since each resistor gets the full voltage. Thus,

Equation:

$$I_1 = \frac{V}{R_1} = \frac{12.0 \text{ V}}{1.00 \Omega} = 12.0 \text{ A.}$$

Similarly,

Equation:

$$I_2 = \frac{V}{R_2} = \frac{12.0 \text{ V}}{6.00 \Omega} = 2.00 \text{ A}$$

and

Equation:

$$I_3 = \frac{V}{R_3} = \frac{12.0 \text{ V}}{13.0 \Omega} = 0.92 \text{ A.}$$

Discussion for (c)

The total current is the sum of the individual currents:

Equation:

$$I_1 + I_2 + I_3 = 14.92 \text{ A.}$$

This is consistent with conservation of charge.

Strategy and Solution for (d)

The power dissipated by each resistor can be found using any of the equations relating power to current, voltage, and resistance, since all three

are known. Let us use $P = \frac{V^2}{R}$, since each resistor gets full voltage. Thus,
Equation:

$$P_1 = \frac{V^2}{R_1} = \frac{(12.0 \text{ V})^2}{1.00 \, \Omega} = 144 \text{ W}.$$

Similarly,
Equation:

$$P_2 = \frac{V^2}{R_2} = \frac{(12.0 \text{ V})^2}{6.00 \, \Omega} = 24.0 \text{ W}$$

and
Equation:

$$P_3 = \frac{V^2}{R_3} = \frac{(12.0 \text{ V})^2}{13.0 \, \Omega} = 11.1 \text{ W}.$$

Discussion for (d)

The power dissipated by each resistor is considerably higher in parallel than when connected in series to the same voltage source.

Strategy and Solution for (e)

The total power can also be calculated in several ways. Choosing $P = IV$, and entering the total current, yields

Equation:

$$P = IV = (14.92 \text{ A})(12.0 \text{ V}) = 179 \text{ W}.$$

Discussion for (e)

Total power dissipated by the resistors is also 179 W:

Equation:

$$P_1 + P_2 + P_3 = 144 \text{ W} + 24.0 \text{ W} + 11.1 \text{ W} = 179 \text{ W}.$$

This is consistent with the law of conservation of energy.

Overall Discussion

Note that both the currents and powers in parallel connections are greater than for the same devices in series.

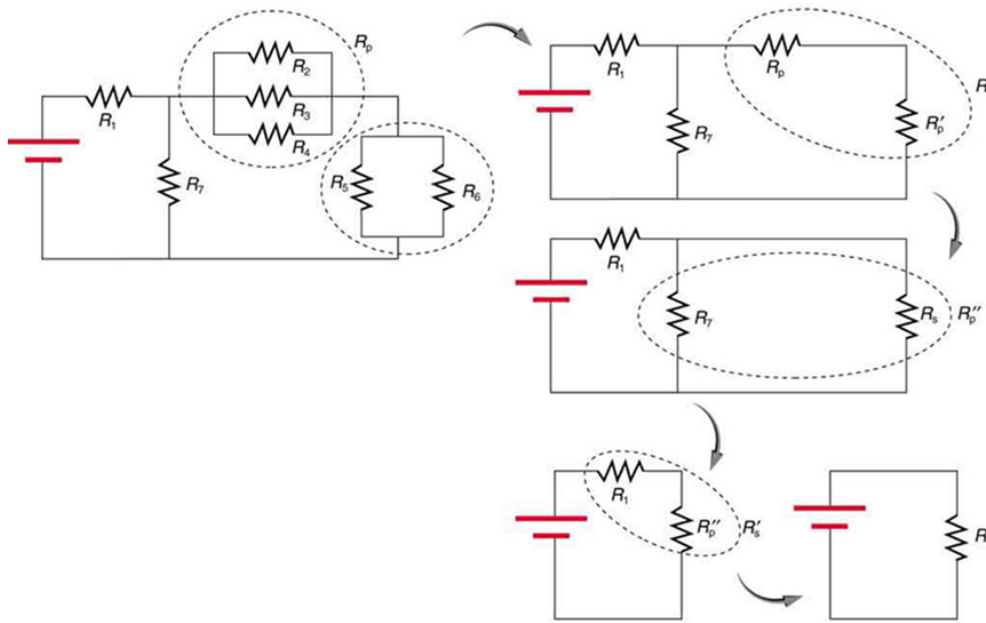
Note:**Major Features of Resistors in Parallel**

1. Parallel resistance is found from $\frac{1}{R_p} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \dots$, and it is smaller than any individual resistance in the combination.
2. Each resistor in parallel has the same full voltage of the source applied to it. (Power distribution systems most often use parallel connections to supply the myriad devices served with the same voltage and to allow them to operate independently.)
3. Parallel resistors do not each get the total current; they divide it.

Combinations of Series and Parallel

More complex connections of resistors are sometimes just combinations of series and parallel. These are commonly encountered, especially when wire resistance is considered. In that case, wire resistance is in series with other resistances that are in parallel.

Combinations of series and parallel can be reduced to a single equivalent resistance using the technique illustrated in [\[link\]](#). Various parts are identified as either series or parallel, reduced to their equivalents, and further reduced until a single resistance is left. The process is more time consuming than difficult.



This combination of seven resistors has both series and parallel parts. Each is identified and reduced to an equivalent resistance, and these are further reduced until a single equivalent resistance is reached.

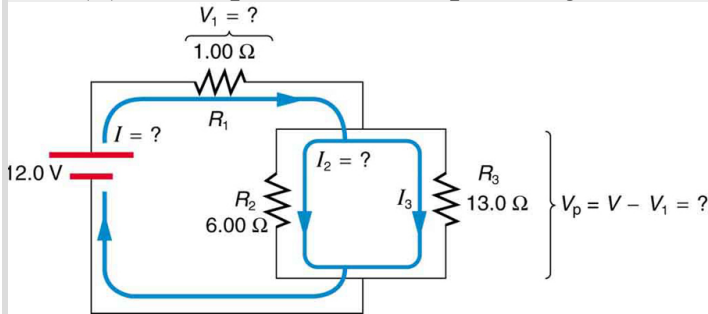
The simplest combination of series and parallel resistance, shown in [\[link\]](#), is also the most instructive, since it is found in many applications. For example, R_1 could be the resistance of wires from a car battery to its electrical devices, which are in parallel. R_2 and R_3 could be the starter motor and a passenger compartment light. We have previously assumed that wire resistance is negligible, but, when it is not, it has important effects, as the next example indicates.

Example:

Calculating Resistance, IR Drop, Current, and Power Dissipation: Combining Series and Parallel Circuits

[\[link\]](#) shows the resistors from the previous two examples wired in a different way—a combination of series and parallel. We can consider R_1 to be the resistance of wires leading to R_2 and R_3 . (a) Find the total

resistance. (b) What is the IR drop in R_1 ? (c) Find the current I_2 through R_2 . (d) What power is dissipated by R_2 ?



These three resistors are connected to a voltage source so that R_2 and R_3 are in parallel with one another and that combination is in series with R_1 .

Strategy and Solution for (a)

To find the total resistance, we note that R_2 and R_3 are in parallel and their combination R_p is in series with R_1 . Thus the total (equivalent) resistance of this combination is

Equation:

$$R_{\text{tot}} = R_1 + R_p.$$

First, we find R_p using the equation for resistors in parallel and entering known values:

Equation:

$$\frac{1}{R_p} = \frac{1}{R_2} + \frac{1}{R_3} = \frac{1}{6.00 \, \Omega} + \frac{1}{13.0 \, \Omega} = \frac{0.2436}{\Omega}.$$

Inverting gives

Equation:

$$R_p = \frac{1}{0.2436} \, \Omega = 4.11 \, \Omega.$$

So the total resistance is

Equation:

$$R_{\text{tot}} = R_1 + R_p = 1.00 \, \Omega + 4.11 \, \Omega = 5.11 \, \Omega.$$

Discussion for (a)

The total resistance of this combination is intermediate between the pure series and pure parallel values ($20.0 \, \Omega$ and $0.804 \, \Omega$, respectively) found for the same resistors in the two previous examples.

Strategy and Solution for (b)

To find the IR drop in R_1 , we note that the full current I flows through R_1 . Thus its IR drop is

Equation:

$$V_1 = IR_1.$$

We must find I before we can calculate V_1 . The total current I is found using Ohm's law for the circuit. That is,

Equation:

$$I = \frac{V}{R_{\text{tot}}} = \frac{12.0 \, \text{V}}{5.11 \, \Omega} = 2.35 \, \text{A}.$$

Entering this into the expression above, we get

Equation:

$$V_1 = IR_1 = (2.35 \, \text{A})(1.00 \, \Omega) = 2.35 \, \text{V}.$$

Discussion for (b)

The voltage applied to R_2 and R_3 is less than the total voltage by an amount V_1 . When wire resistance is large, it can significantly affect the operation of the devices represented by R_2 and R_3 .

Strategy and Solution for (c)

To find the current through R_2 , we must first find the voltage applied to it. We call this voltage V_p , because it is applied to a parallel combination of resistors. The voltage applied to both R_2 and R_3 is reduced by the amount V_1 , and so it is

Equation:

$$V_p = V - V_1 = 12.0 \text{ V} - 2.35 \text{ V} = 9.65 \text{ V}.$$

Now the current I_2 through resistance R_2 is found using Ohm's law:

Equation:

$$I_2 = \frac{V_p}{R_2} = \frac{9.65 \text{ V}}{6.00 \Omega} = 1.61 \text{ A}.$$

Discussion for (c)

The current is less than the 2.00 A that flowed through R_2 when it was connected in parallel to the battery in the previous parallel circuit example.

Strategy and Solution for (d)

The power dissipated by R_2 is given by

Equation:

$$P_2 = (I_2)^2 R_2 = (1.61 \text{ A})^2 (6.00 \Omega) = 15.5 \text{ W}.$$

Discussion for (d)

The power is less than the 24.0 W this resistor dissipated when connected in parallel to the 12.0-V source.

Note:

Applying the Science Practices: Circuit Construction Kit (DC only)

Plan an experiment to analyze the effect on currents and potential differences due to rearrangement of resistors and variations in voltage sources. Your experimental investigation should include data collection for at least five different scenarios of rearranged resistors (i.e., several combinations of series and parallel) and three scenarios of different voltage sources.

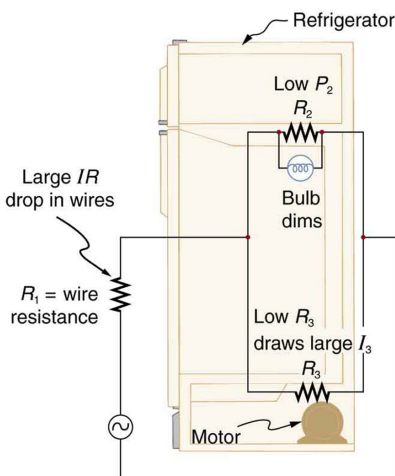
Practical Implications

One implication of this last example is that resistance in wires reduces the current and power delivered to a resistor. If wire resistance is relatively

large, as in a worn (or a very long) extension cord, then this loss can be significant. If a large current is drawn, the IR drop in the wires can also be significant.

For example, when you are rummaging in the refrigerator and the motor comes on, the refrigerator light dims momentarily. Similarly, you can see the passenger compartment light dim when you start the engine of your car (although this may be due to resistance inside the battery itself).

What is happening in these high-current situations is illustrated in [\[link\]](#). The device represented by R_3 has a very low resistance, and so when it is switched on, a large current flows. This increased current causes a larger IR drop in the wires represented by R_1 , reducing the voltage across the light bulb (which is R_2), which then dims noticeably.



Why do lights dim
when a large
appliance is
switched on? The
answer is that the
large current the
appliance motor
draws causes a
significant IR drop

in the wires and
reduces the voltage
across the light.

Exercise:

Check Your Understanding

Problem:

Can any arbitrary combination of resistors be broken down into series and parallel combinations? See if you can draw a circuit diagram of resistors that cannot be broken down into combinations of series and parallel.

Solution:

No, there are many ways to connect resistors that are not combinations of series and parallel, including loops and junctions. In such cases Kirchhoff's rules, to be introduced in [Kirchhoff's Rules](#), will allow you to analyze the circuit.

Note:

Problem-Solving Strategies for Series and Parallel Resistors

1. Draw a clear circuit diagram, labeling all resistors and voltage sources. This step includes a list of the knowns for the problem, since they are labeled in your circuit diagram.
2. Identify exactly what needs to be determined in the problem (identify the unknowns). A written list is useful.
3. Determine whether resistors are in series, parallel, or a combination of both series and parallel. Examine the circuit diagram to make this assessment. Resistors are in series if the same current must pass sequentially through them.
4. Use the appropriate list of major features for series or parallel connections to solve for the unknowns. There is one list for series and

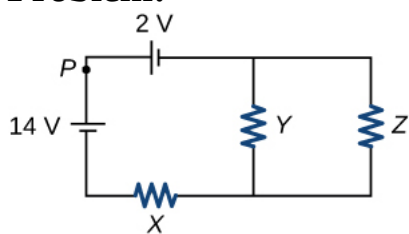
another for parallel. If your problem has a combination of series and parallel, reduce it in steps by considering individual groups of series or parallel connections, as done in this module and the examples. Special note: When finding R_p , the reciprocal must be taken with care.

5. Check to see whether the answers are reasonable and consistent. Units and numerical results must be reasonable. Total series resistance should be greater, whereas total parallel resistance should be smaller, for example. Power should be greater for the same devices in parallel compared with series, and so on.

Test Prep for AP Courses

Exercise:

Problem:



The figure above shows a circuit containing two batteries and three identical resistors with resistance R . Which of the following changes to the circuit will result in an increase in the current at point P ? Select *two* answers.

- a. Reversing the connections to the 14 V battery.
- b. Removing the 2 V battery and connecting the wires to close the left loop.
- c. Rearranging the resistors so all three are in series.
- d. Removing the branch containing resistor Z .

Solution:

(a), (b)

Exercise:

Problem:

In a circuit, a parallel combination of six $1.6\text{-k}\Omega$ resistors is connected in series with a parallel combination of four $2.4\text{-k}\Omega$ resistors. If the source voltage is 24 V , what will be the percentage of total current in one of the $2.4\text{-k}\Omega$ resistors?

- a. 10%
- b. 12%
- c. 20%
- d. 25%

Exercise:

Problem:

If the circuit in the previous question is modified by removing some of the $1.6\text{ k}\Omega$ resistors, the total current in the circuit is 24 mA . How many resistors were removed?

- a. 1
- b. 2
- c. 3
- d. 4

Solution:

(b)

Exercise:

Problem:



Two resistors, with resistances R and $2R$ are connected to a voltage source as shown in this figure. If the power dissipated in R is 10 W, what is the power dissipated in $2R$?

- a. 1 W
- b. 2.5 W
- c. 5 W
- d. 10 W

Exercise:

Problem:

In a circuit, a parallel combination of two 20- Ω and one 10- Ω resistors is connected in series with a 4- Ω resistor. The source voltage is 36 V.

- a. Find the resistor(s) with the maximum current.
- b. Find the resistor(s) with the maximum voltage drop.
- c. Find the power dissipated in each resistor and hence the total power dissipated in all the resistors. Also find the power output of the source. Are they equal or not? Justify your answer.
- d. Will the answers for questions (a) and (b) differ if a 3 Ω resistor is added in series to the 4 Ω resistor? If yes, repeat the question(s) for the new resistor combination.
- e. If the values of all the resistors and the source voltage are doubled, what will be the effect on the current?

Solution:

(a) 4- Ω resistor; (b) combination of 20- Ω , 20- Ω , and 10- Ω resistors; (c) 20 W in each 20- Ω resistor, 40 W in 10- Ω resistor, 64 W in 4- Ω resistor, total 144W total in resistors, output power is 144 W, yes they are equal (law of conservation of energy); (d) 4 Ω and 3 Ω for part (a) and no change for part (b); (e) no effect, it will remain the same.

Section Summary

- The total resistance of an electrical circuit with resistors wired in a series is the sum of the individual resistances:
 $R_s = R_1 + R_2 + R_3 + \dots$
- Each resistor in a series circuit has the same amount of current flowing through it.
- The voltage drop, or power dissipation, across each individual resistor in a series is different, and their combined total adds up to the power source input.
- The total resistance of an electrical circuit with resistors wired in parallel is less than the lowest resistance of any of the components and can be determined using the formula:

Equation:

$$\frac{1}{R_p} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \dots$$

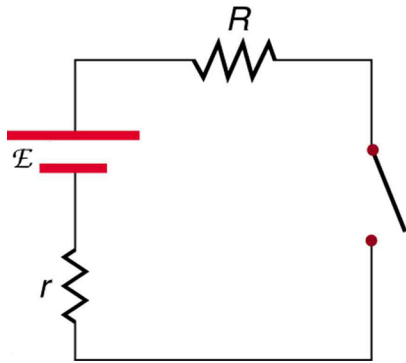
- Each resistor in a parallel circuit has the same full voltage of the source applied to it.
- The current flowing through each resistor in a parallel circuit is different, depending on the resistance.
- If a more complex connection of resistors is a combination of series and parallel, it can be reduced to a single equivalent resistance by identifying its various parts as series or parallel, reducing each to its equivalent, and continuing until a single resistance is eventually reached.

Conceptual Questions

Exercise:

Problem:

A switch has a variable resistance that is nearly zero when closed and extremely large when open, and it is placed in series with the device it controls. Explain the effect the switch in [\[link\]](#) has on current when open and when closed.



A switch is ordinarily in series with a resistance and voltage source. Ideally, the switch has nearly zero resistance when closed but has an extremely large resistance when open. (Note that in this diagram, the script E represents the voltage (or electromotive force) of the battery.)

Exercise:

Problem: What is the voltage across the open switch in [\[link\]](#)?

Exercise:

Problem:

There is a voltage across an open switch, such as in [\[link\]](#). Why, then, is the power dissipated by the open switch small?

Exercise:

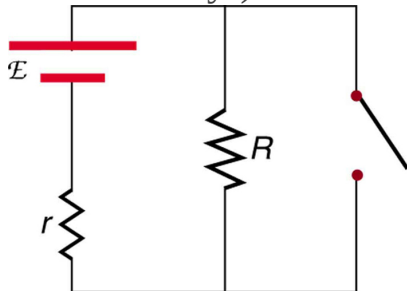
Problem:

Why is the power dissipated by a closed switch, such as in [\[link\]](#), small?

Exercise:

Problem:

A student in a physics lab mistakenly wired a light bulb, battery, and switch as shown in [\[link\]](#). Explain why the bulb is on when the switch is open, and off when the switch is closed. (Do not try this—it is hard on the battery!)



A wiring mistake put this switch in parallel with the device represented by R . (Note that in this diagram, the script E represents the voltage (or

electromotive
force) of the
battery.)

Exercise:

Problem:

Knowing that the severity of a shock depends on the magnitude of the current through your body, would you prefer to be in series or parallel with a resistance, such as the heating element of a toaster, if shocked by it? Explain.

Exercise:

Problem:

Would your headlights dim when you start your car's engine if the wires in your automobile were superconductors? (Do not neglect the battery's internal resistance.) Explain.

Exercise:

Problem:

Some strings of holiday lights are wired in series to save wiring costs. An old version utilized bulbs that break the electrical connection, like an open switch, when they burn out. If one such bulb burns out, what happens to the others? If such a string operates on 120 V and has 40 identical bulbs, what is the normal operating voltage of each? Newer versions use bulbs that short circuit, like a closed switch, when they burn out. If one such bulb burns out, what happens to the others? If such a string operates on 120 V and has 39 remaining identical bulbs, what is then the operating voltage of each?

Exercise:

Problem:

If two household lightbulbs rated 60 W and 100 W are connected in series to household power, which will be brighter? Explain.

Exercise:**Problem:**

Suppose you are doing a physics lab that asks you to put a resistor into a circuit, but all the resistors supplied have a larger resistance than the requested value. How would you connect the available resistances to attempt to get the smaller value asked for?

Exercise:**Problem:**

Before World War II, some radios got power through a “resistance cord” that had a significant resistance. Such a resistance cord reduces the voltage to a desired level for the radio’s tubes and the like, and it saves the expense of a transformer. Explain why resistance cords become warm and waste energy when the radio is on.

Exercise:**Problem:**

Some light bulbs have three power settings (not including zero), obtained from multiple filaments that are individually switched and wired in parallel. What is the minimum number of filaments needed for three power settings?

Problem Exercises

Note: Data taken from figures can be assumed to be accurate to three significant digits.

Exercise:

Problem:

- (a) What is the resistance of ten $275\text{-}\Omega$ resistors connected in series?
(b) In parallel?
-

Solution:

- (a) $2.75\text{ k}\Omega$
(b) $27.5\text{ }\Omega$

Exercise:**Problem:**

- (a) What is the resistance of a $1.00 \times 10^2\text{-}\Omega$, a $2.50\text{-k}\Omega$, and a $4.00\text{-k}\Omega$ resistor connected in series? (b) In parallel?

Exercise:**Problem:**

What are the largest and smallest resistances you can obtain by connecting a $36.0\text{-}\Omega$, a $50.0\text{-}\Omega$, and a $700\text{-}\Omega$ resistor together?

Solution:

- (a) $786\text{ }\Omega$
(b) $20.3\text{ }\Omega$

Exercise:**Problem:**

An 1800-W toaster, a 1400-W electric frying pan, and a 75-W lamp are plugged into the same outlet in a 15-A , 120-V circuit. (The three devices are in parallel when plugged into the same socket.). (a) What current is drawn by each device? (b) Will this combination blow the 15-A fuse?

Exercise:

Problem:

Your car's 30.0-W headlight and 2.40-kW starter are ordinarily connected in parallel in a 12.0-V system. What power would one headlight and the starter consume if connected in series to a 12.0-V battery? (Neglect any other resistance in the circuit and any change in resistance in the two devices.)

Solution:

29.6 W

Exercise:**Problem:**

(a) Given a 48.0-V battery and $24.0\text{-}\Omega$ and $96.0\text{-}\Omega$ resistors, find the current and power for each when connected in series. (b) Repeat when the resistances are in parallel.

Exercise:**Problem:**

Referring to the example combining series and parallel circuits and [\[link\]](#), calculate I_3 in the following two different ways: (a) from the known values of I and I_2 ; (b) using Ohm's law for R_3 . In both parts explicitly show how you follow the steps in the [Problem-Solving Strategies for Series and Parallel Resistors](#).

Solution:

(a) 0.74 A

(b) 0.742 A

Exercise:

Problem:

Referring to [\[link\]](#): (a) Calculate P_3 and note how it compares with P_3 found in the first two example problems in this module. (b) Find the total power supplied by the source and compare it with the sum of the powers dissipated by the resistors.

Exercise:**Problem:**

Refer to [\[link\]](#) and the discussion of lights dimming when a heavy appliance comes on. (a) Given the voltage source is 120 V, the wire resistance is $0.400\ \Omega$, and the bulb is nominally 75.0 W, what power will the bulb dissipate if a total of 15.0 A passes through the wires when the motor comes on? Assume negligible change in bulb resistance. (b) What power is consumed by the motor?

Solution:

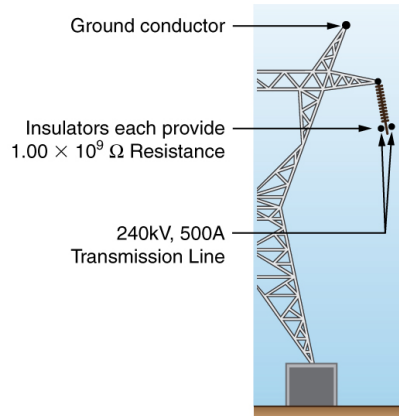
(a) 60.8 W

(b) 3.18 kW

Exercise:**Problem:**

A 240-kV power transmission line carrying $5.00 \times 10^2\ \text{A}$ is hung from grounded metal towers by ceramic insulators, each having a $1.00 \times 10^9\ \Omega$ resistance. [\[link\]](#). (a) What is the resistance to ground of 100 of these insulators? (b) Calculate the power dissipated by 100 of them. (c) What fraction of the power carried by the line is this?

Explicitly show how you follow the steps in the [Problem-Solving Strategies for Series and Parallel Resistors](#).



High-voltage (240-kV) transmission line carrying $5.00 \times 10^2 \text{ A}$ is hung from a grounded metal transmission tower. The row of ceramic insulators provide $1.00 \times 10^9 \Omega$ of resistance each.

Exercise:

Problem:

Show that if two resistors R_1 and R_2 are combined and one is much greater than the other ($R_1 \gg R_2$): (a) Their series resistance is very nearly equal to the greater resistance R_1 . (b) Their parallel resistance is very nearly equal to smaller resistance R_2 .

Solution:

$$R_s = R_1 + R_2$$

(a) $\Rightarrow R_s \approx R_1 (R_1 \gg R_2)$

$$(b) \frac{1}{R_p} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{R_1 + R_2}{R_1 R_2},$$

so that

$$R_p = \frac{R_1 R_2}{R_1 + R_2} \approx \frac{R_1 R_2}{R_1} = R_2 (R_1 \gg R_2).$$

Exercise:

Problem: Unreasonable Results

Two resistors, one having a resistance of $145 \, \Omega$, are connected in parallel to produce a total resistance of $150 \, \Omega$. (a) What is the value of the second resistance? (b) What is unreasonable about this result? (c) Which assumptions are unreasonable or inconsistent?

Exercise:

Problem: Unreasonable Results

Two resistors, one having a resistance of $900 \, \text{k}\Omega$, are connected in series to produce a total resistance of $0.500 \, \text{M}\Omega$. (a) What is the value of the second resistance? (b) What is unreasonable about this result? (c) Which assumptions are unreasonable or inconsistent?

Solution:

(a) $-400 \, \text{k}\Omega$

(b) Resistance cannot be negative.

(c) Series resistance is said to be less than one of the resistors, but it must be greater than any of the resistors.

Glossary

series

a sequence of resistors or other components wired into a circuit one after the other

resistor

a component that provides resistance to the current flowing through an electrical circuit

resistance

causing a loss of electrical power in a circuit

Ohm's law

the relationship between current, voltage, and resistance within an electrical circuit: $V = IR$

voltage

the electrical potential energy per unit charge; electric pressure created by a power source, such as a battery

voltage drop

the loss of electrical power as a current travels through a resistor, wire or other component

current

the flow of charge through an electric circuit past a given point of measurement

Joule's law

the relationship between potential electrical power, voltage, and resistance in an electrical circuit, given by: $P_e = IV$

parallel

the wiring of resistors or other components in an electrical circuit such that each component receives an equal voltage from the power source; often pictured in a ladder-shaped diagram, with each component on a rung of the ladder

Electromotive Force: Terminal Voltage

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Compare and contrast the voltage and the electromagnetic force of an electric power source.
- Describe what happens to the terminal voltage, current, and power delivered to a load as internal resistance of the voltage source increases.
- Explain why it is beneficial to use more than one voltage source connected in parallel.

The information presented in this section supports the following AP® learning objectives and science practices:

- **5.B.9.7** The student is able to refine and analyze a scientific question for an experiment using Kirchhoff's loop rule for circuits that includes determination of internal resistance of the battery and analysis of a nonohmic resistor. **(S.P. 4.1, 4.2, 5.1, 5.3)**

When you forget to turn off your car lights, they slowly dim as the battery runs down. Why don't they simply blink off when the battery's energy is gone? Their gradual dimming implies that battery output voltage decreases as the battery is depleted.

Furthermore, if you connect an excessive number of 12-V lights in parallel to a car battery, they will be dim even when the battery is fresh and even if

the wires to the lights have very low resistance. This implies that the battery's output voltage is reduced by the overload.

The reason for the decrease in output voltage for depleted or overloaded batteries is that all voltage sources have two fundamental parts—a source of electrical energy and an **internal resistance**. Let us examine both.

Electromotive Force

You can think of many different types of voltage sources. Batteries themselves come in many varieties. There are many types of mechanical/electrical generators, driven by many different energy sources, ranging from nuclear to wind. Solar cells create voltages directly from light, while thermoelectric devices create voltage from temperature differences.

A few voltage sources are shown in [\[link\]](#). All such devices create a **potential difference** and can supply current if connected to a resistance. On the small scale, the potential difference creates an electric field that exerts force on charges, causing current. We thus use the name **electromotive force**, abbreviated emf.

Emf is not a force at all; it is a special type of potential difference. To be precise, the electromotive force (emf) is the potential difference of a source when no current is flowing. Units of emf are volts.



A variety of voltage sources

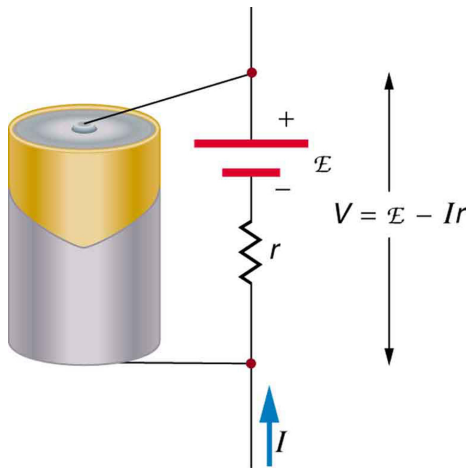
(clockwise from top left):
the Brazos Wind Farm in
Fluvanna, Texas (credit:
Leaflet, Wikimedia
Commons); the Krasnoyarsk
Dam in Russia (credit: Alex
Polezhaev); a solar farm
(credit: U.S. Department of
Energy); and a group of
nickel metal hydride
batteries (credit: Tiaa
Monto). The voltage output
of each depends on its
construction and load, and
equals emf only if there is
no load.

Electromotive force is directly related to the source of potential difference, such as the particular combination of chemicals in a battery. However, emf differs from the voltage output of the device when current flows. The voltage across the terminals of a battery, for example, is less than the emf when the battery supplies current, and it declines further as the battery is depleted or loaded down. However, if the device's output voltage can be measured without drawing current, then output voltage will equal emf (even for a very depleted battery).

Internal Resistance

As noted before, a 12-V truck battery is physically larger, contains more charge and energy, and can deliver a larger current than a 12-V motorcycle battery. Both are lead-acid batteries with identical emf, but, because of its size, the truck battery has a smaller internal resistance r . Internal resistance is the inherent resistance to the flow of current within the source itself.

[\[link\]](#) is a schematic representation of the two fundamental parts of any voltage source. The emf (represented by a script \mathcal{E} in the figure) and internal resistance r are in series. The smaller the internal resistance for a given emf, the more current and the more power the source can supply.



Any voltage source (in this case, a carbon-zinc dry cell) has an emf related to its source of potential difference, and an internal resistance r related to its construction. (Note that the script \mathcal{E} stands for emf.). Also shown are the output terminals across which the terminal voltage V is measured. Since $V = \text{emf} - Ir$, terminal voltage equals emf only if there is no current flowing.

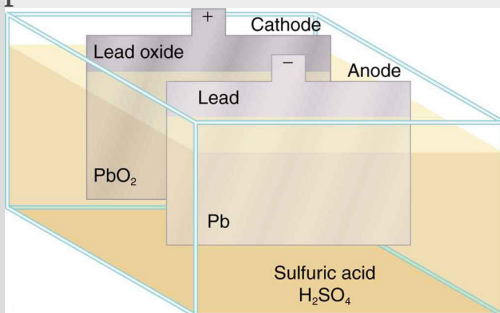
The internal resistance r can behave in complex ways. As noted, r increases as a battery is depleted. But internal resistance may also depend on the magnitude and direction of the current through a voltage source, its temperature, and even its history. The internal resistance of rechargeable nickel-cadmium cells, for example, depends on how many times and how deeply they have been depleted.

Note:

Things Great and Small: The Submicroscopic Origin of Battery Potential

Various types of batteries are available, with emfs determined by the combination of chemicals involved. We can view this as a molecular reaction (what much of chemistry is about) that separates charge.

The lead-acid battery used in cars and other vehicles is one of the most common types. A single cell (one of six) of this battery is seen in [\[link\]](#). The cathode (positive) terminal of the cell is connected to a lead oxide plate, while the anode (negative) terminal is connected to a lead plate. Both plates are immersed in sulfuric acid, the electrolyte for the system.

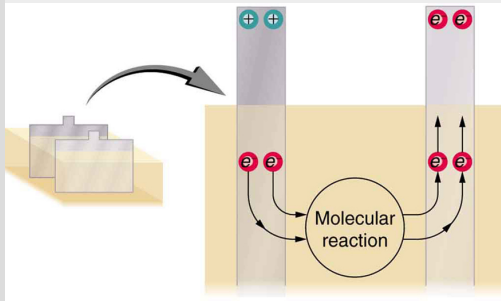


Artist's conception of a lead-acid cell. Chemical reactions in a lead-acid cell separate charge, sending negative charge to the anode, which is connected to the lead plates. The lead oxide plates are connected to

the positive or cathode terminal of the cell. Sulfuric acid conducts the charge as well as participating in the chemical reaction.

The details of the chemical reaction are left to the reader to pursue in a chemistry text, but their results at the molecular level help explain the potential created by the battery. [\[link\]](#) shows the result of a single chemical reaction. Two electrons are placed on the anode, making it negative, provided that the cathode supplied two electrons. This leaves the cathode positively charged, because it has lost two electrons. In short, a separation of charge has been driven by a chemical reaction.

Note that the reaction will not take place unless there is a complete circuit to allow two electrons to be supplied to the cathode. Under many circumstances, these electrons come from the anode, flow through a resistance, and return to the cathode. Note also that since the chemical reactions involve substances with resistance, it is not possible to create the emf without an internal resistance.



Artist's conception of two electrons being forced onto the anode of a cell and two electrons being removed from the cathode of the cell. The chemical reaction in a lead-acid battery places

two electrons on the anode and removes two from the cathode. It requires a closed circuit to proceed, since the two electrons must be supplied to the cathode.

Why are the chemicals able to produce a unique potential difference? Quantum mechanical descriptions of molecules, which take into account the types of atoms and numbers of electrons in them, are able to predict the energy states they can have and the energies of reactions between them.

In the case of a lead-acid battery, an energy of 2 eV is given to each electron sent to the anode. Voltage is defined as the electrical potential energy divided by charge: $V = \frac{P_E}{q}$. An electron volt is the energy given to a single electron by a voltage of 1 V. So the voltage here is 2 V, since 2 eV is given to each electron. It is the energy produced in each molecular reaction that produces the voltage. A different reaction produces a different energy and, hence, a different voltage.

Terminal Voltage

The voltage output of a device is measured across its terminals and, thus, is called its **terminal voltage** V . Terminal voltage is given by

Equation:

$$V = \text{emf} - Ir,$$

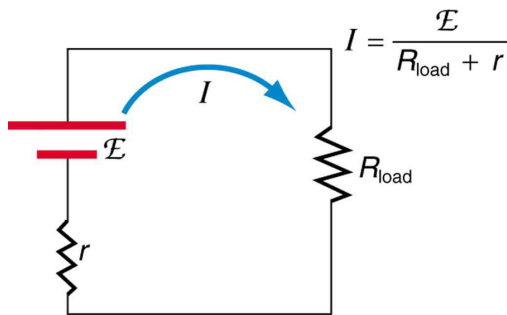
where r is the internal resistance and I is the current flowing at the time of the measurement.

I is positive if current flows away from the positive terminal, as shown in [\[link\]](#). You can see that the larger the current, the smaller the terminal voltage. And it is likewise true that the larger the internal resistance, the smaller the terminal voltage.

Suppose a load resistance R_{load} is connected to a voltage source, as in [\[link\]](#). Since the resistances are in series, the total resistance in the circuit is $R_{\text{load}} + r$. Thus the current is given by Ohm's law to be

Equation:

$$I = \frac{\text{emf}}{R_{\text{load}} + r}.$$



Schematic of a voltage source and its load R_{load} .

Since the internal resistance r is in series with the load, it can significantly affect the terminal voltage and current delivered to the load. (Note that the script E stands for emf.)

We see from this expression that the smaller the internal resistance r , the greater the current the voltage source supplies to its load R_{load} . As batteries are depleted, r increases. If r becomes a significant fraction of the load resistance, then the current is significantly reduced, as the following example illustrates.

Example:

Calculating Terminal Voltage, Power Dissipation, Current, and Resistance: Terminal Voltage and Load

A certain battery has a 12.0-V emf and an internal resistance of 0.100 Ω .

(a) Calculate its terminal voltage when connected to a 10.0- Ω load. (b) What is the terminal voltage when connected to a 0.500- Ω load? (c) What power does the 0.500- Ω load dissipate? (d) If the internal resistance grows to 0.500 Ω , find the current, terminal voltage, and power dissipated by a 0.500- Ω load.

Strategy

The analysis above gave an expression for current when internal resistance is taken into account. Once the current is found, the terminal voltage can be calculated using the equation $V = \text{emf} - Ir$. Once current is found, the power dissipated by a resistor can also be found.

Solution for (a)

Entering the given values for the emf, load resistance, and internal resistance into the expression above yields

Equation:

$$I = \frac{\text{emf}}{R_{\text{load}} + r} = \frac{12.0 \text{ V}}{10.1 \Omega} = 1.188 \text{ A.}$$

Enter the known values into the equation $V = \text{emf} - Ir$ to get the terminal voltage:

Equation:

$$\begin{aligned} V &= \text{emf} - Ir = 12.0 \text{ V} - (1.188 \text{ A})(0.100 \Omega) \\ &= 11.9 \text{ V.} \end{aligned}$$

Discussion for (a)

The terminal voltage here is only slightly lower than the emf, implying that $10.0\ \Omega$ is a light load for this particular battery.

Solution for (b)

Similarly, with $R_{\text{load}} = 0.500\ \Omega$, the current is

Equation:

$$I = \frac{\text{emf}}{R_{\text{load}} + r} = \frac{12.0\ \text{V}}{0.600\ \Omega} = 20.0\ \text{A}.$$

The terminal voltage is now

Equation:

$$\begin{aligned} V &= \text{emf} - Ir = 12.0\ \text{V} - (20.0\ \text{A})(0.100\ \Omega) \\ &= 10.0\ \text{V}. \end{aligned}$$

Discussion for (b)

This terminal voltage exhibits a more significant reduction compared with emf, implying $0.500\ \Omega$ is a heavy load for this battery.

Solution for (c)

The power dissipated by the $0.500\text{-}\Omega$ load can be found using the formula $P = I^2 R$. Entering the known values gives

Equation:

$$P_{\text{load}} = I^2 R_{\text{load}} = (20.0\ \text{A})^2 (0.500\ \Omega) = 2.00 \times 10^2\ \text{W}.$$

Discussion for (c)

Note that this power can also be obtained using the expressions $\frac{V^2}{R}$ or IV , where V is the terminal voltage ($10.0\ \text{V}$ in this case).

Solution for (d)

Here the internal resistance has increased, perhaps due to the depletion of the battery, to the point where it is as great as the load resistance. As before, we first find the current by entering the known values into the expression, yielding

Equation:

$$I = \frac{\text{emf}}{R_{\text{load}} + r} = \frac{12.0\ \text{V}}{1.00\ \Omega} = 12.0\ \text{A}.$$

Now the terminal voltage is

Equation:

$$\begin{aligned} V &= \text{emf} - Ir = 12.0 \text{ V} - (12.0 \text{ A})(0.500 \Omega) \\ &= 6.00 \text{ V}, \end{aligned}$$

and the power dissipated by the load is

Equation:

$$P_{\text{load}} = I^2 R_{\text{load}} = (12.0 \text{ A})^2 (0.500 \Omega) = 72.0 \text{ W}.$$

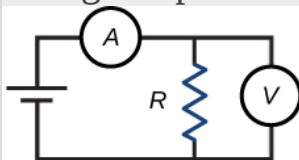
Discussion for (d)

We see that the increased internal resistance has significantly decreased terminal voltage, current, and power delivered to a load.

Note:

Applying the Science Practices: Internal Resistance

The internal resistance of a battery can be estimated using a simple activity. The circuit shown in the figure below includes a resistor R in series with a battery along with an ammeter and voltmeter to measure the current and voltage respectively.



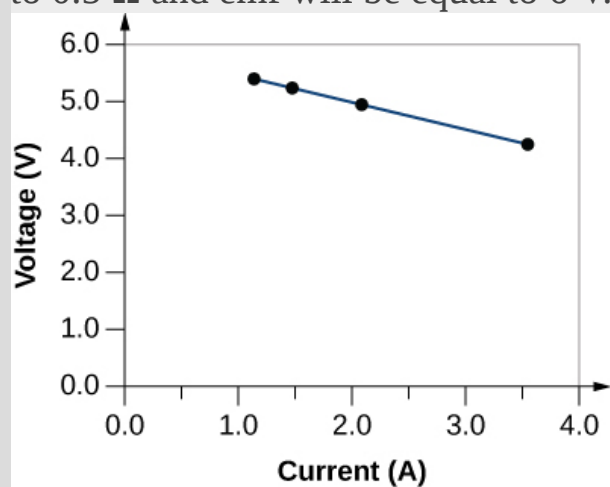
The currents and voltages measured for several R values are shown in the table below. Using the data given in the table and applying graphical analysis, determine the emf and internal resistance of the battery. Your response should clearly explain the method used to obtain the result.

Resistance	Current (A)	Voltage (V)
R_1	3.53	4.24
R_2	2.07	4.97
R_3	1.46	5.27
R_4	1.13	5.43

Note:

Answer

Plot the measured currents and voltages on a graph. The terminal voltage of a battery is equal to the emf of the battery minus the voltage drop across the internal resistance of the battery or $V = emf - Ir$. Using this linear relationship and the plotted graph, the internal resistance and emf of the battery can be found. The graph for this case is shown below. The equation is $V = -0.50I + 6.0$ and hence the internal resistance will be equal to 0.5Ω and emf will be equal to 6 V.



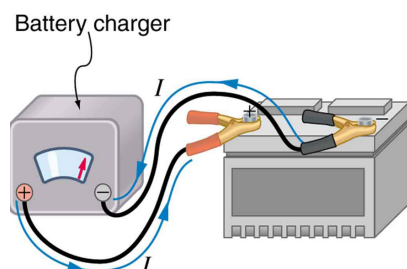
Battery testers, such as those in [\[link\]](#), use small load resistors to intentionally draw current to determine whether the terminal voltage drops below an acceptable level. They really test the internal resistance of the battery. If internal resistance is high, the battery is weak, as evidenced by its low terminal voltage.



These two battery testers measure terminal voltage under a load to determine the condition of a battery. The large device is being used by a U.S. Navy electronics technician to test large batteries aboard the aircraft carrier USS *Nimitz* and has a small resistance that can dissipate large amounts of power. (credit: U.S. Navy photo by Photographer's Mate Airman Jason A. Johnston) The small device is used on small batteries and has a digital display to indicate the acceptability of their terminal voltage. (credit: Keith Williamson)

Some batteries can be recharged by passing a current through them in the direction opposite to the current they supply to a resistance. This is done routinely in cars and batteries for small electrical appliances and electronic devices, and is represented pictorially in [\[link\]](#). The voltage output of the battery charger must be greater than the emf of the battery to reverse current

through it. This will cause the terminal voltage of the battery to be greater than the emf, since $V = \text{emf} - Ir$, and I is now negative.



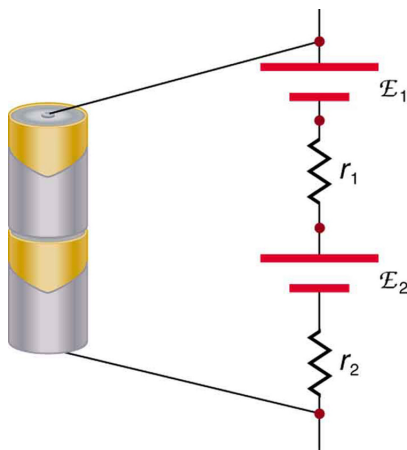
A car battery charger reverses the normal direction of current through a battery, reversing its chemical reaction and replenishing its chemical potential.

Multiple Voltage Sources

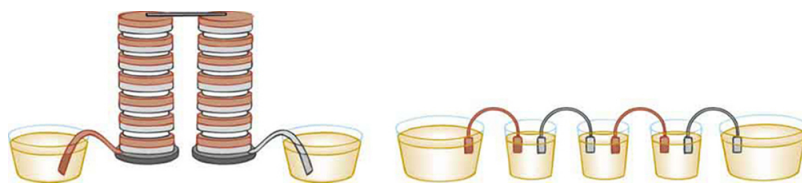
There are two voltage sources when a battery charger is used. Voltage sources connected in series are relatively simple. When voltage sources are in series, their internal resistances add and their emfs add algebraically. (See [\[link\]](#).) Series connections of voltage sources are common—for example, in flashlights, toys, and other appliances. Usually, the cells are in series in order to produce a larger total emf.

But if the cells oppose one another, such as when one is put into an appliance backward, the total emf is less, since it is the algebraic sum of the individual emfs.

A battery is a multiple connection of voltaic cells, as shown in [\[link\]](#). The disadvantage of series connections of cells is that their internal resistances add. One of the authors once owned a 1957 MGA that had two 6-V batteries in series, rather than a single 12-V battery. This arrangement produced a large internal resistance that caused him many problems in starting the engine.

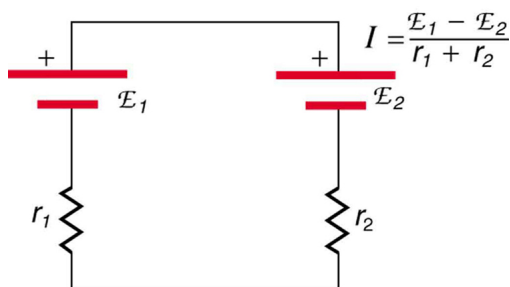


A series connection
of two voltage
sources. The emfs
(each labeled with
a script E) and
internal resistances
add, giving a total
emf of
 $\text{emf}_1 + \text{emf}_2$ and a
total internal
resistance of
 $r_1 + r_2$.



Batteries are multiple connections of individual cells, as shown in this modern rendition of an old print. Single cells, such as AA or C cells, are commonly called batteries, although this is technically incorrect.

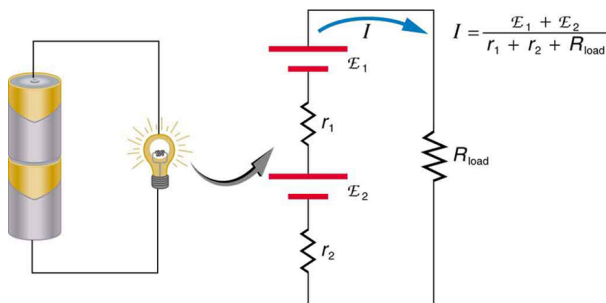
If the *series* connection of two voltage sources is made into a complete circuit with the emfs in opposition, then a current of magnitude $I = \frac{(\text{emf}_1 - \text{emf}_2)}{r_1 + r_2}$ flows. See [\[link\]](#), for example, which shows a circuit exactly analogous to the battery charger discussed above. If two voltage sources in series with emfs in the same sense are connected to a load R_{load} , as in [\[link\]](#), then $I = \frac{(\text{emf}_1 + \text{emf}_2)}{r_1 + r_2 + R_{\text{load}}}$ flows.



These two voltage sources are connected in series with their emfs in opposition. Current flows in the direction of the greater emf and is limited

to $I = \frac{(\text{emf}_1 - \text{emf}_2)}{r_1 + r_2}$ by the sum of the internal resistances. (Note that each emf is represented by script E in the figure.)

A battery charger connected to a battery is an example of such a connection. The charger must have a larger emf than the battery to reverse current through it.



This schematic represents a flashlight with two cells (voltage sources) and a single bulb (load resistance) in series.

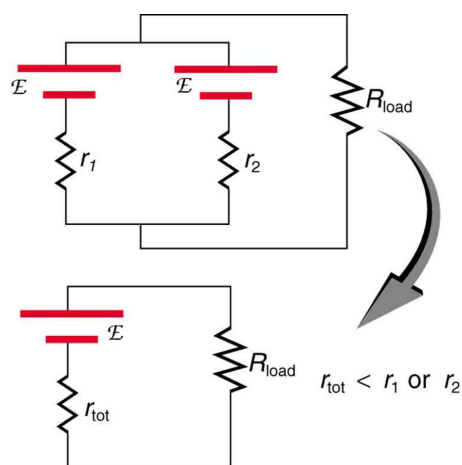
The current that flows is $I = \frac{(\text{emf}_1 + \text{emf}_2)}{r_1 + r_2 + R_{\text{load}}}$. (Note that each emf is represented by script E in the figure.)

Note:**Take-Home Experiment: Flashlight Batteries**

Find a flashlight that uses several batteries and find new and old batteries. Based on the discussions in this module, predict the brightness of the flashlight when different combinations of batteries are used. Do your predictions match what you observe? Now place new batteries in the flashlight and leave the flashlight switched on for several hours. Is the flashlight still quite bright? Do the same with the old batteries. Is the flashlight as bright when left on for the same length of time with old and new batteries? What does this say for the case when you are limited in the number of available new batteries?

[\[link\]](#) shows two voltage sources with identical emfs in parallel and connected to a load resistance. In this simple case, the total emf is the same as the individual emfs. But the total internal resistance is reduced, since the internal resistances are in parallel. The parallel connection thus can produce a larger current.

Here, $I = \frac{\text{emf}}{(r_{\text{tot}} + R_{\text{load}})}$ flows through the load, and r_{tot} is less than those of the individual batteries. For example, some diesel-powered cars use two 12-V batteries in parallel; they produce a total emf of 12 V but can deliver the larger current needed to start a diesel engine.



Two voltage sources
with identical emfs
(each labeled by script
E) connected in
parallel produce the
same emf but have a
smaller total internal
resistance than the
individual sources.
Parallel combinations
are often used to
deliver more current.
Here $I = \frac{\text{emf}}{(r_{\text{tot}} + R_{\text{load}})}$
flows through the
load.

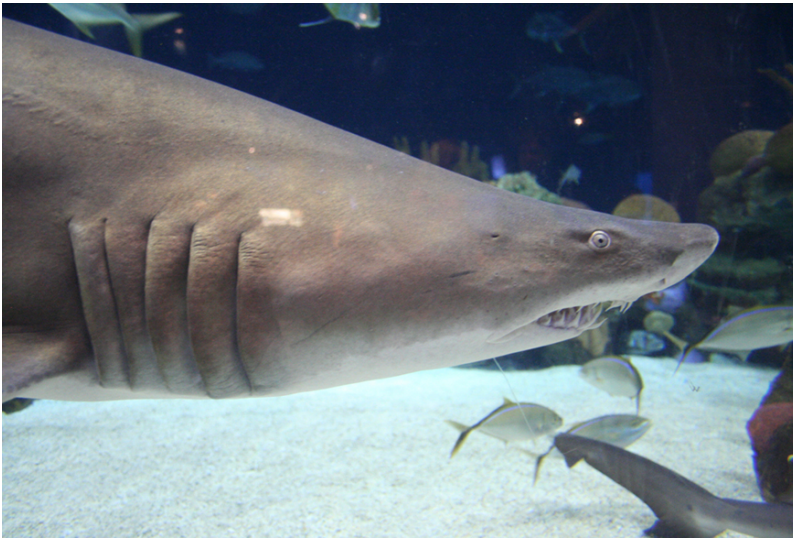
Animals as Electrical Detectors

A number of animals both produce and detect electrical signals. Fish, sharks, platypuses, and echidnas (spiny anteaters) all detect electric fields generated by nerve activity in prey. Electric eels produce their own emf through biological cells (electric organs) called electroplaques, which are arranged in both series and parallel as a set of batteries.

Electroplaques are flat, disk-like cells; those of the electric eel have a voltage of 0.15 V across each one. These cells are usually located toward the head or tail of the animal, although in the case of the electric eel, they are found along the entire body. The electroplaques in the South American eel are arranged in 140 rows, with each row stretching horizontally along the body and containing 5,000 electroplaques. This can yield an emf of approximately 600 V, and a current of 1 A—deadly.

The mechanism for detection of external electric fields is similar to that for producing nerve signals in the cell through depolarization and

repolarization—the movement of ions across the cell membrane. Within the fish, weak electric fields in the water produce a current in a gel-filled canal that runs from the skin to sensing cells, producing a nerve signal. The Australian platypus, one of the very few mammals that lay eggs, can detect fields of $30 \frac{\text{mV}}{\text{m}}$, while sharks have been found to be able to sense a field in their snouts as small as $100 \frac{\text{mV}}{\text{m}}$ ([link](#)). Electric eels use their own electric fields produced by the electroplaques to stun their prey or enemies.



Sand tiger sharks (*Carcharias taurus*), like this one at the Minnesota Zoo, use electroreceptors in their snouts to locate prey. (credit: Jim Winstead, Flickr)

Solar Cell Arrays

Another example dealing with multiple voltage sources is that of combinations of solar cells—wired in both series and parallel combinations to yield a desired voltage and current. Photovoltaic generation (PV), the conversion of sunlight directly into electricity, is based upon the

photoelectric effect, in which photons hitting the surface of a solar cell create an electric current in the cell.

Most solar cells are made from pure silicon—either as single-crystal silicon, or as a thin film of silicon deposited upon a glass or metal backing. Most single cells have a voltage output of about 0.5 V, while the current output is a function of the amount of sunlight upon the cell (the incident solar radiation—the insolation). Under bright noon sunlight, a current of about 100 mA/cm^2 of cell surface area is produced by typical single-crystal cells.

Individual solar cells are connected electrically in modules to meet electrical-energy needs. They can be wired together in series or in parallel—connected like the batteries discussed earlier. A solar-cell array or module usually consists of between 36 and 72 cells, with a power output of 50 W to 140 W.

The output of the solar cells is direct current. For most uses in a home, AC is required, so a device called an inverter must be used to convert the DC to AC. Any extra output can then be passed on to the outside electrical grid for sale to the utility.

Note:

Take-Home Experiment: Virtual Solar Cells

One can assemble a “virtual” solar cell array by using playing cards, or business or index cards, to represent a solar cell. Combinations of these cards in series and/or parallel can model the required array output. Assume each card has an output of 0.5 V and a current (under bright light) of 2 A. Using your cards, how would you arrange them to produce an output of 6 A at 3 V (18 W)?

Suppose you were told that you needed only 18 W (but no required voltage). Would you need more cards to make this arrangement?

Test Prep for AP Courses

Exercise:**Problem:**

Suppose there are two voltage sources – Sources A and B – with the same emfs but different internal resistances, i.e., the internal resistance of Source A is lower than Source B. If they both supply the same current in their circuits, which of the following statements is true?

- a. External resistance in Source A's circuit is more than Source B's circuit.
- b. External resistance in Source A's circuit is less than Source B's circuit.
- c. External resistance in Source A's circuit is the same as Source B's circuit.
- d. The relationship between external resistances in the two circuits can't be determined.

Exercise:**Problem:**

Calculate the internal resistance of a voltage source if the terminal voltage of the source increases by 1 V when the current supplied decreases by 4 A? Suppose this source is connected in series (in the same direction) to another source with a different voltage but same internal resistance. What will be the total internal resistance? How will the total internal resistance change if the sources are connected in the opposite direction?

Solution:

0.25 Ω , 0.50 Ω , no change

Section Summary

- All voltage sources have two fundamental parts—a source of electrical energy that has a characteristic electromotive force (emf), and an

internal resistance r .

- The emf is the potential difference of a source when no current is flowing.
- The numerical value of the emf depends on the source of potential difference.
- The internal resistance r of a voltage source affects the output voltage when a current flows.
- The voltage output of a device is called its terminal voltage V and is given by $V = \text{emf} - Ir$, where I is the electric current and is positive when flowing away from the positive terminal of the voltage source.
- When multiple voltage sources are in series, their internal resistances add and their emfs add algebraically.
- Solar cells can be wired in series or parallel to provide increased voltage or current, respectively.

Conceptual Questions

Exercise:

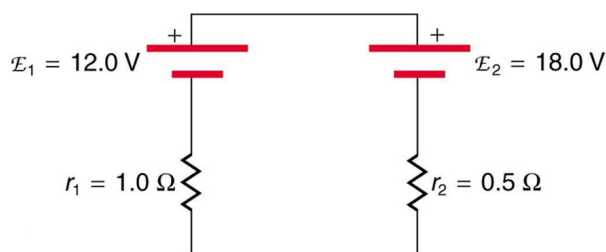
Problem:

Is every emf a potential difference? Is every potential difference an emf? Explain.

Exercise:

Problem:

Explain which battery is doing the charging and which is being charged in [\[link\]](#).



Exercise:**Problem:**

Given a battery, an assortment of resistors, and a variety of voltage and current measuring devices, describe how you would determine the internal resistance of the battery.

Exercise:**Problem:**

Two different 12-V automobile batteries on a store shelf are rated at 600 and 850 “cold cranking amps.” Which has the smallest internal resistance?

Exercise:**Problem:**

What are the advantages and disadvantages of connecting batteries in series? In parallel?

Exercise:**Problem:**

Semitractor trucks use four large 12-V batteries. The starter system requires 24 V, while normal operation of the truck’s other electrical components utilizes 12 V. How could the four batteries be connected to produce 24 V? To produce 12 V? Why is 24 V better than 12 V for starting the truck’s engine (a very heavy load)?

Problem Exercises**Exercise:**

Problem:

Standard automobile batteries have six lead-acid cells in series, creating a total emf of 12.0 V. What is the emf of an individual lead-acid cell?

Solution:

2.00 V

Exercise:**Problem:**

Carbon-zinc dry cells (sometimes referred to as non-alkaline cells) have an emf of 1.54 V, and they are produced as single cells or in various combinations to form other voltages. (a) How many 1.54-V cells are needed to make the common 9-V battery used in many small electronic devices? (b) What is the actual emf of the approximately 9-V battery? (c) Discuss how internal resistance in the series connection of cells will affect the terminal voltage of this approximately 9-V battery.

Exercise:**Problem:**

What is the output voltage of a 3.0000-V lithium cell in a digital wristwatch that draws 0.300 mA, if the cell's internal resistance is $2.00\ \Omega$?

Solution:

2.9994 V

Exercise:

Problem:

(a) What is the terminal voltage of a large 1.54-V carbon-zinc dry cell used in a physics lab to supply 2.00 A to a circuit, if the cell's internal resistance is $0.100\ \Omega$? (b) How much electrical power does the cell produce? (c) What power goes to its load?

Exercise:**Problem:**

What is the internal resistance of an automobile battery that has an emf of 12.0 V and a terminal voltage of 15.0 V while a current of 8.00 A is charging it?

Solution:

$0.375\ \Omega$

Exercise:**Problem:**

(a) Find the terminal voltage of a 12.0-V motorcycle battery having a $0.600\text{-}\Omega$ internal resistance, if it is being charged by a current of 10.0 A. (b) What is the output voltage of the battery charger?

Exercise:**Problem:**

A car battery with a 12-V emf and an internal resistance of $0.050\ \Omega$ is being charged with a current of 60 A. Note that in this process the battery is being charged. (a) What is the potential difference across its terminals? (b) At what rate is thermal energy being dissipated in the battery? (c) At what rate is electric energy being converted to chemical energy? (d) What are the answers to (a) and (b) when the battery is used to supply 60 A to the starter motor?

Exercise:

Problem:

The hot resistance of a flashlight bulb is $2.30\ \Omega$, and it is run by a 1.58-V alkaline cell having a $0.100\text{-}\Omega$ internal resistance. (a) What current flows? (b) Calculate the power supplied to the bulb using $I^2 R_{\text{bulb}}$. (c) Is this power the same as calculated using $\frac{V^2}{R_{\text{bulb}}}$?

Solution:

(a) 0.658 A

(b) 0.997 W

(c) 0.997 W; yes

Exercise:**Problem:**

The label on a portable radio recommends the use of rechargeable nickel-cadmium cells (nicads), although they have a 1.25-V emf while alkaline cells have a 1.58-V emf. The radio has a $3.20\text{-}\Omega$ resistance. (a) Draw a circuit diagram of the radio and its batteries. Now, calculate the power delivered to the radio. (b) When using Nicad cells each having an internal resistance of $0.0400\ \Omega$. (c) When using alkaline cells each having an internal resistance of $0.200\ \Omega$. (d) Does this difference seem significant, considering that the radio's effective resistance is lowered when its volume is turned up?

Exercise:

Problem:

An automobile starter motor has an equivalent resistance of $0.0500\ \Omega$ and is supplied by a 12.0-V battery with a $0.0100\text{-}\Omega$ internal resistance. (a) What is the current to the motor? (b) What voltage is applied to it? (c) What power is supplied to the motor? (d) Repeat these calculations for when the battery connections are corroded and add $0.0900\ \Omega$ to the circuit. (Significant problems are caused by even small amounts of unwanted resistance in low-voltage, high-current applications.)

Solution:

- (a) 200 A
- (b) 10.0 V
- (c) 2.00 kW
- (d) $0.1000\ \Omega$; 80.0 A, 4.0 V, 320 W

Exercise:**Problem:**

A child's electronic toy is supplied by three 1.58-V alkaline cells having internal resistances of $0.0200\ \Omega$ in series with a 1.53-V carbon-zinc dry cell having a $0.100\text{-}\Omega$ internal resistance. The load resistance is $10.0\ \Omega$. (a) Draw a circuit diagram of the toy and its batteries. (b) What current flows? (c) How much power is supplied to the load? (d) What is the internal resistance of the dry cell if it goes bad, resulting in only 0.500 W being supplied to the load?

Exercise:

Problem:

(a) What is the internal resistance of a voltage source if its terminal voltage drops by 2.00 V when the current supplied increases by 5.00 A? (b) Can the emf of the voltage source be found with the information supplied?

Solution:

(a) $0.400\ \Omega$

(b) No, there is only one independent equation, so only r can be found.

Exercise:**Problem:**

A person with body resistance between his hands of $10.0\ \text{k}\Omega$ accidentally grasps the terminals of a 20.0-kV power supply. (Do NOT do this!) (a) Draw a circuit diagram to represent the situation. (b) If the internal resistance of the power supply is $2000\ \Omega$, what is the current through his body? (c) What is the power dissipated in his body? (d) If the power supply is to be made safe by increasing its internal resistance, what should the internal resistance be for the maximum current in this situation to be 1.00 mA or less? (e) Will this modification compromise the effectiveness of the power supply for driving low-resistance devices? Explain your reasoning.

Exercise:**Problem:**

Electric fish generate current with biological cells called electroplaques, which are physiological emf devices. The electroplaques in the South American eel are arranged in 140 rows, each row stretching horizontally along the body and each containing 5000 electroplaques. Each electroplaque has an emf of 0.15 V and internal resistance of $0.25\ \Omega$. If the water surrounding the fish has resistance of $800\ \Omega$, how much current can the eel produce in water from near its head to near its tail?

Exercise:**Problem: Integrated Concepts**

A 12.0-V emf automobile battery has a terminal voltage of 16.0 V when being charged by a current of 10.0 A. (a) What is the battery's internal resistance? (b) What power is dissipated inside the battery? (c) At what rate (in $^{\circ}\text{C}/\text{min}$) will its temperature increase if its mass is 20.0 kg and it has a specific heat of $0.300 \text{ kcal}/\text{kg} \cdot ^{\circ}\text{C}$, assuming no heat escapes?

Exercise:**Problem: Unreasonable Results**

A 1.58-V alkaline cell with a $0.200\text{-}\Omega$ internal resistance is supplying 8.50 A to a load. (a) What is its terminal voltage? (b) What is the value of the load resistance? (c) What is unreasonable about these results? (d) Which assumptions are unreasonable or inconsistent?

Solution:

(a) -0.120 V

(b) $-1.41 \times 10^{-2} \Omega$

(c) Negative terminal voltage; negative load resistance.

(d) The assumption that such a cell could provide 8.50 A is inconsistent with its internal resistance.

Exercise:**Problem: Unreasonable Results**

(a) What is the internal resistance of a 1.54-V dry cell that supplies 1.00 W of power to a $15.0\text{-}\Omega$ bulb? (b) What is unreasonable about this result? (c) Which assumptions are unreasonable or inconsistent?

Glossary

electromotive force (emf)

the potential difference of a source of electricity when no current is flowing; measured in volts

internal resistance

the amount of resistance within the voltage source

potential difference

the difference in electric potential between two points in an electric circuit, measured in volts

terminal voltage

the voltage measured across the terminals of a source of potential difference

Kirchhoff's Rules

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Analyze a complex circuit using Kirchhoff's rules, applying the conventions for determining the correct signs of various terms.

The information presented in this section supports the following AP® learning objectives and science practices:

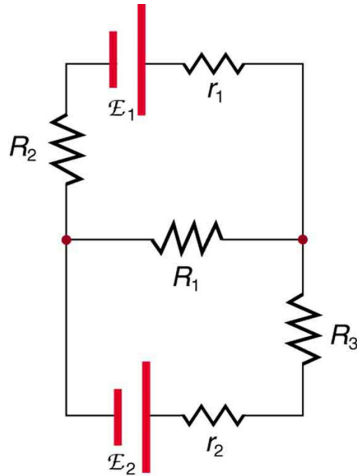
- **5.B.9.1** The student is able to construct or interpret a graph of the energy changes within an electrical circuit with only a single battery and resistors in series and/or in, at most, one parallel branch as an application of the conservation of energy (Kirchhoff's loop rule). **(S.P. 1.1, 1.4)**
- **5.B.9.2** The student is able to apply conservation of energy concepts to the design of an experiment that will demonstrate the validity of Kirchhoff's loop rule in a circuit with only a battery and resistors either in series or in, at most, one pair of parallel branches. **(S.P. 4.2, 6.4, 7.2)**
- **5.B.9.3** The student is able to apply conservation of energy (Kirchhoff's loop rule) in calculations involving the total electric potential difference for complete circuit loops with only a single battery and resistors in series and/or in, at most, one parallel branch. **(S.P. 2.2, 6.4, 7.2)**
- **5.B.9.4** The student is able to analyze experimental data including an analysis of experimental uncertainty that will demonstrate the validity of Kirchhoff's loop rule. **(S.P. 5.1)**
- **5.B.9.5** The student is able to use conservation of energy principles (Kirchhoff's loop rule) to describe and make

predictions regarding electrical potential difference, charge, and current in steady-state circuits composed of various combinations of resistors and capacitors. **(S.P. 6.4)**

- **5.C.3.1** The student is able to apply conservation of electric charge (Kirchhoff's junction rule) to the comparison of electric current in various segments of an electrical circuit with a single battery and resistors in series and in, at most, one parallel branch and predict how those values would change if configurations of the circuit are changed. **(S.P. 6.4, 7.2)**
- **5.C.3.2** The student is able to design an investigation of an electrical circuit with one or more resistors in which evidence of conservation of electric charge can be collected and analyzed. **(S.P. 4.1, 4.2, 5.1)**
- **5.C.3.3** The student is able to use a description or schematic diagram of an electrical circuit to calculate unknown values of current in various segments or branches of the circuit. **(S.P. 1.4, 2.2)**
- **5.C.3.4** The student is able to predict or explain current values in series and parallel arrangements of resistors and other branching circuits using Kirchhoff's junction rule and relate the rule to the law of charge conservation. **(S.P. 6.4, 7.2)**
- **5.C.3.5** The student is able to determine missing values and direction of electric current in branches of a circuit with resistors and NO capacitors from values and directions of current in other branches of the circuit through appropriate selection of nodes and application of the junction rule. **(S.P. 1.4, 2.2)**

Many complex circuits, such as the one in [\[link\]](#), cannot be analyzed with the series-parallel techniques developed in [Resistors in Series and Parallel](#) and [Electromotive Force: Terminal Voltage](#). There are, however, two circuit analysis rules that can be used to analyze any circuit, simple or complex. These rules are special cases of the laws of conservation of charge and

conservation of energy. The rules are known as **Kirchhoff's rules**, after their inventor Gustav Kirchhoff (1824–1887).



This circuit cannot be reduced to a combination of series and parallel connections. Kirchhoff's rules, special applications of the laws of conservation of charge and energy, can be used to analyze it. (Note: The script E in the figure represents electromotive force, emf.)

Note:**Kirchhoff's Rules**

- Kirchhoff's first rule—the junction rule. The sum of all currents entering a junction must equal the sum of all currents leaving the junction.
- Kirchhoff's second rule—the loop rule. The algebraic sum of changes in potential around any closed circuit path (loop) must be zero.

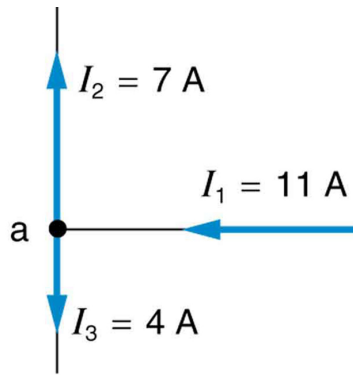
Explanations of the two rules will now be given, followed by problem-solving hints for applying Kirchhoff's rules, and a worked example that uses them.

Kirchhoff's First Rule

Kirchhoff's first rule (the **junction rule**) is an application of the conservation of charge to a junction; it is illustrated in [\[link\]](#). Current is the flow of charge, and charge is conserved; thus, whatever charge flows into the junction must flow out. Kirchhoff's first rule requires that $I_1 = I_2 + I_3$ (see figure). Equations like this can and will be used to analyze circuits and to solve circuit problems.

Note:**Making Connections: Conservation Laws**

Kirchhoff's rules for circuit analysis are applications of **conservation laws** to circuits. The first rule is the application of conservation of charge, while the second rule is the application of conservation of energy. Conservation laws, even used in a specific application, such as circuit analysis, are so basic as to form the foundation of that application.



$$I_1 = I_2 + I_3$$

The junction rule. The diagram shows an example of Kirchhoff's first rule where the sum of the currents into a junction equals the sum of the currents out of a junction. In this case, the current going into the junction splits and comes out as two currents, so that

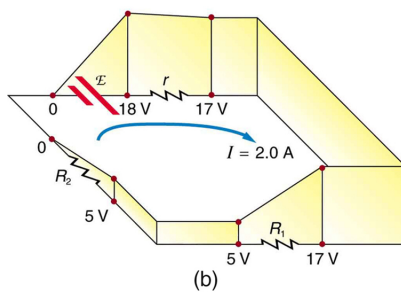
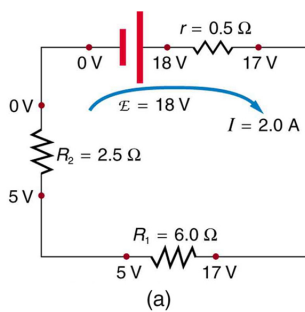
$$I_1 = I_2 + I_3.$$

Here I_1 must be 11 A, since I_2 is 7 A and I_3 is 4 A.

Kirchhoff's Second Rule

Kirchhoff's second rule (the **loop rule**) is an application of conservation of energy. The loop rule is stated in terms of potential, V , rather than potential energy, but the two are related since $PE_{\text{elec}} = qV$. Recall that **emf** is the potential difference of a source when no current is flowing. In a closed loop, whatever energy is supplied by emf must be transferred into other forms by devices in the loop, since there are no other ways in which energy can be transferred into or out of the circuit. [\[link\]](#) illustrates the changes in potential in a simple series circuit loop.

Kirchhoff's second rule requires $\text{emf} - Ir - IR_1 - IR_2 = 0$. Rearranged, this is $\text{emf} = Ir + IR_1 + IR_2$, which means the emf equals the sum of the IR (voltage) drops in the loop.



The loop rule. An example of Kirchhoff's second rule where the sum of the changes in potential around a

closed loop must be zero. (a) In this standard schematic of a simple series circuit, the emf supplies 18 V, which is reduced to zero by the resistances, with 1 V across the internal resistance, and 12 V and 5 V across the two load resistances, for a total of 18 V. (b) This perspective view represents the potential as something like a roller coaster, where charge is raised in potential by the emf and lowered by the resistances. (Note that the script E stands for emf.)

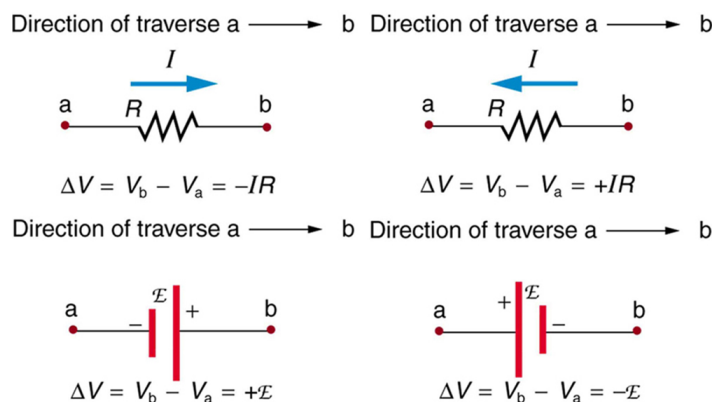
Applying Kirchhoff's Rules

By applying Kirchhoff's rules, we generate equations that allow us to find the unknowns in circuits. The unknowns may be currents, emfs, or resistances. Each time a rule is applied, an equation is produced. If there are as many independent equations as unknowns, then the problem can be solved. There are two decisions you must make when applying Kirchhoff's

rules. These decisions determine the signs of various quantities in the equations you obtain from applying the rules.

1. When applying Kirchhoff's first rule, the junction rule, you must label the current in each branch and decide in what direction it is going. For example, in [\[link\]](#), [\[link\]](#), and [\[link\]](#), currents are labeled I_1 , I_2 , I_3 , and I , and arrows indicate their directions. There is no risk here, for if you choose the wrong direction, the current will be of the correct magnitude but negative.
2. When applying Kirchhoff's second rule, the loop rule, you must identify a closed loop and decide in which direction to go around it, clockwise or counterclockwise. For example, in [\[link\]](#) the loop was traversed in the same direction as the current (clockwise). Again, there is no risk; going around the circuit in the opposite direction reverses the sign of every term in the equation, which is like multiplying both sides of the equation by -1 .

[\[link\]](#) and the following points will help you get the plus or minus signs right when applying the loop rule. Note that the resistors and emfs are traversed by going from a to b. In many circuits, it will be necessary to construct more than one loop. In traversing each loop, one needs to be consistent for the sign of the change in potential. (See [\[link\]](#).)



Each of these resistors and voltage sources is traversed from a to b. The potential changes are shown beneath

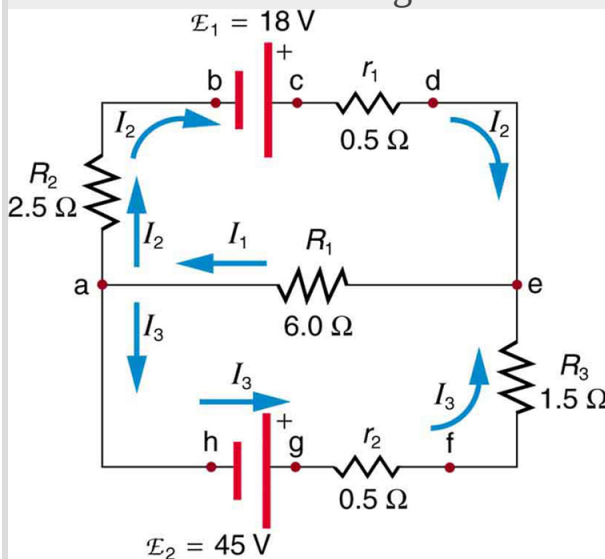
each element and are explained in the text. (Note that the script E stands for emf.)

- When a resistor is traversed in the same direction as the current, the change in potential is $-IR$. (See [\[link\]](#).)
- When a resistor is traversed in the direction opposite to the current, the change in potential is $+IR$. (See [\[link\]](#).)
- When an emf is traversed from $-$ to $+$ (the same direction it moves positive charge), the change in potential is $+\text{emf}$. (See [\[link\]](#).)
- When an emf is traversed from $+$ to $-$ (opposite to the direction it moves positive charge), the change in potential is $-\text{emf}$. (See [\[link\]](#).)

Example:

Calculating Current: Using Kirchhoff's Rules

Find the currents flowing in the circuit in [\[link\]](#).



This circuit is similar to that in [\[link\]](#), but the resistances and emfs are specified. (Each emf is denoted by script E.) The

currents in each branch are labeled and assumed to move in the directions shown. This example uses Kirchhoff's rules to find the currents.

Strategy

This circuit is sufficiently complex that the currents cannot be found using Ohm's law and the series-parallel techniques—it is necessary to use Kirchhoff's rules. Currents have been labeled I_1 , I_2 , and I_3 in the figure and assumptions have been made about their directions. Locations on the diagram have been labeled with letters a through h. In the solution we will apply the junction and loop rules, seeking three independent equations to allow us to solve for the three unknown currents.

Solution

We begin by applying Kirchhoff's first or junction rule at point a. This gives

Equation:

$$I_1 = I_2 + I_3,$$

since I_1 flows into the junction, while I_2 and I_3 flow out. Applying the junction rule at e produces exactly the same equation, so that no new information is obtained. This is a single equation with three unknowns—three independent equations are needed, and so the loop rule must be applied.

Now we consider the loop abcdea. Going from a to b, we traverse R_2 in the same (assumed) direction of the current I_2 , and so the change in potential is $-I_2R_2$. Then going from b to c, we go from $-$ to $+$, so that the change in potential is $+\text{emf}_1$. Traversing the internal resistance r_1 from c to d gives $-I_2r_1$. Completing the loop by going from d to a again traverses a resistor in the same direction as its current, giving a change in potential of $-I_1R_1$.

The loop rule states that the changes in potential sum to zero. Thus,

Equation:

$$-I_2 R_2 + \text{emf}_1 - I_2 r_1 - I_1 R_1 = -I_2(R_2 + r_1) + \text{emf}_1 - I_1 R_1 = 0.$$

Substituting values from the circuit diagram for the resistances and emf, and canceling the ampere unit gives

Equation:

$$-3I_2 + 18 - 6I_1 = 0.$$

Now applying the loop rule to aefgha (we could have chosen abcdefgha as well) similarly gives

Equation:

$$+ I_1 R_1 + I_3 R_3 + I_3 r_2 - \text{emf}_2 = +I_1 R_1 + I_3(R_3 + r_2) - \text{emf}_2 = 0.$$

Note that the signs are reversed compared with the other loop, because elements are traversed in the opposite direction. With values entered, this becomes

Equation:

$$+ 6I_1 + 2I_3 - 45 = 0.$$

These three equations are sufficient to solve for the three unknown currents. First, solve the second equation for I_2 :

Equation:

$$I_2 = 6 - 2I_1.$$

Now solve the third equation for I_3 :

Equation:

$$I_3 = 22.5 - 3I_1.$$

Substituting these two new equations into the first one allows us to find a value for I_1 :

Equation:

$$I_1 = I_2 + I_3 = (6 - 2I_1) + (22.5 - 3I_1) = 28.5 - 5I_1.$$

Combining terms gives

Equation:

$$6I_1 = 28.5, \text{ and}$$

Equation:

$$I_1 = 4.75 \text{ A.}$$

Substituting this value for I_1 back into the fourth equation gives

Equation:

$$I_2 = 6 - 2I_1 = 6 - 9.50$$

Equation:

$$I_2 = -3.50 \text{ A.}$$

The minus sign means I_2 flows in the direction opposite to that assumed in [\[link\]](#).

Finally, substituting the value for I_1 into the fifth equation gives

Equation:

$$I_3 = 22.5 - 3I_1 = 22.5 - 14.25$$

Equation:

$$I_3 = 8.25 \text{ A.}$$

Discussion

Just as a check, we note that indeed $I_1 = I_2 + I_3$. The results could also have been checked by entering all of the values into the equation for the abcdefgha loop.

Note:

Problem-Solving Strategies for Kirchhoff's Rules

1. Make certain there is a clear circuit diagram on which you can label all known and unknown resistances, emfs, and currents. If a current is unknown, you must assign it a direction. This is necessary for determining the signs of potential changes. If you assign the direction

incorrectly, the current will be found to have a negative value—no harm done.

2. Apply the junction rule to any junction in the circuit. Each time the junction rule is applied, you should get an equation with a current that does not appear in a previous application—if not, then the equation is redundant.
3. Apply the loop rule to as many loops as needed to solve for the unknowns in the problem. (There must be as many independent equations as unknowns.) To apply the loop rule, you must choose a direction to go around the loop. Then carefully and consistently determine the signs of the potential changes for each element using the four bulleted points discussed above in conjunction with [\[link\]](#).
4. Solve the simultaneous equations for the unknowns. This may involve many algebraic steps, requiring careful checking and rechecking.
5. Check to see whether the answers are reasonable and consistent. The numbers should be of the correct order of magnitude, neither exceedingly large nor vanishingly small. The signs should be reasonable—for example, no resistance should be negative. Check to see that the values obtained satisfy the various equations obtained from applying the rules. The currents should satisfy the junction rule, for example.

The material in this section is correct in theory. We should be able to verify it by making measurements of current and voltage. In fact, some of the devices used to make such measurements are straightforward applications of the principles covered so far and are explored in the next modules. As we shall see, a very basic, even profound, fact results—making a measurement alters the quantity being measured.

Exercise:

Check Your Understanding

Problem:

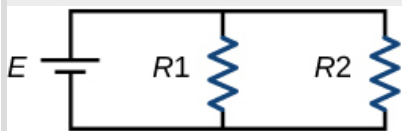
Can Kirchhoff's rules be applied to simple series and parallel circuits or are they restricted for use in more complicated circuits that are not combinations of series and parallel?

Solution:

Kirchhoff's rules can be applied to any circuit since they are applications to circuits of two conservation laws. Conservation laws are the most broadly applicable principles in physics. It is usually mathematically simpler to use the rules for series and parallel in simpler circuits so we emphasize Kirchhoff's rules for use in more complicated situations. But the rules for series and parallel can be derived from Kirchhoff's rules. Moreover, Kirchhoff's rules can be expanded to devices other than resistors and emfs, such as capacitors, and are one of the basic analysis devices in circuit analysis.

Note:**Making Connections: Parallel Resistors**

A simple circuit shown below – with two parallel resistors and a voltage source – is implemented in a laboratory experiment with $\mathcal{E} = 6.00 \pm 0.02 \text{ V}$ and $R_1 = 4.8 \pm 0.1 \, \Omega$ and $R_2 = 9.6 \pm 0.1 \, \Omega$. The values include an allowance for experimental uncertainties as they cannot be measured with perfect certainty. For example if you measure the value for a resistor a few times, you may get slightly different results. Hence values are expressed with some level of uncertainty.



In the laboratory experiment the currents measured in the two resistors are $I_1 = 1.27 \text{ A}$ and $I_2 = 0.62 \text{ A}$ respectively. Let us examine these values using Kirchhoff's laws.

For the two loops,

$$E - I_1 R_1 = 0 \text{ or } I_1 = E/R_1$$

$$E - I_2 R_2 = 0 \text{ or } I_2 = E/R_2$$

Converting the given uncertainties for voltage and resistances into percentages, we get

$$E = 6.00 \text{ V} \pm 0.33\%$$

$$R_1 = 4.8 \, \Omega \pm 2.08\%$$

$$R_2 = 9.6 \, \Omega \pm 1.04\%$$

We now find the currents for the two loops. While the voltage is divided by the resistance to find the current, uncertainties in voltage and resistance are directly added to find the uncertainty in the current value.

$$I_1 = (6.00/4.8) \pm (0.33\% + 2.08\%)$$

$$= 1.25 \pm 2.4\%$$

$$= 1.25 \pm 0.03 \text{ A}$$

$$I_2 = (6.00/9.6) \pm (0.33\% + 1.04\%)$$

$$= 0.63 \pm 1.4\%$$

$$= 0.63 \pm 0.01 \text{ A}$$

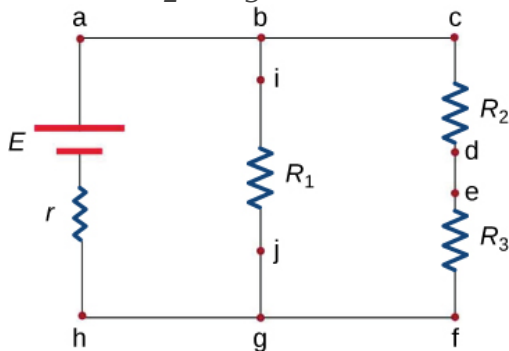
Finally you can check that the two measured values in this case are within the uncertainty ranges found for the currents. However there can also be additional experimental uncertainty in the measurements of currents.

Test Prep for AP Courses

Exercise:

Problem:

An experiment was set up with the circuit diagram shown. Assume $R_1 = 10 \, \Omega$, $R_2 = R_3 = 5 \, \Omega$, $r = 0 \, \Omega$ and $E = 6 \text{ V}$.



One of the steps to examine the set-up is to test points with the
a. same potential. Which of the following points can be tested?

- a. Points b , c and d .
- b. Points d , e and f .
- c. Points f , h and j .
- d. Points a , h and i .

At which three points should the currents be measured so that
b. Kirchhoff's junction rule can be directly confirmed?

- a. Points b , c and d .
- b. Points d , e and f .
- c. Points f , h and j .
- d. Points a , h and i .

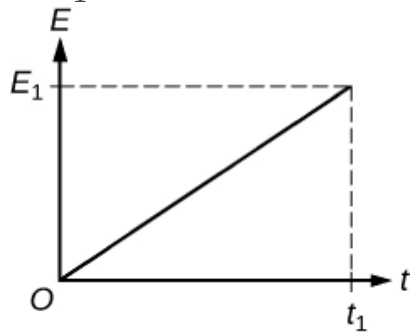
If the current in the branch with the voltage source is upward and
currents in the other two branches are downward, i.e. $I_a = I_i + I_c$,
c. identify which of the following can be true? Select *two* answers.

- a. $I_i = I_j - I_f$
- b. $I_e = I_h - I_i$
- c. $I_c = I_j - I_a$
- d. $I_d = I_h - I_j$

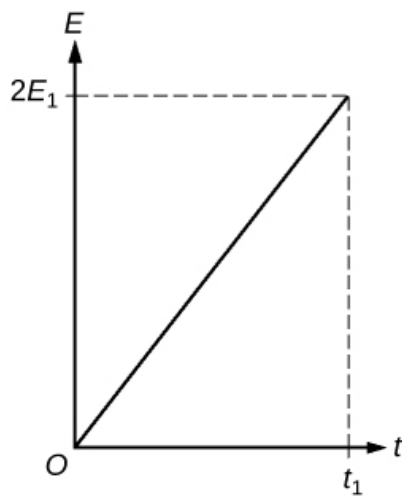
The measurements reveal that the current through R_1 is 0.5 A and
 R_3 is 0.6 A. Based on your knowledge of Kirchhoff's laws, confirm
d. which of the following statements are true.

- a. The measured current for R_1 is correct but for R_3 is incorrect.
- b. The measured current for R_3 is correct but for R_1 is incorrect.
- c. Both the measured currents are correct.
- d. Both the measured currents are incorrect.

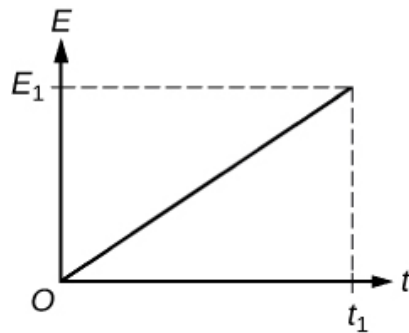
The graph shown in the following figure is the energy dissipated
e. at R_1 as a function of time.



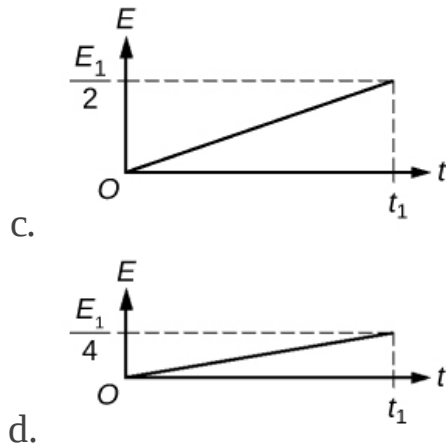
Which of the following shows the graph for energy dissipated at
 R_2 as a function of time?



a.



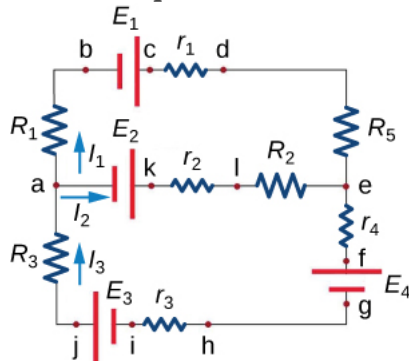
b.



Exercise:

Problem:

For this question, consider the circuit shown in the following figure.



Assuming that none of the three currents (I_1 , I_2 , and I_3) are equal
a. to zero, which of the following statements is false?

- $I_3 = I_1 + I_2$ at point a .
- $I_2 = I_3 - I_1$ at point e .
- The current through R_3 is equal to the current through R_5 .
- The current through R_1 is equal to the current through R_5 .

b. Which of the following statements is true?

- $E_1 + E_2 + I_1 R_1 - I_2 R_2 + I_1 r_1 - I_2 r_2 + I_1 R_5 = 0$

- b. $-E_1 + E_2 + I_1R_1 - I_2R_2 + I_1r_1 - I_2r_2 - I_1R_5 = 0$
- c. $E_1 - E_2 - I_1R_1 + I_2R_2 - I_1r_1 + I_2r_2 - I_1R_5 = 0$
- d. $E_1 + E_2 - I_1R_1 + I_2R_2 - I_1r_1 + I_2r_2 + I_1R_5 = 0$

If $I_1 = 5$ A and $I_3 = -2$ A, which of the following statements is false?

- a. The current through R_1 will flow from a to b and will be equal to 5 A.
- b. The current through R_3 will flow from a to j and will be equal to 2 A.
- c. The current through R_5 will flow from d to e and will be equal to 5 A.
- d. None of the above.

d. If $I_1 = 5$ A and $I_3 = -2$ A, I_2 will be equal to

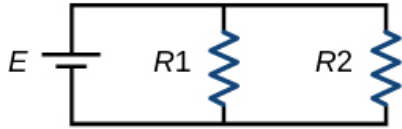
- a. 3 A
- b. -3 A
- c. 7 A
- d. -7 A

Solution:

- a. (c)
- b. (c)
- c. (d)
- d. (d)

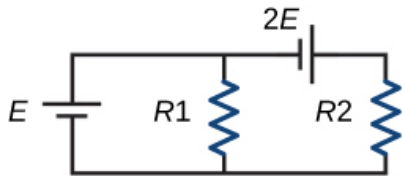
Exercise:

Problem:



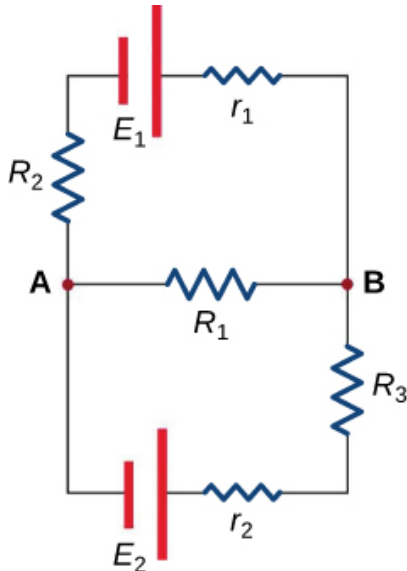
In an experiment this circuit is set up. Three ammeters are used to record the currents in the three vertical branches (with R_1 , R_2 , and E). The readings of the ammeters in the resistor branches (i.e. currents in R_1 and R_2) are 2 A and 3 A respectively.

- Find the equation obtained by applying Kirchhoff's loop rule in the loop involving R_1 and R_2 .
- What will be the reading of the third ammeter (i.e. the branch with E)? If E were replaced by $3E$, how would this reading change?
- If the original circuit is modified by adding another voltage source (as shown in the following circuit), find the readings of the three ammeters.



Exercise:

Problem:



In this circuit, assume the currents through R_1 , R_2 and R_3 are I_1 , I_2 and I_3 respectively and all are flowing in the clockwise direction.

- Find the equation obtained by applying Kirchhoff's junction rule at point A.
- Find the equations obtained by applying Kirchhoff's loop rule in the upper and lower loops.
- Assume $R_1 = R_2 = 6\ \Omega$, $R_3 = 12\ \Omega$, $r_1 = r_2 = 0\ \Omega$, $E_1 = 6\ \text{V}$ and $E_2 = 4\ \text{V}$. Calculate I_1 , I_2 and I_3 .
- For the situation in which E_2 is replaced by a closed switch, repeat parts (a) and (b). Using the values for R_1 , R_2 , R_3 , r_1 and E_1 from part (c) calculate the currents through the three resistors.
- For the circuit in part (d) calculate the output power of the voltage source and across all the resistors. Examine if energy is conserved in the circuit.
- A student implemented the circuit of part (d) in the lab and measured the current through one of the resistors as 0.19 A. According to the results calculated in part (d) identify the resistor(s). Justify any difference in measured and calculated value.

Solution:

- $I_1 + I_3 = I_2$
- $E_1 - I_1 R_1 - I_2 R_2 - I_1 r_1 = 0$; $-E_2 + I_1 R_1 - I_3 R_3 - I_3 r_2 = 0$
- $I_1 = 8/15\ \text{A}$, $I_2 = 7/15\ \text{A}$ and $I_3 = -1/15\ \text{A}$
- $I_1 = 2/5\ \text{A}$, $I_2 = 3/5\ \text{A}$ and $I_3 = 1/5\ \text{A}$
- $P_{E1} = 18/5\ \text{W}$ and $P_{R1} = 24/25\ \text{W}$, $P_{R2} = 54/25\ \text{W}$, $P_{R3} = 12/25\ \text{W}$.
Yes, $P_{E1} = P_{R1} + P_{R2} + P_{R3}$
- R_3 , losses in the circuit

Section Summary

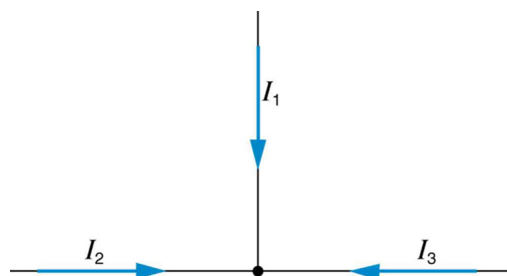
- Kirchhoff's rules can be used to analyze any circuit, simple or complex.
- Kirchhoff's first rule—the junction rule: The sum of all currents entering a junction must equal the sum of all currents leaving the junction.
- Kirchhoff's second rule—the loop rule: The algebraic sum of changes in potential around any closed circuit path (loop) must be zero.
- The two rules are based, respectively, on the laws of conservation of charge and energy.
- When calculating potential and current using Kirchhoff's rules, a set of conventions must be followed for determining the correct signs of various terms.
- The simpler series and parallel rules are special cases of Kirchhoff's rules.

Conceptual Questions

Exercise:

Problem:

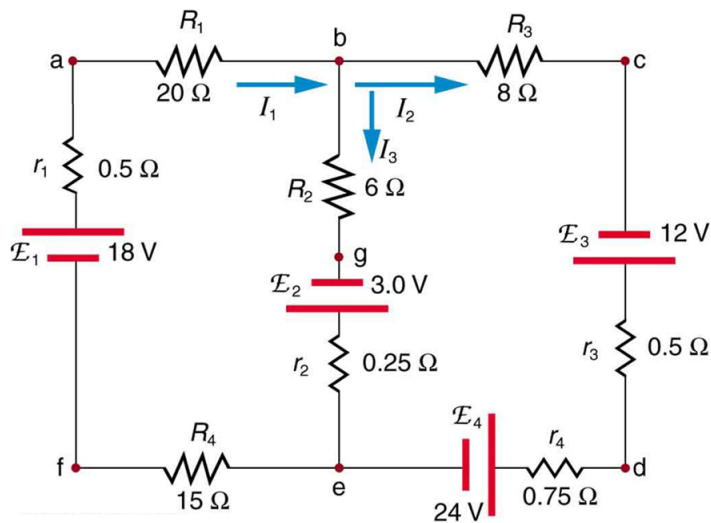
Can all of the currents going into the junction in [\[link\]](#) be positive? Explain.



Exercise:

Problem:

Apply the junction rule to junction b in [\[link\]](#). Is any new information gained by applying the junction rule at e? (In the figure, each emf is represented by script E.)

**Exercise:****Problem:**

(a) What is the potential difference going from point a to point b in [\[link\]](#)? (b) What is the potential difference going from c to b? (c) From e to g? (d) From e to d?

Exercise:

Problem: Apply the loop rule to loop afedcba in [\[link\]](#).

Exercise:

Problem: Apply the loop rule to loops abgefa and cbgedc in [\[link\]](#).

Problem Exercises

Exercise:

Problem: Apply the loop rule to loop abcdefgha in [\[link\]](#).

Solution:

$$-I_2 R_2 + \text{emf}_1 - I_2 r_1 + I_3 R_3 + I_3 r_2 - \text{emf}_2 = 0$$

Exercise:

Problem: Apply the loop rule to loop aedcba in [\[link\]](#).

Exercise:

Problem:

Verify the second equation in [\[link\]](#) by substituting the values found for the currents I_1 and I_2 .

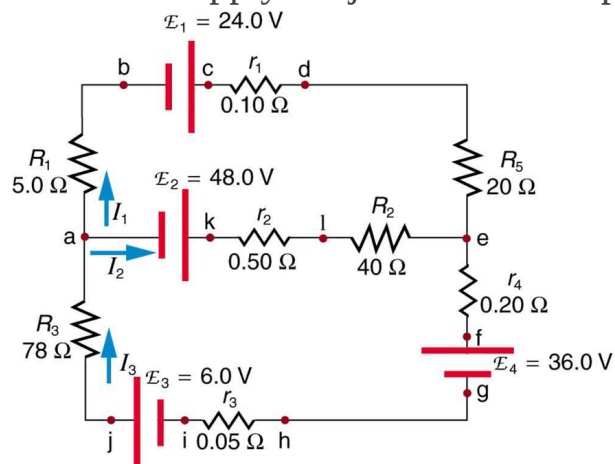
Exercise:

Problem:

Verify the third equation in [\[link\]](#) by substituting the values found for the currents I_1 and I_3 .

Exercise:

Problem: Apply the junction rule at point a in [\[link\]](#).



Solution:

$$I_3 = I_1 + I_2$$

Exercise:

Problem: Apply the loop rule to loop abcdefghija in [\[link\]](#).

Exercise:

Problem: Apply the loop rule to loop akledcba in [\[link\]](#).

Solution:

$$\text{emf}_2 - I_2 r_2 - I_2 R_2 + I_1 R_5 + I_1 r_1 - \text{emf}_1 + I_1 R_1 = 0$$

Exercise:**Problem:**

Find the currents flowing in the circuit in [\[link\]](#). Explicitly show how you follow the steps in the [Problem-Solving Strategies for Series and Parallel Resistors](#).

Exercise:**Problem:**

Solve [\[link\]](#), but use loop abcdefgha instead of loop akledcba. Explicitly show how you follow the steps in the [Problem-Solving Strategies for Series and Parallel Resistors](#).

Solution:

(a) $I_1 = 4.75 \text{ A}$

(b) $I_2 = -3.5 \text{ A}$

(c) $I_3 = 8.25 \text{ A}$

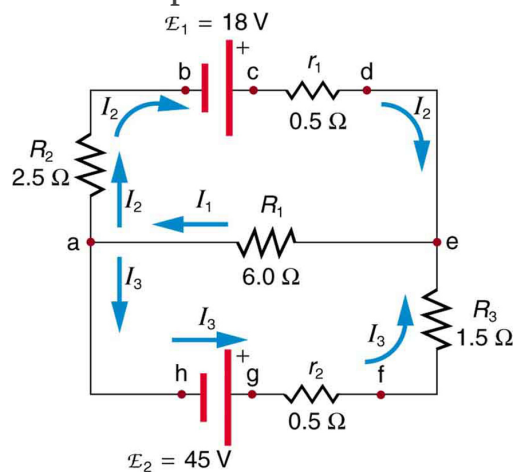
Exercise:

Problem: Find the currents flowing in the circuit in [\[link\]](#).

Exercise:

Problem: Unreasonable Results

Consider the circuit in [\[link\]](#), and suppose that the emfs are unknown and the currents are given to be $I_1 = 5.00$ A, $I_2 = 3.0$ A, and $I_3 = -2.00$ A. (a) Could you find the emfs? (b) What is wrong with the assumptions?



Solution:

(a) No, you would get inconsistent equations to solve.

(b) $I_1 \neq I_2 + I_3$. The assumed currents violate the junction rule.

Glossary

Kirchhoff's rules

a set of two rules, based on conservation of charge and energy, governing current and changes in potential in an electric circuit

junction rule

Kirchhoff's first rule, which applies the conservation of charge to a junction; current is the flow of charge; thus, whatever charge flows into the junction must flow out; the rule can be stated $I_1 = I_2 + I_3$

loop rule

Kirchhoff's second rule, which states that in a closed loop, whatever energy is supplied by emf must be transferred into other forms by devices in the loop, since there are no other ways in which energy can be transferred into or out of the circuit. Thus, the emf equals the sum of the IR (voltage) drops in the loop and can be stated:

$$\text{emf} = Ir + IR_1 + IR_2$$

conservation laws

require that energy and charge be conserved in a system

DC Voltmeters and Ammeters

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain why a voltmeter must be connected in parallel with the circuit.
- Draw a diagram showing an ammeter correctly connected in a circuit.
- Describe how a galvanometer can be used as either a voltmeter or an ammeter.
- Find the resistance that must be placed in series with a galvanometer to allow it to be used as a voltmeter with a given reading.
- Explain why measuring the voltage or current in a circuit can never be exact.

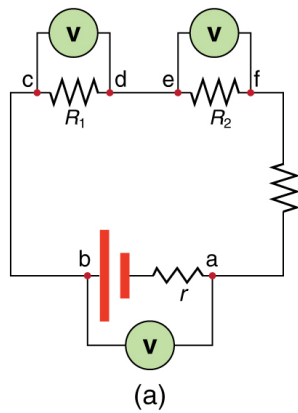
Voltmeters measure voltage, whereas **ammeters** measure current. Some of the meters in automobile dashboards, digital cameras, cell phones, and tuner-amplifiers are voltmeters or ammeters. (See [\[link\]](#).) The internal construction of the simplest of these meters and how they are connected to the system they monitor give further insight into applications of series and parallel connections.



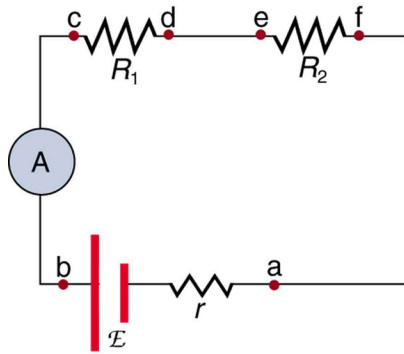
The fuel and temperature gauges (far right and far left, respectively) in this 1996 Volkswagen are voltmeters that register the voltage output of “sender” units, which are hopefully proportional to the amount of gasoline in the tank and the engine temperature. (credit: Christian Giersing)

Voltmeters are connected in parallel with whatever device’s voltage is to be measured. A parallel connection is used because objects in parallel experience the same potential difference. (See [\[link\]](#), where the voltmeter is represented by the symbol V.)

Ammeters are connected in series with whatever device’s current is to be measured. A series connection is used because objects in series have the same current passing through them. (See [\[link\]](#), where the ammeter is represented by the symbol A.)



(a) To measure potential differences in this series circuit, the voltmeter (V) is placed in parallel with the voltage source or either of the resistors. Note that terminal voltage is measured between points a and b. It is not possible to connect the voltmeter directly across the emf without including its internal resistance, r . (b) A digital voltmeter in use.
(credit: Messtechniker, Wikimedia Commons)



An ammeter (A) is placed in series to measure current. All of the current in this circuit flows through the meter.

The ammeter would have the same reading if located between points d and e or between points f and a as it does in the position shown. (Note that the script capital E stands for emf, and r stands for the internal resistance of the source of potential difference.)

Analog Meters: Galvanometers

Analog meters have a needle that swivels to point at numbers on a scale, as opposed to **digital meters**, which have numerical readouts similar to a

hand-held calculator. The heart of most analog meters is a device called a **galvanometer**, denoted by G. Current flow through a galvanometer, I_G , produces a proportional needle deflection. (This deflection is due to the force of a magnetic field upon a current-carrying wire.)

The two crucial characteristics of a given galvanometer are its resistance and current sensitivity. **Current sensitivity** is the current that gives a **full-scale deflection** of the galvanometer's needle, the maximum current that the instrument can measure. For example, a galvanometer with a current sensitivity of $50\ \mu\text{A}$ has a maximum deflection of its needle when $50\ \mu\text{A}$ flows through it, reads half-scale when $25\ \mu\text{A}$ flows through it, and so on.

If such a galvanometer has a $25\text{-}\Omega$ resistance, then a voltage of only $V = IR = (50\ \mu\text{A})(25\ \Omega) = 1.25\ \text{mV}$ produces a full-scale reading. By connecting resistors to this galvanometer in different ways, you can use it as either a voltmeter or ammeter that can measure a broad range of voltages or currents.

Galvanometer as Voltmeter

[\[link\]](#) shows how a galvanometer can be used as a voltmeter by connecting it in series with a large resistance, R . The value of the resistance R is determined by the maximum voltage to be measured. Suppose you want 10 V to produce a full-scale deflection of a voltmeter containing a $25\text{-}\Omega$ galvanometer with a $50\text{-}\mu\text{A}$ sensitivity. Then 10 V applied to the meter must produce a current of $50\ \mu\text{A}$. The total resistance must be

Equation:

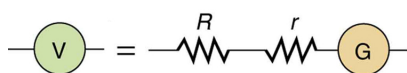
$$R_{\text{tot}} = R + r = \frac{V}{I} = \frac{10\ \text{V}}{50\ \mu\text{A}} = 200\ \text{k}\ \Omega, \text{ or}$$

Equation:

$$R = R_{\text{tot}} - r = 200\ \text{k}\Omega - 25\ \Omega \approx 200\ \text{k}\Omega.$$

(R is so large that the galvanometer resistance, r , is nearly negligible.) Note that 5 V applied to this voltmeter produces a half-scale deflection by producing a $25\text{-}\mu\text{A}$ current through the meter, and so the voltmeter's reading is proportional to voltage as desired.

This voltmeter would not be useful for voltages less than about half a volt, because the meter deflection would be small and difficult to read accurately. For other voltage ranges, other resistances are placed in series with the galvanometer. Many meters have a choice of scales. That choice involves switching an appropriate resistance into series with the galvanometer.



A large resistance
 R placed in series
with a
galvanometer G
produces a
voltmeter, the full-
scale deflection of
which depends on
the choice of R .

The larger the
voltage to be
measured, the
larger R must be.

(Note that r
represents the
internal resistance
of the
galvanometer.)

Galvanometer as Ammeter

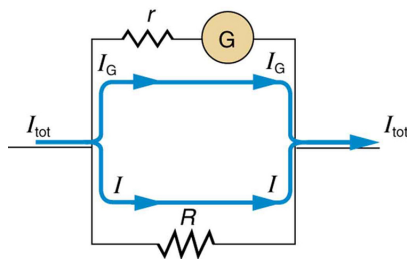
The same galvanometer can also be made into an ammeter by placing it in parallel with a small resistance R , often called the **shunt resistance**, as shown in [\[link\]](#). Since the shunt resistance is small, most of the current passes through it, allowing an ammeter to measure currents much greater than those producing a full-scale deflection of the galvanometer.

Suppose, for example, an ammeter is needed that gives a full-scale deflection for 1.0 A, and contains the same $25\text{-}\Omega$ galvanometer with its $50\text{-}\mu\text{A}$ sensitivity. Since R and r are in parallel, the voltage across them is the same.

These IR drops are $IR = I_G r$ so that $IR = \frac{I_G}{I} = \frac{R}{r}$. Solving for R , and noting that I_G is $50\text{ }\mu\text{A}$ and I is 0.999950 A , we have

Equation:

$$R = r \frac{I_G}{I} = (25\text{ }\Omega) \frac{50\text{ }\mu\text{A}}{0.999950\text{ A}} = 1.25 \times 10^{-3}\text{ }\Omega.$$



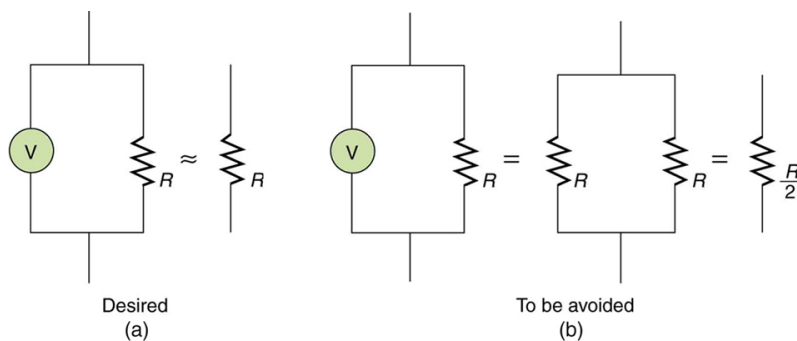
A small shunt resistance R placed in parallel with a galvanometer G produces an ammeter, the full-scale deflection of which depends on

the choice of R .
The larger the current to be measured, the smaller R must be. Most of the current (I) flowing through the meter is shunted through R to protect the galvanometer. (Note that r represents the internal resistance of the galvanometer.)
Ammeters may also have multiple scales for greater flexibility in application. The various scales are achieved by switching various shunt resistances in parallel with the galvanometer—the greater the maximum current to be measured, the smaller the shunt resistance must be.

Taking Measurements Alters the Circuit

When you use a voltmeter or ammeter, you are connecting another resistor to an existing circuit and, thus, altering the circuit. Ideally, voltmeters and ammeters do not appreciably affect the circuit, but it is instructive to examine the circumstances under which they do or do not interfere.

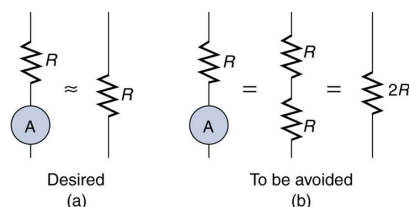
First, consider the voltmeter, which is always placed in parallel with the device being measured. Very little current flows through the voltmeter if its resistance is a few orders of magnitude greater than the device, and so the circuit is not appreciably affected. (See [\[link\]](#)(a).) (A large resistance in parallel with a small one has a combined resistance essentially equal to the small one.) If, however, the voltmeter's resistance is comparable to that of the device being measured, then the two in parallel have a smaller resistance, appreciably affecting the circuit. (See [\[link\]](#)(b).) The voltage across the device is not the same as when the voltmeter is out of the circuit.



(a) A voltmeter having a resistance much larger than the device ($R_{\text{Voltmeter}} \gg R$) with which it is in parallel produces a parallel resistance essentially the same as the device and does not appreciably affect the circuit being measured. (b) Here the voltmeter has the same resistance as the device ($R_{\text{Voltmeter}} \cong R$), so that the parallel resistance is half of what it is when the voltmeter is not connected. This is an example of a significant alteration of the circuit and is to be avoided.

An ammeter is placed in series in the branch of the circuit being measured, so that its resistance adds to that branch. Normally, the ammeter's resistance is very small compared with the resistances of the devices in the circuit, and so the extra resistance is negligible. (See [\[link\]](#)(a).) However, if very small load resistances are involved, or if the ammeter is not as low in resistance as it should be, then the total series resistance is significantly greater, and the current in the branch being measured is reduced. (See [\[link\]](#)(b).)

A practical problem can occur if the ammeter is connected incorrectly. If it was put in parallel with the resistor to measure the current in it, you could possibly damage the meter; the low resistance of the ammeter would allow most of the current in the circuit to go through the galvanometer, and this current would be larger since the effective resistance is smaller.



(a) An ammeter normally has such a small resistance that the total series resistance in the branch being measured is not appreciably increased. The circuit is essentially unaltered compared with when the ammeter is absent.

(b) Here the

ammeter's resistance is the same as that of the branch, so that the total resistance is doubled and the current is half what it is without the ammeter. This significant alteration of the circuit is to be avoided.

One solution to the problem of voltmeters and ammeters interfering with the circuits being measured is to use galvanometers with greater sensitivity. This allows construction of voltmeters with greater resistance and ammeters with smaller resistance than when less sensitive galvanometers are used.

There are practical limits to galvanometer sensitivity, but it is possible to get analog meters that make measurements accurate to a few percent. Note that the inaccuracy comes from altering the circuit, not from a fault in the meter.

Note:**Connections: Limits to Knowledge**

Making a measurement alters the system being measured in a manner that produces uncertainty in the measurement. For macroscopic systems, such as the circuits discussed in this module, the alteration can usually be made negligibly small, but it cannot be eliminated entirely. For submicroscopic systems, such as atoms, nuclei, and smaller particles, measurement alters the system in a manner that cannot be made arbitrarily small. This actually limits knowledge of the system—even limiting what nature can know

about itself. We shall see profound implications of this when the Heisenberg uncertainty principle is discussed in the modules on quantum mechanics.

There is another measurement technique based on drawing no current at all and, hence, not altering the circuit at all. These are called null measurements and are the topic of [Null Measurements](#). Digital meters that employ solid-state electronics and null measurements can attain accuracies of one part in 10^6 .

Exercise:

Check Your Understanding

Problem:

Digital meters are able to detect smaller currents than analog meters employing galvanometers. How does this explain their ability to measure voltage and current more accurately than analog meters?

Solution:

Since digital meters require less current than analog meters, they alter the circuit less than analog meters. Their resistance as a voltmeter can be far greater than an analog meter, and their resistance as an ammeter can be far less than an analog meter. Consult [\[link\]](#) and [\[link\]](#) and their discussion in the text.

Note:

PhET Explorations: Circuit Construction Kit (DC Only), Virtual Lab Stimulate a neuron and monitor what happens. Pause, rewind, and move forward in time in order to observe the ions as they move across the neuron membrane.

[Circuit](#)

Section Summary

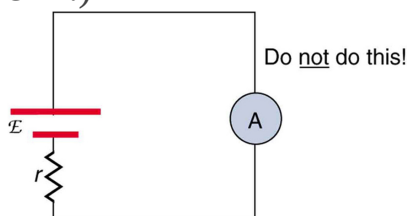
- Voltmeters measure voltage, and ammeters measure current.
- A voltmeter is placed in parallel with the voltage source to receive full voltage and must have a large resistance to limit its effect on the circuit.
- An ammeter is placed in series to get the full current flowing through a branch and must have a small resistance to limit its effect on the circuit.
- Both can be based on the combination of a resistor and a galvanometer, a device that gives an analog reading of current.
- Standard voltmeters and ammeters alter the circuit being measured and are thus limited in accuracy.

Conceptual Questions

Exercise:

Problem:

Why should you not connect an ammeter directly across a voltage source as shown in [\[link\]](#)? (Note that script E in the figure stands for emf.)

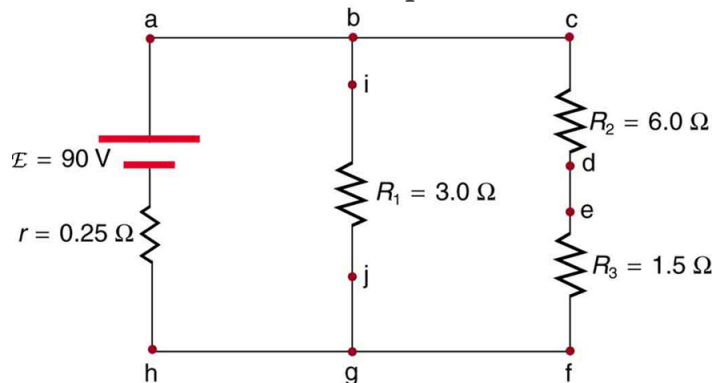


Exercise:**Problem:**

Suppose you are using a multimeter (one designed to measure a range of voltages, currents, and resistances) to measure current in a circuit and you inadvertently leave it in a voltmeter mode. What effect will the meter have on the circuit? What would happen if you were measuring voltage but accidentally put the meter in the ammeter mode?

Exercise:**Problem:**

Specify the points to which you could connect a voltmeter to measure the following potential differences in [\[link\]](#): (a) the potential difference of the voltage source; (b) the potential difference across R_1 ; (c) across R_2 ; (d) across R_3 ; (e) across R_2 and R_3 . Note that there may be more than one answer to each part.

**Exercise:****Problem:**

To measure currents in [\[link\]](#), you would replace a wire between two points with an ammeter. Specify the points between which you would place an ammeter to measure the following: (a) the total current; (b) the current flowing through R_1 ; (c) through R_2 ; (d) through R_3 . Note that there may be more than one answer to each part.

Problem Exercises

Exercise:

Problem:

What is the sensitivity of the galvanometer (that is, what current gives a full-scale deflection) inside a voltmeter that has a $1.00\text{-M}\Omega$ resistance on its 30.0-V scale?

Solution:

$$30\ \mu\text{A}$$

Exercise:

Problem:

What is the sensitivity of the galvanometer (that is, what current gives a full-scale deflection) inside a voltmeter that has a $25.0\text{-k}\Omega$ resistance on its 100-V scale?

Exercise:

Problem:

Find the resistance that must be placed in series with a $25.0\text{-}\Omega$ galvanometer having a $50.0\text{-}\mu\text{A}$ sensitivity (the same as the one discussed in the text) to allow it to be used as a voltmeter with a 0.100-V full-scale reading.

Solution:

$$1.98\ \text{k}\Omega$$

Exercise:

Problem:

Find the resistance that must be placed in series with a $25.0\text{-}\Omega$ galvanometer having a $50.0\text{-}\mu\text{A}$ sensitivity (the same as the one discussed in the text) to allow it to be used as a voltmeter with a 3000-V full-scale reading. Include a circuit diagram with your solution.

Exercise:**Problem:**

Find the resistance that must be placed in parallel with a $25.0\text{-}\Omega$ galvanometer having a $50.0\text{-}\mu\text{A}$ sensitivity (the same as the one discussed in the text) to allow it to be used as an ammeter with a 10.0-A full-scale reading. Include a circuit diagram with your solution.

Solution:

$$1.25 \times 10^{-4} \Omega$$

Exercise:**Problem:**

Find the resistance that must be placed in parallel with a $25.0\text{-}\Omega$ galvanometer having a $50.0\text{-}\mu\text{A}$ sensitivity (the same as the one discussed in the text) to allow it to be used as an ammeter with a 300-mA full-scale reading.

Exercise:**Problem:**

Find the resistance that must be placed in series with a $10.0\text{-}\Omega$ galvanometer having a $100\text{-}\mu\text{A}$ sensitivity to allow it to be used as a voltmeter with: (a) a 300-V full-scale reading, and (b) a 0.300-V full-scale reading.

Solution:

(a) $3.00 \text{ M}\Omega$

(b) $2.99 \text{ k}\Omega$

Exercise:

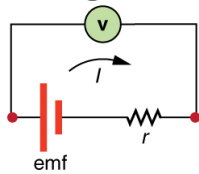
Problem:

Find the resistance that must be placed in parallel with a $10.0\text{-}\Omega$ galvanometer having a $100\text{-}\mu\text{A}$ sensitivity to allow it to be used as an ammeter with: (a) a 20.0-A full-scale reading, and (b) a 100-mA full-scale reading.

Exercise:

Problem:

Suppose you measure the terminal voltage of a 1.585-V alkaline cell having an internal resistance of $0.100 \text{ }\Omega$ by placing a $1.00\text{-k}\Omega$ voltmeter across its terminals. (See [\[link\]](#).) (a) What current flows? (b) Find the terminal voltage. (c) To see how close the measured terminal voltage is to the emf, calculate their ratio.



Solution:

(a) 1.58 mA

(b) 1.5848 V (need four digits to see the difference)

(c) 0.99990 (need five digits to see the difference from unity)

Exercise:

Problem:

Suppose you measure the terminal voltage of a 3.200-V lithium cell having an internal resistance of $5.00\ \Omega$ by placing a $1.00\text{-k}\Omega$ voltmeter across its terminals. (a) What current flows? (b) Find the terminal voltage. (c) To see how close the measured terminal voltage is to the emf, calculate their ratio.

Exercise:**Problem:**

A certain ammeter has a resistance of $5.00 \times 10^{-5}\ \Omega$ on its 3.00-A scale and contains a $10.0\text{-}\Omega$ galvanometer. What is the sensitivity of the galvanometer?

Solution:

$15.0\ \mu\text{A}$

Exercise:**Problem:**

A $1.00\text{-M}\Omega$ voltmeter is placed in parallel with a $75.0\text{-k}\Omega$ resistor in a circuit. (a) Draw a circuit diagram of the connection. (b) What is the resistance of the combination? (c) If the voltage across the combination is kept the same as it was across the $75.0\text{-k}\Omega$ resistor alone, what is the percent increase in current? (d) If the current through the combination is kept the same as it was through the $75.0\text{-k}\Omega$ resistor alone, what is the percentage decrease in voltage? (e) Are the changes found in parts (c) and (d) significant? Discuss.

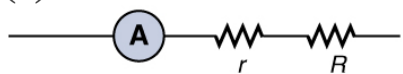
Exercise:

Problem:

A $0.0200\text{-}\Omega$ ammeter is placed in series with a $10.00\text{-}\Omega$ resistor in a circuit. (a) Draw a circuit diagram of the connection. (b) Calculate the resistance of the combination. (c) If the voltage is kept the same across the combination as it was through the $10.00\text{-}\Omega$ resistor alone, what is the percent decrease in current? (d) If the current is kept the same through the combination as it was through the $10.00\text{-}\Omega$ resistor alone, what is the percent increase in voltage? (e) Are the changes found in parts (c) and (d) significant? Discuss.

Solution:

(a)

(b) $10.02\text{ }\Omega$ (c) 0.9980 , or a 2.0×10^{-1} percent decrease(d) 1.002 , or a 2.0×10^{-1} percent increase

(e) Not significant.

Exercise:**Problem: Unreasonable Results**

Suppose you have a $40.0\text{-}\Omega$ galvanometer with a $25.0\text{-}\mu\text{A}$ sensitivity.

(a) What resistance would you put in series with it to allow it to be used as a voltmeter that has a full-scale deflection for 0.500 mV ? (b) What is unreasonable about this result? (c) Which assumptions are responsible?

Exercise:**Problem: Unreasonable Results**

(a) What resistance would you put in parallel with a $40.0\text{-}\Omega$ galvanometer having a $25.0\text{-}\mu\text{A}$ sensitivity to allow it to be used as an ammeter that has a full-scale deflection for $10.0\text{-}\mu\text{A}$? (b) What is unreasonable about this result? (c) Which assumptions are responsible?

Solution:

(a) $-66.7\ \Omega$

(b) You can't have negative resistance.

(c) It is unreasonable that I_G is greater than I_{tot} (see [\[link\]](#)). You cannot achieve a full-scale deflection using a current less than the sensitivity of the galvanometer.

Glossary

voltmeter

an instrument that measures voltage

ammeter

an instrument that measures current

analog meter

a measuring instrument that gives a readout in the form of a needle movement over a marked gauge

digital meter

a measuring instrument that gives a readout in a digital form

galvanometer

an analog measuring device, denoted by G , that measures current flow using a needle deflection caused by a magnetic field force acting upon a current-carrying wire

current sensitivity

the maximum current that a galvanometer can read

full-scale deflection

the maximum deflection of a galvanometer needle, also known as current sensitivity; a galvanometer with a full-scale deflection of $50\ \mu\text{A}$ has a maximum deflection of its needle when $50\ \mu\text{A}$ flows through it

shunt resistance

a small resistance R placed in parallel with a galvanometer G to produce an ammeter; the larger the current to be measured, the smaller R must be; most of the current flowing through the meter is shunted through R to protect the galvanometer

Null Measurements

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain why a null measurement device is more accurate than a standard voltmeter or ammeter.
- Demonstrate how a Wheatstone bridge can be used to accurately calculate the resistance in a circuit.

Standard measurements of voltage and current alter the circuit being measured, introducing uncertainties in the measurements. Voltmeters draw some extra current, whereas ammeters reduce current flow. **Null measurements** balance voltages so that there is no current flowing through the measuring device and, therefore, no alteration of the circuit being measured.

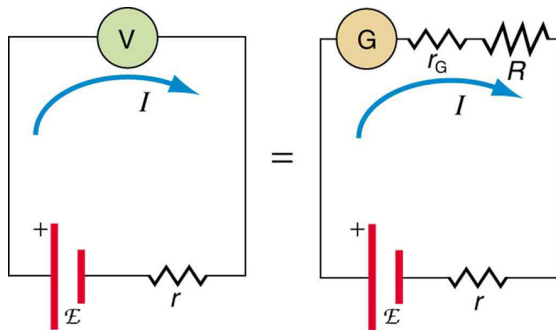
Null measurements are generally more accurate but are also more complex than the use of standard voltmeters and ammeters, and they still have limits to their precision. In this module, we shall consider a few specific types of null measurements, because they are common and interesting, and they further illuminate principles of electric circuits.

The Potentiometer

Suppose you wish to measure the emf of a battery. Consider what happens if you connect the battery directly to a standard voltmeter as shown in [\[link\]](#). (Once we note the problems with this measurement, we will examine

a null measurement that improves accuracy.) As discussed before, the actual quantity measured is the terminal voltage V , which is related to the emf of the battery by $V = \text{emf} - Ir$, where I is the current that flows and r is the internal resistance of the battery.

The emf could be accurately calculated if r were very accurately known, but it is usually not. If the current I could be made zero, then $V = \text{emf}$, and so emf could be directly measured. However, standard voltmeters need a current to operate; thus, another technique is needed.



An analog voltmeter attached to a battery draws a small but nonzero current and measures a terminal voltage that differs from the emf of the battery. (Note that the script capital \mathcal{E} symbolizes electromotive force, or emf.) Since the internal resistance of the battery is not known precisely, it is not possible to calculate the emf precisely.

A **potentiometer** is a null measurement device for measuring potentials (voltages). (See [\[link\]](#).) A voltage source is connected to a resistor R , say, a long wire, and passes a constant current through it. There is a steady drop in potential (an IR drop) along the wire, so that a variable potential can be obtained by making contact at varying locations along the wire.

[\[link\]](#)(b) shows an unknown emf_x (represented by script E_x in the figure) connected in series with a galvanometer. Note that emf_x opposes the other voltage source. The location of the contact point (see the arrow on the drawing) is adjusted until the galvanometer reads zero. When the galvanometer reads zero, $\text{emf}_x = IR_x$, where R_x is the resistance of the section of wire up to the contact point. Since no current flows through the galvanometer, none flows through the unknown emf , and so emf_x is directly sensed.

Now, a very precisely known standard emf_s is substituted for emf_x , and the contact point is adjusted until the galvanometer again reads zero, so that $\text{emf}_s = IR_s$. In both cases, no current passes through the galvanometer, and so the current I through the long wire is the same. Upon taking the ratio $\frac{\text{emf}_x}{\text{emf}_s}$, I cancels, giving

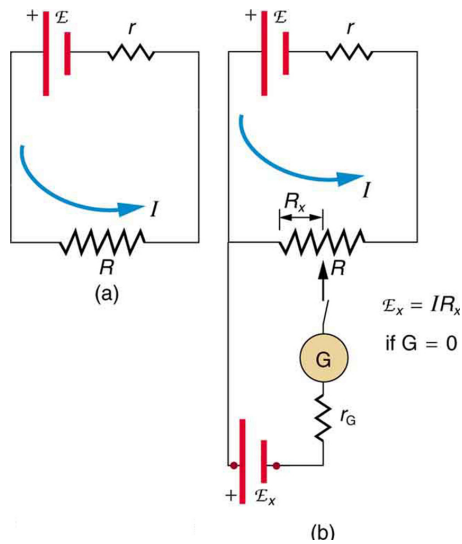
Equation:

$$\frac{\text{emf}_x}{\text{emf}_s} = \frac{IR_x}{IR_s} = \frac{R_x}{R_s}.$$

Solving for emf_x gives

Equation:

$$\text{emf}_x = \text{emf}_s \frac{R_x}{R_s}.$$



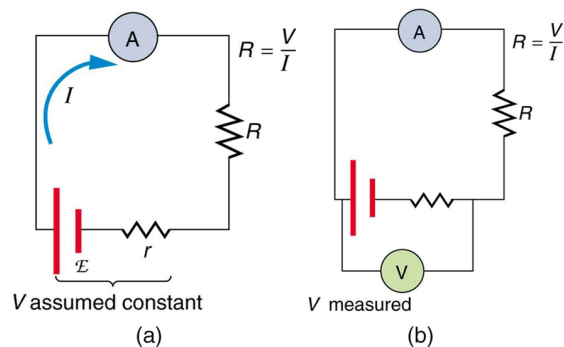
The potentiometer, a null measurement device. (a) A voltage source connected to a long wire resistor passes a constant current I through it. (b) An unknown emf (labeled script \mathcal{E}_x in the figure) is connected as shown, and the point of contact along R is adjusted until the galvanometer reads zero. The segment of wire has a resistance R_x and script $\mathcal{E}_x = IR_x$, where I is unaffected by the connection since no current flows through the galvanometer. The unknown emf is thus

proportional to the
resistance of the wire
segment.

Because a long uniform wire is used for R , the ratio of resistances R_x/R_s is the same as the ratio of the lengths of wire that zero the galvanometer for each emf. The three quantities on the right-hand side of the equation are now known or measured, and emf_x can be calculated. The uncertainty in this calculation can be considerably smaller than when using a voltmeter directly, but it is not zero. There is always some uncertainty in the ratio of resistances R_x/R_s and in the standard emf_s . Furthermore, it is not possible to tell when the galvanometer reads exactly zero, which introduces error into both R_x and R_s , and may also affect the current I .

Resistance Measurements and the Wheatstone Bridge

There is a variety of so-called **ohmmeters** that purport to measure resistance. What the most common ohmmeters actually do is to apply a voltage to a resistance, measure the current, and calculate the resistance using Ohm's law. Their readout is this calculated resistance. Two configurations for ohmmeters using standard voltmeters and ammeters are shown in [\[link\]](#). Such configurations are limited in accuracy, because the meters alter both the voltage applied to the resistor and the current that flows through it.



Two methods for measuring resistance with standard meters. (a) Assuming a known voltage for the source, an ammeter measures current, and resistance is calculated as $R = \frac{V}{I}$. (b) Since the terminal voltage V varies with current, it is better to measure it. V is most accurately known when I is small, but I itself is most accurately known when it is large.

The **Wheatstone bridge** is a null measurement device for calculating resistance by balancing potential drops in a circuit. (See [\[link\]](#).) The device is called a bridge because the galvanometer forms a bridge between two branches. A variety of **bridge devices** are used to make null measurements in circuits.

Resistors R_1 and R_2 are precisely known, while the arrow through R_3 indicates that it is a variable resistance. The value of R_3 can be precisely read. With the unknown resistance R_x in the circuit, R_3 is adjusted until the galvanometer reads zero. The potential difference between points b and d is then zero, meaning that b and d are at the same potential. With no current running through the galvanometer, it has no effect on the rest of the circuit. So the branches abc and adc are in parallel, and each branch has the full voltage of the source. That is, the IR drops along abc and adc are the same. Since b and d are at the same potential, the IR drop along ad must equal the IR drop along ab. Thus,

Equation:

$$I_1 R_1 = I_2 R_3.$$

Again, since b and d are at the same potential, the IR drop along dc must equal the IR drop along bc. Thus,

Equation:

$$I_1 R_2 = I_2 R_x.$$

Taking the ratio of these last two expressions gives

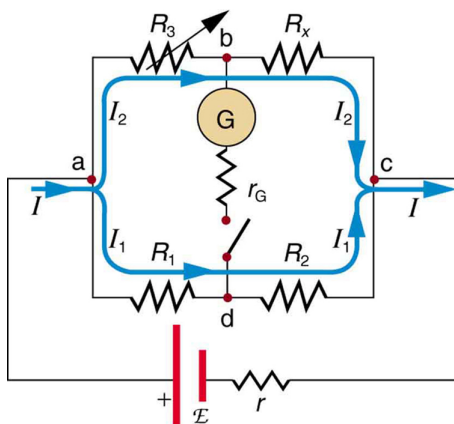
Equation:

$$\frac{I_1 R_1}{I_1 R_2} = \frac{I_2 R_3}{I_2 R_x}.$$

Canceling the currents and solving for R_x yields

Equation:

$$R_x = R_3 \frac{R_2}{R_1}.$$



The Wheatstone bridge is used to calculate unknown resistances. The

variable resistance R_3
is adjusted until the
galvanometer reads
zero with the switch
closed. This simplifies
the circuit, allowing
 R_x to be calculated
based on the IR drops
as discussed in the
text.

This equation is used to calculate the unknown resistance when current through the galvanometer is zero. This method can be very accurate (often to four significant digits), but it is limited by two factors. First, it is not possible to get the current through the galvanometer to be exactly zero. Second, there are always uncertainties in R_1 , R_2 , and R_3 , which contribute to the uncertainty in R_x .

Exercise:

Check Your Understanding

Problem:

Identify other factors that might limit the accuracy of null measurements. Would the use of a digital device that is more sensitive than a galvanometer improve the accuracy of null measurements?

Solution:

One factor would be resistance in the wires and connections in a null measurement. These are impossible to make zero, and they can change over time. Another factor would be temperature variations in resistance, which can be reduced but not completely eliminated by choice of material. Digital devices sensitive to smaller currents than analog devices do improve the accuracy of null measurements because they allow you to get the current closer to zero.

Section Summary

- Null measurement techniques achieve greater accuracy by balancing a circuit so that no current flows through the measuring device.
- One such device, for determining voltage, is a potentiometer.
- Another null measurement device, for determining resistance, is the Wheatstone bridge.
- Other physical quantities can also be measured with null measurement techniques.

Conceptual questions

Exercise:

Problem:

Why can a null measurement be more accurate than one using standard voltmeters and ammeters? What factors limit the accuracy of null measurements?

Exercise:

Problem:

If a potentiometer is used to measure cell emfs on the order of a few volts, why is it most accurate for the standard emf_s to be the same order of magnitude and the resistances to be in the range of a few ohms?

Problem Exercises

Exercise:

Problem:

What is the emf_x of a cell being measured in a potentiometer, if the standard cell's emf is 12.0 V and the potentiometer balances for $R_x = 5.000\ \Omega$ and $R_s = 2.500\ \Omega$?

Solution:

24.0 V

Exercise:**Problem:**

Calculate the emf_x of a dry cell for which a potentiometer is balanced when $R_x = 1.200\ \Omega$, while an alkaline standard cell with an emf of 1.600 V requires $R_s = 1.247\ \Omega$ to balance the potentiometer.

Exercise:**Problem:**

When an unknown resistance R_x is placed in a Wheatstone bridge, it is possible to balance the bridge by adjusting R_3 to be $2500\ \Omega$. What is R_x if $\frac{R_2}{R_1} = 0.625$?

Solution:

1.56 k Ω

Exercise:**Problem:**

To what value must you adjust R_3 to balance a Wheatstone bridge, if the unknown resistance R_x is $100\ \Omega$, R_1 is $50.0\ \Omega$, and R_2 is $175\ \Omega$?

Exercise:**Problem:**

(a) What is the unknown emf_x in a potentiometer that balances when R_x is $10.0\ \Omega$, and balances when R_s is $15.0\ \Omega$ for a standard 3.000-V emf? (b) The same emf_x is placed in the same potentiometer, which now balances when R_s is $15.0\ \Omega$ for a standard emf of 3.100 V. At what resistance R_x will the potentiometer balance?

Solution:

(a) 2.00 V

(b) 9.68 Ω

Exercise:**Problem:**

Suppose you want to measure resistances in the range from 10.0 Ω to 10.0 k Ω using a Wheatstone bridge that has $\frac{R_2}{R_1} = 2.000$. Over what range should R_3 be adjustable?

Solution:**Equation:**

$$\text{Range} = 5.00 \, \Omega \text{ to } 5.00 \, \text{k}\Omega$$

Glossary

null measurements

methods of measuring current and voltage more accurately by balancing the circuit so that no current flows through the measurement device

potentiometer

a null measurement device for measuring potentials (voltages)

ohmmeter

an instrument that applies a voltage to a resistance, measures the current, calculates the resistance using Ohm's law, and provides a readout of this calculated resistance

bridge device

a device that forms a bridge between two branches of a circuit; some bridge devices are used to make null measurements in circuits

Wheatstone bridge

a null measurement device for calculating resistance by balancing potential drops in a circuit

DC Circuits Containing Resistors and Capacitors

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain the importance of the time constant τ , and calculate the time constant for a given resistance and capacitance.
- Explain why batteries in a flashlight gradually lose power and the light dims over time.
- Describe what happens to a graph of the voltage across a capacitor over time as it charges.
- Explain how a timing circuit works and list some applications.
- Calculate the necessary speed of a strobe flash needed to “stop” the movement of an object over a particular length.

The information presented in this section supports the following AP® learning objectives and science practices:

- **5.C.3.6** The student is able to determine missing values and direction of electric current in branches of a circuit with both resistors and capacitors from values and directions of current in other branches of the circuit through appropriate selection of nodes and application of the junction rule. **(S.P. 1.4, 2.2)**
- **5.C.3.7** The student is able to determine missing values, direction of electric current, charge of capacitors at steady state, and potential differences within a circuit with resistors and capacitors from values and directions of current in other branches of the circuit. **(S.P. 1.4, 2.2)**

When you use a flash camera, it takes a few seconds to charge the capacitor that powers the flash. The light flash discharges the capacitor in a tiny fraction of a second. Why does charging take longer than discharging? This question and a number of other phenomena that involve charging and discharging capacitors are discussed in this module.

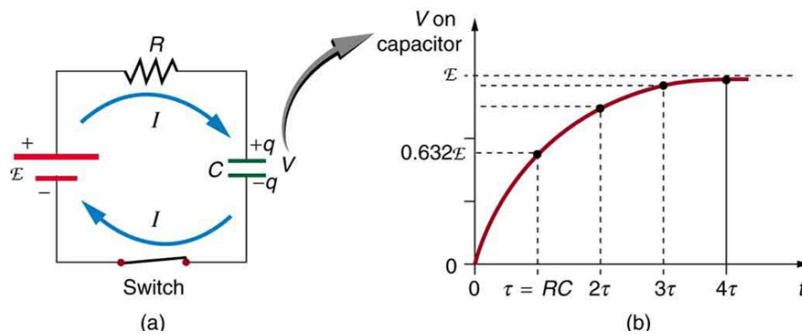
RC Circuits

An **RC circuit** is one containing a **resistor** R and a **capacitor** C . The capacitor is an electrical component that stores electric charge.

[\[link\]](#) shows a simple RC circuit that employs a DC (direct current) voltage source. The capacitor is initially uncharged. As soon as the switch is closed, current flows to and from the initially uncharged capacitor. As charge increases on the capacitor plates, there is increasing opposition to the flow of charge by the repulsion of like charges on each plate.

In terms of voltage, this is because voltage across the capacitor is given by $V_c = Q/C$, where Q is the amount of charge stored on each plate and C is the **capacitance**. This voltage opposes the battery, growing from zero to the maximum emf when fully charged. The current thus decreases from its initial value of $I_0 = \frac{\text{emf}}{R}$ to zero as the voltage on the capacitor reaches the same value as the emf. When there is no current, there is no IR drop, and so the voltage on the capacitor must then equal the emf of the voltage source. This can also be explained with Kirchhoff's second rule (the loop rule), discussed in [Kirchhoff's Rules](#), which says that the algebraic sum of changes in potential around any closed loop must be zero.

The initial current is $I_0 = \frac{\text{emf}}{R}$, because all of the IR drop is in the resistance. Therefore, the smaller the resistance, the faster a given capacitor will be charged. Note that the internal resistance of the voltage source is included in R , as are the resistances of the capacitor and the connecting wires. In the flash camera scenario above, when the batteries powering the camera begin to wear out, their internal resistance rises, reducing the current and lengthening the time it takes to get ready for the next flash.



- (a) An RC circuit with an initially uncharged capacitor. Current flows in the direction shown (opposite of electron flow) as soon as the switch is closed. Mutual repulsion of like charges in the capacitor progressively slows the flow as the capacitor is charged, stopping the current when the capacitor is fully charged and $Q = C \cdot \text{emf}$.
- (b) A graph of voltage across the capacitor versus time, with the switch closing at time $t = 0$. (Note that in the two parts of the figure, the capital script E stands for emf, q stands for the charge stored on the capacitor, and τ is the RC time constant.)

Voltage on the capacitor is initially zero and rises rapidly at first, since the initial current is a maximum. [\[link\]](#)(b) shows a graph of capacitor voltage versus time (t) starting when the switch is closed at $t = 0$. The voltage approaches emf asymptotically, since the closer it gets to emf the less current flows. The equation for voltage versus time when charging a capacitor C through a resistor R , derived using calculus, is

Equation:

$$V = \text{emf}(1 - e^{-t/RC}) \text{ (charging),}$$

where V is the voltage across the capacitor, emf is equal to the emf of the DC voltage source, and the exponential $e = 2.718 \dots$ is the base of the natural logarithm. Note that the units of RC are seconds. We define

Equation:

$$\tau = RC,$$

where τ (the Greek letter tau) is called the time constant for an RC circuit. As noted before, a small resistance R allows the capacitor to charge faster. This is reasonable, since a larger current flows through a smaller resistance. It is also reasonable that the smaller the capacitor C , the less time needed to charge it. Both factors are contained in $\tau = RC$.

More quantitatively, consider what happens when $t = \tau = RC$. Then the voltage on the capacitor is

Equation:

$$V = \text{emf}(1 - e^{-1}) = \text{emf}(1 - 0.368) = 0.632 \cdot \text{emf}.$$

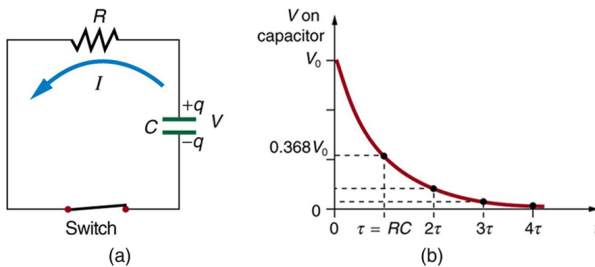
This means that in the time $\tau = RC$, the voltage rises to 0.632 of its final value. The voltage will rise 0.632 of the remainder in the next time τ . It is a characteristic of the exponential function that the final value is never reached, but 0.632 of the remainder to that value is achieved in every time, τ . In just a few multiples of the time constant τ , then, the final value is very nearly achieved, as the graph in [\[link\]](#)(b) illustrates.

Discharging a Capacitor

Discharging a capacitor through a resistor proceeds in a similar fashion, as [\[link\]](#) illustrates. Initially, the current is $I_0 = \frac{V_0}{R}$, driven by the initial voltage V_0 on the capacitor. As the voltage decreases, the current and hence the rate of discharge decreases, implying another exponential formula for V . Using calculus, the voltage V on a capacitor C being discharged through a resistor R is found to be

Equation:

$$V = V_0 e^{-t/RC} \text{ (discharging).}$$



(a) Closing the switch discharges the capacitor C through the resistor R . Mutual repulsion of like charges on each plate drives the current. (b) A graph of voltage across the capacitor versus time, with $V = V_0$ at $t = 0$. The voltage decreases exponentially, falling a fixed fraction of the way to zero in each subsequent time constant τ .

The graph in [\[link\]](#)(b) is an example of this exponential decay. Again, the time constant is $\tau = RC$. A small resistance R allows the capacitor to discharge in a small time, since the current is larger. Similarly, a small capacitance requires less time to discharge, since less charge is stored. In the first time interval $\tau = RC$ after the switch is closed, the voltage falls to 0.368 of its initial value, since $V = V_0 \cdot e^{-1} = 0.368V_0$.

During each successive time τ , the voltage falls to 0.368 of its preceding value. In a few multiples of τ , the voltage becomes very close to zero, as indicated by the graph in [\[link\]](#)(b).

Now we can explain why the flash camera in our scenario takes so much longer to charge than discharge; the resistance while charging is significantly greater than while discharging. The internal resistance of the battery accounts for most of the resistance while charging. As the battery ages, the increasing internal resistance makes the charging process even slower. (You may have noticed this.)

The flash discharge is through a low-resistance ionized gas in the flash tube and proceeds very rapidly. Flash photographs, such as in [\[link\]](#), can capture a brief instant of a rapid motion because the flash can be less than a microsecond in duration. Such flashes can be made extremely intense.

During World War II, nighttime reconnaissance photographs were made from the air with a single flash illuminating more than a square kilometer of enemy territory. The brevity of the flash eliminated blurring due to the surveillance aircraft's motion. Today, an important use of intense flash lamps is to pump energy into a laser. The short intense flash can rapidly energize a laser and allow it to reemit the energy in another form.



This stop-motion photograph of a rufous hummingbird (*Selasphorus rufus*) feeding on a flower was obtained with an extremely brief and intense flash of light powered by the discharge of a capacitor through a gas.

(credit: Dean E. Biggins, U.S.
Fish and Wildlife Service)

Example:

Integrated Concept Problem: Calculating Capacitor Size—Strobe Lights

High-speed flash photography was pioneered by Doc Edgerton in the 1930s, while he was a professor of electrical engineering at MIT. You might have seen examples of his work in the amazing shots of hummingbirds in motion, a drop of milk splattering on a table, or a bullet penetrating an apple (see [\[link\]](#)). To stop the motion and capture these pictures, one needs a high-intensity, very short pulsed flash, as mentioned earlier in this module.

Suppose one wished to capture the picture of a bullet (moving at 5.0×10^2 m/s) that was passing through an apple. The duration of the flash is related to the RC time constant, τ . What size capacitor would one need in the RC circuit to succeed, if the resistance of the flash tube was $10.0 \, \Omega$? Assume the apple is a sphere with a diameter of 8.0×10^{-2} m.

Strategy

We begin by identifying the physical principles involved. This example deals with the strobe light, as discussed above. [\[link\]](#) shows the circuit for this probe. The characteristic time τ of the strobe is given as $\tau = RC$.

Solution

We wish to find C , but we don't know τ . We want the flash to be on only while the bullet traverses the apple. So we need to use the kinematic equations that describe the relationship between distance x , velocity v , and time t :

Equation:

$$x = vt \text{ or } t = \frac{x}{v}.$$

The bullet's velocity is given as 5.0×10^2 m/s, and the distance x is 8.0×10^{-2} m. The traverse time, then, is

Equation:

$$t = \frac{x}{v} = \frac{8.0 \times 10^{-2} \text{ m}}{5.0 \times 10^2 \text{ m/s}} = 1.6 \times 10^{-4} \text{ s}.$$

We set this value for the crossing time t equal to τ . Therefore,

Equation:

$$C = \frac{t}{R} = \frac{1.6 \times 10^{-4} \text{ s}}{10.0 \, \Omega} = 16 \, \mu\text{F}.$$

(Note: Capacitance C is typically measured in farads, F , defined as Coulombs per volt. From the equation, we see that C can also be stated in units of seconds per ohm.)

Discussion

The flash interval of $160 \, \mu\text{s}$ (the traverse time of the bullet) is relatively easy to obtain today. Strobe lights have opened up new worlds from science to entertainment. The information from the picture of the apple and bullet was used in the Warren Commission Report on the assassination of President John F. Kennedy in 1963 to confirm that only one bullet was fired.

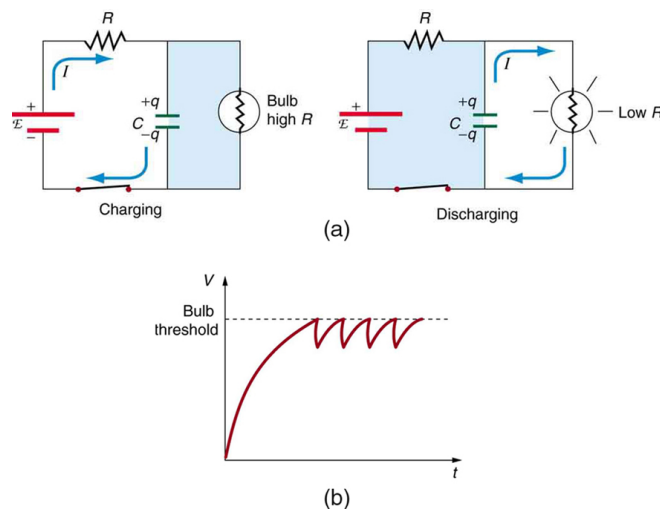
RC Circuits for Timing

RC circuits are commonly used for timing purposes. A mundane example of this is found in the ubiquitous intermittent wiper systems of modern cars. The time between wipes is varied by adjusting the resistance in an RC circuit. Another example of an RC circuit is found in novelty jewelry, Halloween costumes, and various toys that have battery-powered flashing lights. (See [\[link\]](#) for a timing circuit.)

A more crucial use of RC circuits for timing purposes is in the artificial pacemaker, used to control heart rate. The heart rate is normally controlled

by electrical signals generated by the sino-atrial (SA) node, which is on the wall of the right atrium chamber. This causes the muscles to contract and pump blood. Sometimes the heart rhythm is abnormal and the heartbeat is too high or too low.

The artificial pacemaker is inserted near the heart to provide electrical signals to the heart when needed with the appropriate time constant. Pacemakers have sensors that detect body motion and breathing to increase the heart rate during exercise to meet the body's increased needs for blood and oxygen.



(a) The lamp in this RC circuit ordinarily has a very high resistance, so that the battery charges the capacitor as if the lamp were not there. When the voltage reaches a threshold value, a current flows through the lamp that dramatically reduces its resistance, and the capacitor discharges through the lamp as if the battery and charging resistor were not there. Once discharged, the process starts again, with the flash period

determined by the RC constant τ .

(b) A graph of voltage versus time for this circuit.

Example:

Calculating Time: RC Circuit in a Heart Defibrillator

A heart defibrillator is used to resuscitate an accident victim by discharging a capacitor through the trunk of her body. A simplified version of the circuit is seen in [\[link\]](#). (a) What is the time constant if an $8.00\text{-}\mu\text{F}$ capacitor is used and the path resistance through her body is $1.00 \times 10^3 \Omega$? (b) If the initial voltage is 10.0 kV , how long does it take to decline to $5.00 \times 10^2 \text{ V}$?

Strategy

Since the resistance and capacitance are given, it is straightforward to multiply them to give the time constant asked for in part (a). To find the time for the voltage to decline to $5.00 \times 10^2 \text{ V}$, we repeatedly multiply the initial voltage by 0.368 until a voltage less than or equal to $5.00 \times 10^2 \text{ V}$ is obtained. Each multiplication corresponds to a time of τ seconds.

Solution for (a)

The time constant τ is given by the equation $\tau = RC$. Entering the given values for resistance and capacitance (and remembering that units for a farad can be expressed as s/Ω) gives

Equation:

$$\tau = RC = (1.00 \times 10^3 \Omega)(8.00 \mu\text{F}) = 8.00 \text{ ms.}$$

Solution for (b)

In the first 8.00 ms , the voltage (10.0 kV) declines to 0.368 of its initial value. That is:

Equation:

$$V = 0.368V_0 = 3.680 \times 10^3 \text{ V at } t = 8.00 \text{ ms.}$$

(Notice that we carry an extra digit for each intermediate calculation.) After another 8.00 ms, we multiply by 0.368 again, and the voltage is

Equation:

$$\begin{aligned}V_I &= 0.368V \\&= (0.368)(3.680 \times 10^3 \text{ V}) \\&= 1.354 \times 10^3 \text{ V at } t = 16.0 \text{ ms.}\end{aligned}$$

Similarly, after another 8.00 ms, the voltage is

Equation:

$$\begin{aligned}V_{II} &= 0.368V_I = (0.368)(1.354 \times 10^3 \text{ V}) \\&= 498 \text{ V at } t = 24.0 \text{ ms.}\end{aligned}$$

Discussion

So after only 24.0 ms, the voltage is down to 498 V, or 4.98% of its original value. Such brief times are useful in heart defibrillation, because the brief but intense current causes a brief but effective contraction of the heart. The actual circuit in a heart defibrillator is slightly more complex than the one in [\[link\]](#), to compensate for magnetic and AC effects that will be covered in [Magnetism](#).

Exercise:

Check Your Understanding

Problem: When is the potential difference across a capacitor an emf?

Solution:

Only when the current being drawn from or put into the capacitor is zero. Capacitors, like batteries, have internal resistance, so their output voltage is not an emf unless current is zero. This is difficult to measure in practice so we refer to a capacitor's voltage rather than its emf. But the source of potential difference in a capacitor is fundamental and it is an emf.

Note:**PhET Explorations: Circuit Construction Kit (DC only)**

An electronics kit in your computer! Build circuits with resistors, light bulbs, batteries, and switches. Take measurements with the realistic ammeter and voltmeter. View the circuit as a schematic diagram, or switch to a life-like view.

[Circuit
Construction
Kit \(DC
only\)](#)

Test Prep for AP Courses**Exercise:****Problem:**

A battery is connected to a resistor and an uncharged capacitor. The switch for the circuit is closed at $t = 0$ s.

While the capacitor is being charged, which of the following is true?

- a. Current through and voltage across the resistor increase.
- b. Current through and voltage across the resistor decrease.
- c. Current through and voltage across the resistor first increase and then decrease.

- d. Current through and voltage across the resistor first decrease and then increase.

When the capacitor is fully charged, which of the following is
b. NOT zero?

- a. Current in the resistor.
- b. Voltage across the resistor.
- c. Current in the capacitor.
- d. None of the above.

Exercise:

Problem:

An uncharged capacitor C is connected in series (with a switch) to a resistor R_1 and a voltage source E . Assume $E = 24\text{ V}$, $R_1 = 1.2\text{ k}\Omega$ and $C = 1\text{ mF}$.

- a. What will be the current through the circuit as the switch is closed? Draw a circuit diagram and show the direction of current after the switch is closed. How long will it take for the capacitor to be 99% charged?
- b. After full charging, this capacitor is connected in series to another resistor, $R_2 = 1\text{ k}\Omega$. What will be the current in the circuit as soon as it's connected? Draw a circuit diagram and show the direction of current. How long will it take for the capacitor voltage to reach 3.24 V?

Solution:

(a) 20 mA, [\[link\]](#), 5.5 s; (b) 24 mA, [\[link\]](#), 2 s

Section Summary

- An RC circuit is one that has both a resistor and a capacitor.
- The time constant τ for an RC circuit is $\tau = RC$.
- When an initially uncharged ($V_0 = 0$ at $t = 0$) capacitor in series with a resistor is charged by a DC voltage source, the voltage rises, asymptotically approaching the emf of the voltage source; as a function of time,

Equation:

$$V = \text{emf}(1 - e^{-t/RC})(\text{charging}).$$

- Within the span of each time constant τ , the voltage rises by 0.632 of the remaining value, approaching the final voltage asymptotically.
- If a capacitor with an initial voltage V_0 is discharged through a resistor starting at $t = 0$, then its voltage decreases exponentially as given by

Equation:

$$V = V_0 e^{-t/RC}(\text{discharging}).$$

- In each time constant τ , the voltage falls by 0.368 of its remaining initial value, approaching zero asymptotically.

Conceptual questions

Exercise:

Problem:

Regarding the units involved in the relationship $\tau = RC$, verify that the units of resistance times capacitance are time, that is, $\Omega \cdot \text{F} = \text{s}$.

Exercise:

Problem:

The RC time constant in heart defibrillation is crucial to limiting the time the current flows. If the capacitance in the defibrillation unit is fixed, how would you manipulate resistance in the circuit to adjust the RC constant τ ? Would an adjustment of the applied voltage also be needed to ensure that the current delivered has an appropriate value?

Exercise:**Problem:**

When making an ECG measurement, it is important to measure voltage variations over small time intervals. The time is limited by the RC constant of the circuit—it is not possible to measure time variations shorter than RC. How would you manipulate R and C in the circuit to allow the necessary measurements?

Exercise:**Problem:**

Draw two graphs of charge versus time on a capacitor. Draw one for charging an initially uncharged capacitor in series with a resistor, as in the circuit in [\[link\]](#), starting from $t = 0$. Draw the other for discharging a capacitor through a resistor, as in the circuit in [\[link\]](#), starting at $t = 0$, with an initial charge Q_0 . Show at least two intervals of τ .

Exercise:**Problem:**

When charging a capacitor, as discussed in conjunction with [\[link\]](#), how long does it take for the voltage on the capacitor to reach emf? Is this a problem?

Exercise:**Problem:**

When discharging a capacitor, as discussed in conjunction with [\[link\]](#), how long does it take for the voltage on the capacitor to reach zero? Is this a problem?

Exercise:

Problem:

Referring to [\[link\]](#), draw a graph of potential difference across the resistor versus time, showing at least two intervals of τ . Also draw a graph of current versus time for this situation.

Exercise:**Problem:**

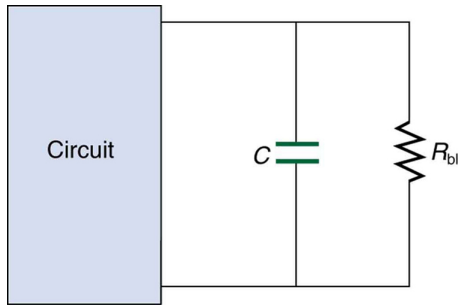
A long, inexpensive extension cord is connected from inside the house to a refrigerator outside. The refrigerator doesn't run as it should. What might be the problem?

Exercise:**Problem:**

In [\[link\]](#), does the graph indicate the time constant is shorter for discharging than for charging? Would you expect ionized gas to have low resistance? How would you adjust R to get a longer time between flashes? Would adjusting R affect the discharge time?

Exercise:**Problem:**

An electronic apparatus may have large capacitors at high voltage in the power supply section, presenting a shock hazard even when the apparatus is switched off. A “bleeder resistor” is therefore placed across such a capacitor, as shown schematically in [\[link\]](#), to bleed the charge from it after the apparatus is off. Why must the bleeder resistance be much greater than the effective resistance of the rest of the circuit? How does this affect the time constant for discharging the capacitor?



A bleeder resistor R_{bl} discharges the capacitor in this electronic device once it is switched off.

Problem Exercises

Exercise:

Problem:

The timing device in an automobile's intermittent wiper system is based on an RC time constant and utilizes a $0.500\text{-}\mu\text{F}$ capacitor and a variable resistor. Over what range must R be made to vary to achieve time constants from 2.00 to 15.0 s ?

Solution:

range 4.00 to $30.0\text{ M}\Omega$

Exercise:

Problem:

A heart pacemaker fires 72 times a minute, each time a 25.0-nF capacitor is charged (by a battery in series with a resistor) to 0.632 of its full voltage. What is the value of the resistance?

Exercise:**Problem:**

The duration of a photographic flash is related to an RC time constant, which is $0.100\ \mu\text{s}$ for a certain camera. (a) If the resistance of the flash lamp is $0.0400\ \Omega$ during discharge, what is the size of the capacitor supplying its energy? (b) What is the time constant for charging the capacitor, if the charging resistance is $800\ \text{k}\Omega$?

Solution:

(a) $2.50\ \mu\text{F}$

(b) $2.00\ \text{s}$

Exercise:**Problem:**

A $2.00\text{-}\mu\text{F}$ and a $7.50\text{-}\mu\text{F}$ capacitor can be connected in series or parallel, as can a $25.0\text{-k}\Omega$ and a $100\text{-k}\Omega$ resistor. Calculate the four RC time constants possible from connecting the resulting capacitance and resistance in series.

Exercise:**Problem:**

After two time constants, what percentage of the final voltage, emf, is on an initially uncharged capacitor C , charged through a resistance R ?

Solution:

86.5%

Exercise:

Problem:

A $500\text{-}\Omega$ resistor, an uncharged $1.50\text{-}\mu\text{F}$ capacitor, and a 6.16-V emf are connected in series. (a) What is the initial current? (b) What is the RC time constant? (c) What is the current after one time constant? (d) What is the voltage on the capacitor after one time constant?

Exercise:**Problem:**

A heart defibrillator being used on a patient has an RC time constant of 10.0 ms due to the resistance of the patient and the capacitance of the defibrillator. (a) If the defibrillator has an $8.00\text{-}\mu\text{F}$ capacitance, what is the resistance of the path through the patient? (You may neglect the capacitance of the patient and the resistance of the defibrillator.) (b) If the initial voltage is 12.0 kV , how long does it take to decline to $6.00 \times 10^2\text{ V}$?

Solution:

(a) $1.25\text{ k}\Omega$

(b) 30.0 ms

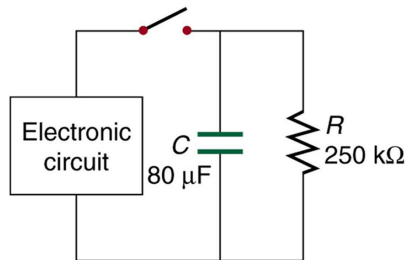
Exercise:**Problem:**

An ECG monitor must have an RC time constant less than $1.00 \times 10^2\text{ }\mu\text{s}$ to be able to measure variations in voltage over small time intervals. (a) If the resistance of the circuit (due mostly to that of the patient's chest) is $1.00\text{ k}\Omega$, what is the maximum capacitance of the circuit? (b) Would it be difficult in practice to limit the capacitance to less than the value found in (a)?

Exercise:

Problem:

[\[link\]](#) shows how a bleeder resistor is used to discharge a capacitor after an electronic device is shut off, allowing a person to work on the electronics with less risk of shock. (a) What is the time constant? (b) How long will it take to reduce the voltage on the capacitor to 0.250% (5% of 5%) of its full value once discharge begins? (c) If the capacitor is charged to a voltage V_0 through a $100\text{-}\Omega$ resistance, calculate the time it takes to rise to $0.865V_0$ (This is about two time constants.)

**Solution:**

- (a) 20.0 s
- (b) 120 s
- (c) 16.0 ms

Exercise:**Problem:**

Using the exact exponential treatment, find how much time is required to discharge a $250\text{-}\mu\text{F}$ capacitor through a $500\text{-}\Omega$ resistor down to 1.00% of its original voltage.

Exercise:**Problem:**

Using the exact exponential treatment, find how much time is required to charge an initially uncharged 100-pF capacitor through a $75.0\text{-M}\Omega$ resistor to 90.0% of its final voltage.

Solution:

$$1.73 \times 10^{-2} \text{ s}$$

Exercise:**Problem: Integrated Concepts**

If you wish to take a picture of a bullet traveling at 500 m/s, then a very brief flash of light produced by an RC discharge through a flash tube can limit blurring. Assuming 1.00 mm of motion during one RC constant is acceptable, and given that the flash is driven by a 600- μF capacitor, what is the resistance in the flash tube?

Solution:

$$3.33 \times 10^{-3} \Omega$$

Exercise:**Problem: Integrated Concepts**

A flashing lamp in a Christmas earring is based on an RC discharge of a capacitor through its resistance. The effective duration of the flash is 0.250 s, during which it produces an average 0.500 W from an average 3.00 V. (a) What energy does it dissipate? (b) How much charge moves through the lamp? (c) Find the capacitance. (d) What is the resistance of the lamp?

Exercise:**Problem: Integrated Concepts**

A 160- μF capacitor charged to 450 V is discharged through a 31.2-k Ω resistor. (a) Find the time constant. (b) Calculate the temperature increase of the resistor, given that its mass is 2.50 g and its specific heat is $1.67 \frac{\text{kJ}}{\text{kg} \cdot ^\circ\text{C}}$, noting that most of the thermal energy is retained in

the short time of the discharge. (c) Calculate the new resistance, assuming it is pure carbon. (d) Does this change in resistance seem significant?

Solution:

(a) 4.99 s

(b) 3.87°C

(c) 31.1 kΩ

(d) No

Exercise:

Problem: Unreasonable Results

(a) Calculate the capacitance needed to get an RC time constant of 1.00×10^3 s with a 0.100-Ω resistor. (b) What is unreasonable about this result? (c) Which assumptions are responsible?

Exercise:

Problem: Construct Your Own Problem

Consider a camera's flash unit. Construct a problem in which you calculate the size of the capacitor that stores energy for the flash lamp. Among the things to be considered are the voltage applied to the capacitor, the energy needed in the flash and the associated charge needed on the capacitor, the resistance of the flash lamp during discharge, and the desired RC time constant.

Exercise:

Problem: Construct Your Own Problem

Consider a rechargeable lithium cell that is to be used to power a camcorder. Construct a problem in which you calculate the internal

resistance of the cell during normal operation. Also, calculate the minimum voltage output of a battery charger to be used to recharge your lithium cell. Among the things to be considered are the emf and useful terminal voltage of a lithium cell and the current it should be able to supply to a camcorder.

Glossary

RC circuit

a circuit that contains both a resistor and a capacitor

capacitor

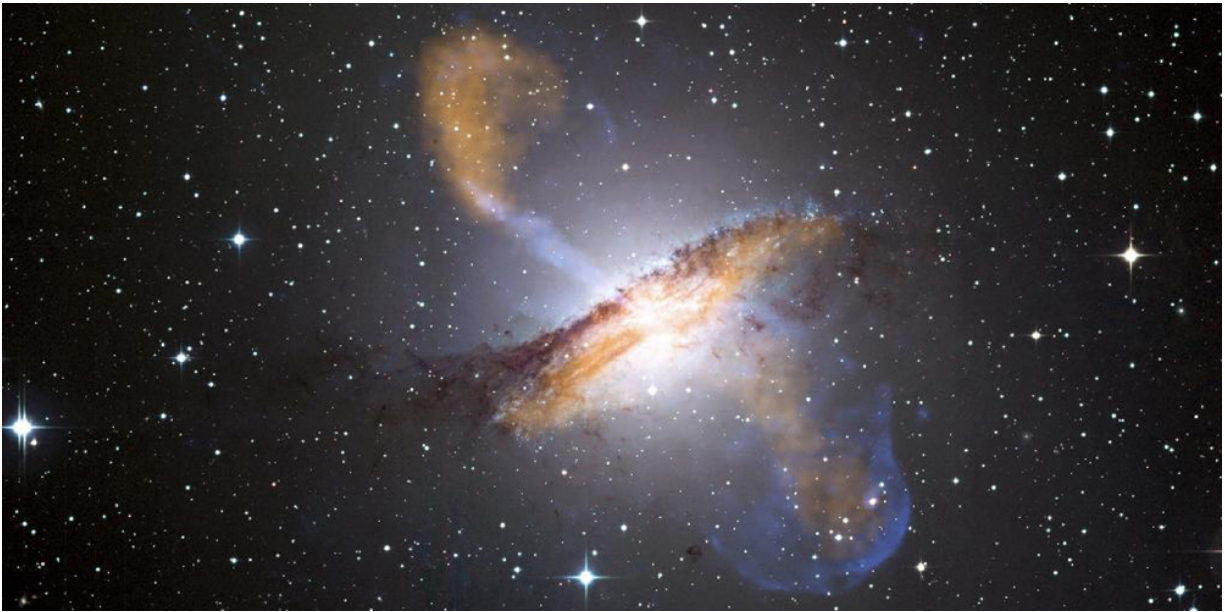
an electrical component used to store energy by separating electric charge on two opposing plates

capacitance

the maximum amount of electric potential energy that can be stored (or separated) for a given electric potential

Connection for AP® Courses
class="introduction"

This galaxy is
ejecting huge jets of
matter, powered by
an immensely
massive black hole
at its center. (credit:
X-ray:
NASA/CXC/CfA/R.
Kraft et al.)



There is mystery, surprise, adventure, and discovery in exploring new frontiers. The search for answers is that much more intriguing because the answer to any question always leads to new questions. As our understanding of nature becomes more complete, nature still retains its sense of mystery and never loses its ability to awe us.

Looking through the lens of physics allows us to look both backward and forward in time, and we can discern marvelous patterns in nature with its myriad rules and complex connections. Moreover, we continue looking ever

deeper and ever further, probing the basic structure of matter, energy, space, and time, and wondering about the scope of the universe, its beginnings, and its future.

The Big Ideas that we have been supporting and justifying throughout the previous chapters will now be used as a framework to investigate and justify new ideas. With the concepts, qualitative and quantitative problem-solving skills, the connections among topics, and all the rest of the coursework you have mastered, you will be more able to deeply appreciate the treatments that follow.

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Discuss the expansion of the universe.
- Explain how the Big Bang gave rise to the universe we see today.

Look at the sky on some clear night when you are away from city lights. There you will see thousands of individual stars and a faint glowing background of millions more. The Milky Way, as it has been called since ancient times, is an arm of our galaxy of stars—the word *galaxy* coming from the Greek word *galaxias*, meaning milky. We know a great deal about our Milky Way galaxy and of the billions of other galaxies beyond its fringes. But they still provoke wonder and awe (see [\[link\]](#)). And there are still many questions to be answered. Most remarkable when we view the universe on the large scale is that once again explanations of its character and evolution are tied to the very small scale. Particle physics and the questions being asked about the very small scales may also have their answers in the very large scales.

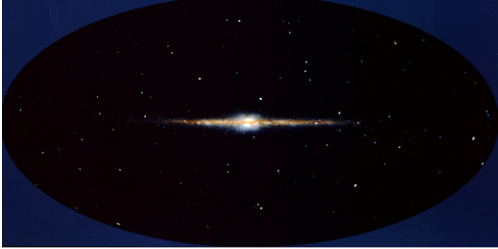


Take a moment to contemplate these clusters of galaxies, photographed by the Hubble Space Telescope. Trillions of stars linked by gravity in fantastic forms, glowing with light and showing evidence of undiscovered matter. What are they like, these myriad stars? How did they evolve? What can they tell us of matter, energy, space, and time? (credit: NASA, ESA, K. Sharon (Tel Aviv University) and E. Ofek (Caltech))

As has been noted in numerous Things Great and Small vignettes, this is not the first time the large has been explained by the small and vice versa. Newton realized that the nature of gravity on Earth that pulls an apple to the ground could explain the motion of the moon and planets so much farther away. Minute atoms and molecules explain the chemistry of substances on a much larger scale. Decays of tiny nuclei explain the hot interior of the Earth. Fusion of nuclei likewise explains the energy of stars. Today, the patterns in particle physics seem to be explaining the evolution and

character of the universe. And the nature of the universe has implications for unexplored regions of particle physics.

Cosmology is the study of the character and evolution of the universe. What are the major characteristics of the universe as we know them today? First, there are approximately 10^{11} galaxies in the observable part of the universe. An average galaxy contains more than 10^{11} stars, with our Milky Way galaxy being larger than average, both in its number of stars and its dimensions. Ours is a spiral-shaped galaxy with a diameter of about 100,000 light years and a thickness of about 2000 light years in the arms with a central bulge about 10,000 light years across. The Sun lies about 30,000 light years from the center near the galactic plane. There are significant clouds of gas, and there is a halo of less-dense regions of stars surrounding the main body. (See [\[link\]](#).) Evidence strongly suggests the existence of a large amount of additional matter in galaxies that does not produce light—the mysterious dark matter we shall later discuss.



(a)



(b)



(c)

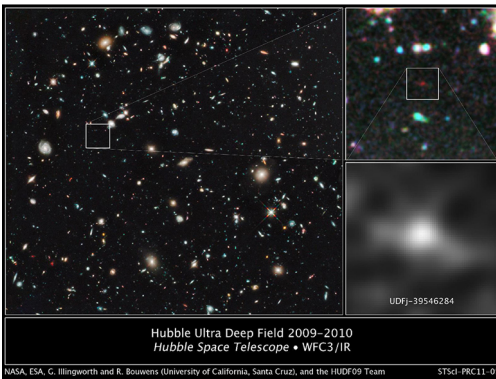
The Milky Way galaxy is typical of large spiral galaxies in its size, its shape, and the presence of gas and dust. We are fortunate to be in a location where we can see out of the galaxy and observe the vastly larger and fascinating universe

around us. (a) Side view.
(b) View from above. (c)
The Milky Way as seen
from Earth. (credits: (a)
NASA, (b) Nick Risinger,
(c) Andy)

Distances are great even within our galaxy and are measured in light years (the distance traveled by light in one year). The average distance between galaxies is on the order of a million light years, but it varies greatly with galaxies forming clusters such as shown in [\[link\]](#). The Magellanic Clouds, for example, are small galaxies close to our own, some 160,000 light years from Earth. The Andromeda galaxy is a large spiral galaxy like ours and lies 2 million light years away. It is just visible to the naked eye as an extended glow in the Andromeda constellation. Andromeda is the closest large galaxy in our local group, and we can see some individual stars in it with our larger telescopes. The most distant known galaxy is 14 billion light years from Earth—a truly incredible distance. (See [\[link\]](#).)



(a)



(b)

(a) Andromeda is the closest large galaxy, at 2 million light years distance, and is very similar to our Milky Way. The blue regions harbor young and emerging stars, while dark streaks are vast clouds of gas and dust. A smaller satellite galaxy is clearly visible.

(b) The box indicates what may be the most distant known galaxy, estimated to be 13 billion light years from us. It exists in a much older part of the universe.

(credit: NASA, ESA, G.

Illingworth (University of
California, Santa Cruz),
R. Bouwens (University
of California, Santa Cruz
and Leiden University),
and the HUDF09 Team)

Consider the fact that the light we receive from these vast distances has been on its way to us for a long time. In fact, the time in years is the same as the distance in light years. For example, the Andromeda galaxy is 2 million light years away, so that the light now reaching us left it 2 million years ago. If we could be there now, Andromeda would be different. Similarly, light from the most distant galaxy left it 14 billion years ago. We have an incredible view of the past when looking great distances. We can try to see if the universe was different then—if distant galaxies are more tightly packed or have younger-looking stars, for example, than closer galaxies, in which case there has been an evolution in time. But the problem is that the uncertainties in our data are great. Cosmology is almost typified by these large uncertainties, so that we must be especially cautious in drawing conclusions. One consequence is that there are more questions than answers, and so there are many competing theories. Another consequence is that any hard data produce a major result. Discoveries of some importance are being made on a regular basis, the hallmark of a field in its golden age.

Perhaps the most important characteristic of the universe is that all galaxies except those in our local cluster seem to be moving away from us at speeds proportional to their distance from our galaxy. It looks as if a gigantic explosion, universally called the **Big Bang**, threw matter out some billions of years ago. This amazing conclusion is based on the pioneering work of Edwin Hubble (1889–1953), the American astronomer. In the 1920s, Hubble first demonstrated conclusively that other galaxies, many previously called nebulae or clouds of stars, were outside our own. He then found that all but the closest galaxies have a red shift in their hydrogen spectra that is proportional to their distance. The explanation is that there is a **cosmological red shift** due to the expansion of space itself. The photon

wavelength is stretched in transit from the source to the observer. Double the distance, and the red shift is doubled. While this cosmological red shift is often called a Doppler shift, it is not—space itself is expanding. There is no center of expansion in the universe. All observers see themselves as stationary; the other objects in space appear to be moving away from them. Hubble was directly responsible for discovering that the universe was much larger than had previously been imagined and that it had this amazing characteristic of rapid expansion.

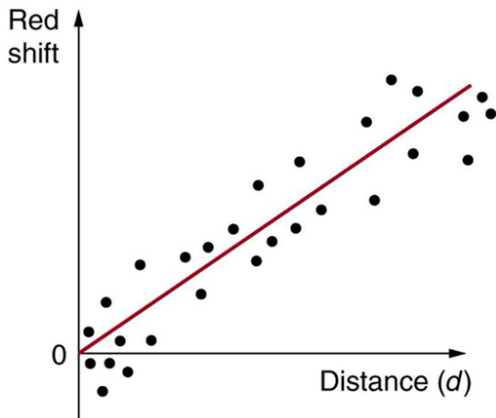
Universal expansion on the scale of galactic clusters (that is, galaxies at smaller distances are not uniformly receding from one another) is an integral part of modern cosmology. For galaxies farther away than about 50 Mly (50 million light years), the expansion is uniform with variations due to local motions of galaxies within clusters. A representative recession velocity v can be obtained from the simple formula

Equation:

$$v = H_0 d,$$

where d is the distance to the galaxy and H_0 is the **Hubble constant**. The Hubble constant is a central concept in cosmology. Its value is determined by taking the slope of a graph of velocity versus distance, obtained from red shift measurements, such as shown in [\[link\]](#). We shall use an approximate value of $H_0 = 20 \text{ km/s} \cdot \text{Mly}$. Thus, $v = H_0 d$ is an average behavior for all but the closest galaxies. For example, a galaxy 100 Mly away (as determined by its size and brightness) typically moves away from us at a speed of $v = (20 \text{ km/s} \cdot \text{Mly})(100 \text{ Mly}) = 2000 \text{ km/s}$. There can be variations in this speed due to so-called local motions or interactions with neighboring galaxies. Conversely, if a galaxy is found to be moving away from us at speed of 100,000 km/s based on its red shift, it is at a distance

$d = v/H_0 = (10,000 \text{ km/s})/(20 \text{ km/s} \cdot \text{Mly}) = 5000 \text{ Mly} = 5 \text{ Gly}$ or $5 \times 10^9 \text{ ly}$. This last calculation is approximate, because it assumes the expansion rate was the same 5 billion years ago as now. A similar calculation in Hubble's measurement changed the notion that the universe is in a steady state.

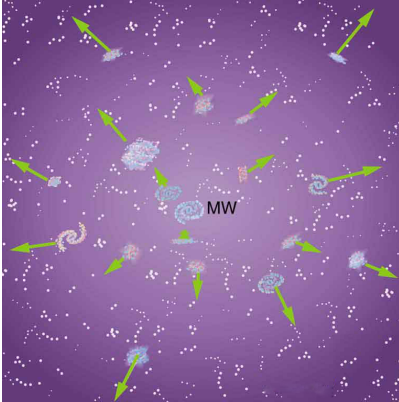


This graph of red shift versus distance for galaxies shows a linear relationship, with larger red shifts at greater distances, implying an expanding universe. The slope gives an approximate value for the expansion rate. (credit: John Cub).

One of the most intriguing developments recently has been the discovery that the expansion of the universe may be *faster now* than in the past, rather than slowing due to gravity as expected. Various groups have been looking, in particular, at supernovas in moderately distant galaxies (less than 1 Gly) to get improved distance measurements. Those distances are larger than expected for the observed galactic red shifts, implying the expansion was slower when that light was emitted. This has cosmological consequences that are discussed in [Dark Matter and Closure](#). The first results, published in 1999, are only the beginning of emerging data, with astronomy now entering a data-rich era.

[\[link\]](#) shows how the recession of galaxies looks like the remnants of a gigantic explosion, the famous Big Bang. Extrapolating backward in time,

the Big Bang would have occurred between 13 and 15 billion years ago when all matter would have been at a point. Questions instantly arise. What caused the explosion? What happened before the Big Bang? Was there a before, or did time start then? Will the universe expand forever, or will gravity reverse it into a Big Crunch? And is there other evidence of the Big Bang besides the well-documented red shifts?

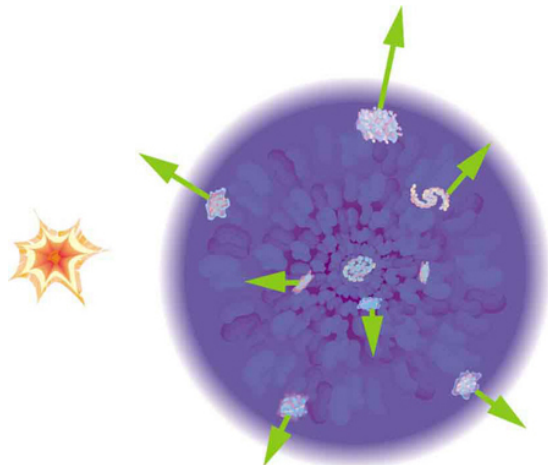


Galaxies are flying
apart from one
another, with the
more distant
moving faster as if
a primordial
explosion expelled
the matter from
which they formed.
The most distant
known galaxies
move nearly at the
speed of light
relative to us.

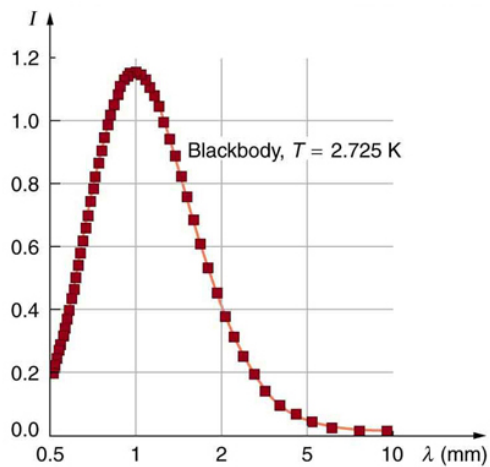
The Russian-born American physicist George Gamow (1904–1968) was among the first to note that, if there was a Big Bang, the remnants of the primordial fireball should still be evident and should be blackbody radiation. Since the radiation from this fireball has been traveling to us

since shortly after the Big Bang, its wavelengths should be greatly stretched. It will look as if the fireball has cooled in the billions of years since the Big Bang. Gamow and collaborators predicted in the late 1940s that there should be blackbody radiation from the explosion filling space with a characteristic temperature of about 7 K. Such blackbody radiation would have its peak intensity in the microwave part of the spectrum. (See [\[link\]](#).) In 1964, Arno Penzias and Robert Wilson, two American scientists working with Bell Telephone Laboratories on a low-noise radio antenna, detected the radiation and eventually recognized it for what it is.

[\[link\]](#)(b) shows the spectrum of this microwave radiation that permeates space and is of cosmic origin. It is the most perfect blackbody spectrum known, and the temperature of the fireball remnant is determined from it to be 2.725 ± 0.002 K. The detection of what is now called the **cosmic microwave background** (CMBR) was so important (generally considered as important as Hubble's detection that the galactic red shift is proportional to distance) that virtually every scientist has accepted the expansion of the universe as fact. Penzias and Wilson shared the 1978 Nobel Prize in Physics for their discovery.



(a)



(b)

(a) The Big Bang is used to explain the present observed expansion of the universe. It was an incredibly energetic explosion some 10 to 20 billion years ago. After expanding and cooling, galaxies form inside the now-cold remnants of the primordial fireball. (b) The spectrum of cosmic microwave radiation is the most perfect blackbody

spectrum ever detected. It is characteristic of a temperature of 2.725 K, the expansion-cooled temperature of the Big Bang's remnant. This radiation can be measured coming from any direction in space not obscured by some other source. It is compelling evidence of the creation of the universe in a gigantic explosion, already indicated by galactic red shifts.

Note:

Making Connections: Cosmology and Particle Physics

There are many connections of cosmology—by definition involving physics on the largest scale—with particle physics—by definition physics on the smallest scale. Among these are the dominance of matter over antimatter, the nearly perfect uniformity of the cosmic microwave background, and the mere existence of galaxies.

Matter versus antimatter

We know from direct observation that antimatter is rare. The Earth and the solar system are nearly pure matter. Space probes and cosmic rays give direct evidence—the landing of the Viking probes on Mars would have been spectacular explosions of mutual annihilation energy if Mars were antimatter. We also know that most of the universe is dominated by matter. This is proven by the lack of annihilation radiation coming to us from space, particularly the relative absence of 0.511-MeV γ rays created by the

mutual annihilation of electrons and positrons. It seemed possible that there could be entire solar systems or galaxies made of antimatter in perfect symmetry with our matter-dominated systems. But the interactions between stars and galaxies would sometimes bring matter and antimatter together in large amounts. The annihilation radiation they would produce is simply not observed. Antimatter in nature is created in particle collisions and in β^+ decays, but only in small amounts that quickly annihilate, leaving almost pure matter surviving.

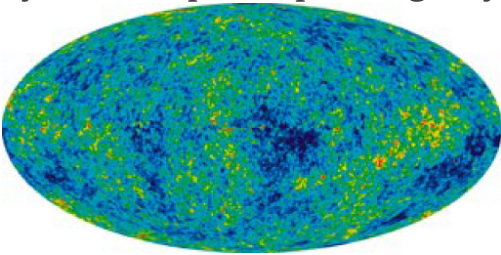
Particle physics seems symmetric in matter and antimatter. Why isn't the cosmos? The answer is that particle physics is not quite perfectly symmetric in this regard. The decay of one of the neutral K -mesons, for example, preferentially creates more matter than antimatter. This is caused by a fundamental small asymmetry in the basic forces. This small asymmetry produced slightly more matter than antimatter in the early universe. If there was only one part in 10^9 more matter (a small asymmetry), the rest would annihilate pair for pair, leaving nearly pure matter to form the stars and galaxies we see today. So the vast number of stars we observe may be only a tiny remnant of the original matter created in the Big Bang. Here at last we see a very real and important asymmetry in nature. Rather than be disturbed by an asymmetry, most physicists are impressed by how small it is. Furthermore, if the universe were completely symmetric, the mutual annihilation would be more complete, leaving far less matter to form us and the universe we know.

How can something so old have so few wrinkles?

A troubling aspect of cosmic microwave background radiation (CMBR) was soon recognized. True, the CMBR verified the Big Bang, had the correct temperature, and had a blackbody spectrum as expected. But the CMBR was *too* smooth—it looked identical in every direction. Galaxies and other similar entities could not be formed without the existence of fluctuations in the primordial stages of the universe and so there should be hot and cool spots in the CMBR, nicknamed wrinkles, corresponding to dense and sparse regions of gas caused by turbulence or early fluctuations. Over time, dense regions would contract under gravity and form stars and galaxies. Why aren't the fluctuations there? (This is a good example of an answer producing more questions.) Furthermore, galaxies are observed very

far from us, so that they formed very long ago. The problem was to explain how galaxies could form so early and so quickly after the Big Bang if its remnant fingerprint is perfectly smooth. The answer is that if you look very closely, the CMBR is not perfectly smooth, only extremely smooth.

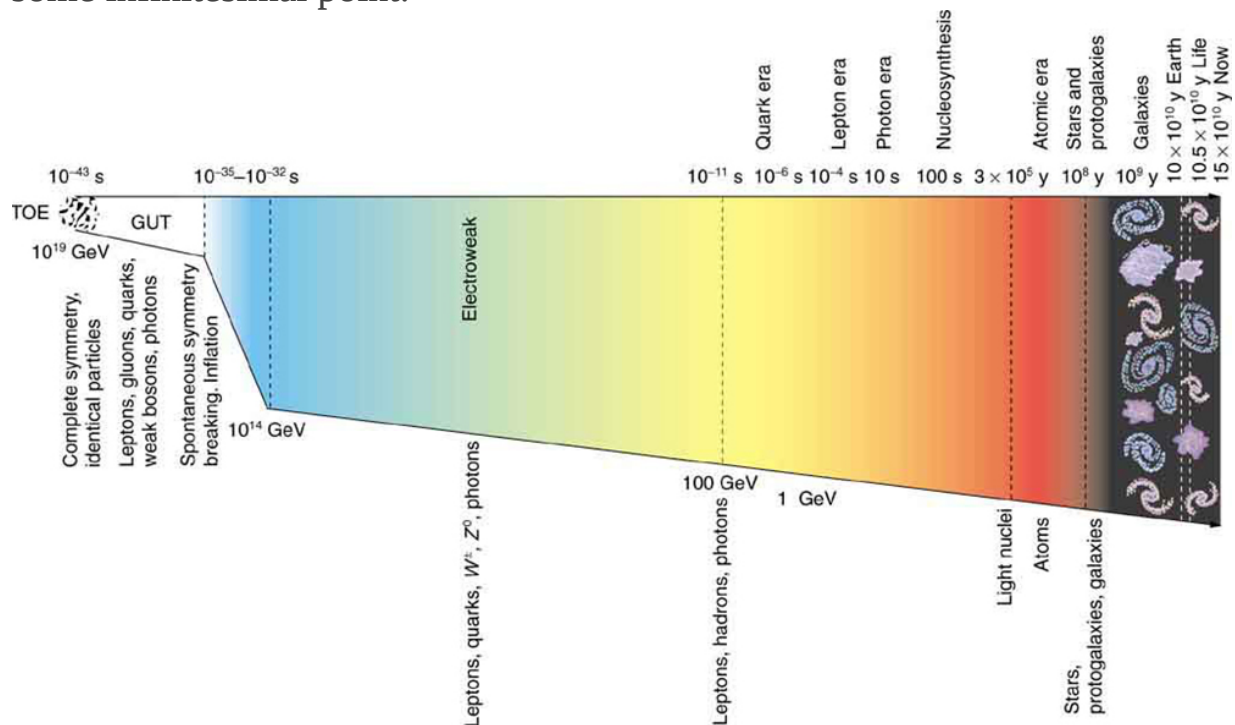
A satellite called the Cosmic Background Explorer (COBE) carried an instrument that made very sensitive and accurate measurements of the CMBR. In April of 1992, there was extraordinary publicity of COBE's first results—there were small fluctuations in the CMBR. Further measurements were carried out by experiments including NASA's Wilkinson Microwave Anisotropy Probe (WMAP), which launched in 2001. Data from WMAP provided a much more detailed picture of the CMBR fluctuations. (See [\[link\]](#).) These amount to temperature fluctuations of only $200\ \mu\text{K}$ out of $2.7\ \text{K}$, better than one part in 1000. The WMAP experiment will be followed up by the European Space Agency's Planck Surveyor, which launched in 2009.



This map of the sky uses color to show fluctuations, or wrinkles, in the cosmic microwave background observed with the WMAP spacecraft. The Milky Way has been removed for clarity. Red represents higher temperature and higher density, while blue is lower temperature and density. The fluctuations are small, less than one part in 1000, but these are still thought to be the

cause of the eventual
formation of galaxies.
(credit: NASA/WMAP
Science Team)

Let us now examine the various stages of the overall evolution of the universe from the Big Bang to the present, illustrated in [\[link\]](#). Note that scientific notation is used to encompass the many orders of magnitude in time, energy, temperature, and size of the universe. Going back in time, the two lines approach but do not cross (there is no zero on an exponential scale). Rather, they extend indefinitely in ever-smaller time intervals to some infinitesimal point.



The evolution of the universe from the Big Bang onward is intimately tied to the laws of physics, especially those of particle physics at the earliest stages. The universe is relativistic throughout its history. Theories of the unification of forces at high energies may be verified by their shaping of the universe and its evolution.

Going back in time is equivalent to what would happen if expansion stopped and gravity pulled all the galaxies together, compressing and heating all matter. At a time long ago, the temperature and density were too high for stars and galaxies to exist. Before then, there was a time when the temperature was too great for atoms to exist. And farther back yet, there was a time when the temperature and density were so great that nuclei could not exist. Even farther back in time, the temperature was so high that average kinetic energy was great enough to create short-lived particles, and the density was high enough to make this likely. When we extrapolate back to the point of W^\pm and Z^0 production (thermal energies reaching 1 TeV, or a temperature of about 10^{15} K), we reach the limits of what we know directly about particle physics. This is at a time about 10^{-12} s after the Big Bang. While 10^{-12} s may seem to be negligibly close to the instant of creation, it is not. There are important stages before this time that are tied to the unification of forces. At those stages, the universe was at extremely high energies and average particle separations were smaller than we can achieve with accelerators. What happened in the early stages before 10^{-12} s is crucial to all later stages and is possibly discerned by observing present conditions in the universe. One of these is the smoothness of the CMBR.

Names are given to early stages representing key conditions. The stage before 10^{-11} s back to 10^{-34} s is called the **electroweak epoch**, because the electromagnetic and weak forces become identical for energies above about 100 GeV. As discussed earlier, theorists expect that the strong force becomes identical to and thus unified with the electroweak force at energies of about 10^{14} GeV. The average particle energy would be this great at 10^{-34} s after the Big Bang, if there are no surprises in the unknown physics at energies above about 1 TeV. At the immense energy of 10^{14} GeV (corresponding to a temperature of about 10^{26} K), the W^\pm and Z^0 carrier particles would be transformed into massless gauge bosons to accomplish the unification. Before 10^{-34} s back to about 10^{-43} s, we have Grand Unification in the **GUT epoch**, in which all forces except gravity are identical. At 10^{-43} s, the average energy reaches the immense 10^{19} GeV needed to unify gravity with the other forces in TOE, the Theory of Everything. Before that time is the **TOE epoch**, but we have almost no idea

as to the nature of the universe then, since we have no workable theory of quantum gravity. We call the hypothetical unified force **superforce**.

Now let us imagine starting at TOE and moving forward in time to see what type of universe is created from various events along the way. As temperatures and average energies decrease with expansion, the universe reaches the stage where average particle separations are large enough to see differences between the strong and electroweak forces (at about 10^{-35} s). After this time, the forces become distinct in almost all interactions—they are no longer unified or symmetric. This transition from GUT to electroweak is an example of **spontaneous symmetry breaking**, in which conditions spontaneously evolved to a point where the forces were no longer unified, breaking that symmetry. This is analogous to a phase transition in the universe, and a clever proposal by American physicist Alan Guth in the early 1980s ties it to the smoothness of the CMBR. Guth proposed that spontaneous symmetry breaking (like a phase transition during cooling of normal matter) released an immense amount of energy that caused the universe to expand extremely rapidly for the brief time from 10^{-35} s to about 10^{-32} s. This expansion may have been by an incredible factor of 10^{50} or more in the size of the universe and is thus called the **inflationary scenario**. One result of this inflation is that it would stretch the wrinkles in the universe nearly flat, leaving an extremely smooth CMBR. While speculative, there is as yet no other plausible explanation for the smoothness of the CMBR. Unless the CMBR is not really cosmic but local in origin, the distances between regions of similar temperatures are too great for any coordination to have caused them, since any coordination mechanism must travel at the speed of light. Again, particle physics and cosmology are intimately entwined. There is little hope that we may be able to test the inflationary scenario directly, since it occurs at energies near 10^{14} GeV, vastly greater than the limits of modern accelerators. But the idea is so attractive that it is incorporated into most cosmological theories.

Characteristics of the present universe may help us determine the validity of this intriguing idea. Additionally, the recent indications that the universe's expansion rate may be *increasing* (see [Dark Matter and Closure](#)) could even imply that we are *in* another inflationary epoch.

It is important to note that, if conditions such as those found in the early universe could be created in the laboratory, we would see the unification of forces directly today. The forces have not changed in time, but the average energy and separation of particles in the universe have. As discussed in [The Four Basic Forces](#), the four basic forces in nature are distinct under most circumstances found today. The early universe and its remnants provide evidence from times when they were unified under most circumstances.

Section Summary

- Cosmology is the study of the character and evolution of the universe.
- The two most important features of the universe are the cosmological red shifts of its galaxies being proportional to distance and its cosmic microwave background (CMBR). Both support the notion that there was a gigantic explosion, known as the Big Bang that created the universe.
- Galaxies farther away than our local group have, on an average, a recessional velocity given by

Equation:

$$v = H_0 d,$$

where d is the distance to the galaxy and H_0 is the Hubble constant, taken to have the average value $H_0 = 20 \text{ km/s} \cdot \text{Mly}$.

- Explanations of the large-scale characteristics of the universe are intimately tied to particle physics.
- The dominance of matter over antimatter and the smoothness of the CMBR are two characteristics that are tied to particle physics.
- The epochs of the universe are known back to very shortly after the Big Bang, based on known laws of physics.
- The earliest epochs are tied to the unification of forces, with the electroweak epoch being partially understood, the GUT epoch being speculative, and the TOE epoch being highly speculative since it involves an unknown single superforce.
- The transition from GUT to electroweak is called spontaneous symmetry breaking. It released energy that caused the inflationary

scenario, which in turn explains the smoothness of the CMBR.

Conceptual Questions

Exercise:

Problem:

Explain why it only *appears* that we are at the center of expansion of the universe and why an observer in another galaxy would see the same relative motion of all but the closest galaxies away from her.

Exercise:

Problem:

If there is no observable edge to the universe, can we determine where its center of expansion is? Explain.

Exercise:

Problem: If the universe is infinite, does it have a center? Discuss.

Exercise:

Problem:

Another known cause of red shift in light is the source being in a high gravitational field. Discuss how this can be eliminated as the source of galactic red shifts, given that the shifts are proportional to distance and not to the size of the galaxy.

Exercise:

Problem:

If some unknown cause of red shift—such as light becoming “tired” from traveling long distances through empty space—is discovered, what effect would there be on cosmology?

Exercise:

Problem:

Olbers's paradox poses an interesting question: If the universe is infinite, then any line of sight should eventually fall on a star's surface. Why then is the sky dark at night? Discuss the commonly accepted evolution of the universe as a solution to this paradox.

Exercise:**Problem:**

If the cosmic microwave background radiation (CMBR) is the remnant of the Big Bang's fireball, we expect to see hot and cold regions in it. What are two causes of these wrinkles in the CMBR? Are the observed temperature variations greater or less than originally expected?

Exercise:**Problem:**

The decay of one type of K -meson is cited as evidence that nature favors matter over antimatter. Since mesons are composed of a quark and an antiquark, is it surprising that they would preferentially decay to one type over another? Is this an asymmetry in nature? Is the predominance of matter over antimatter an asymmetry?

Exercise:**Problem:**

Distances to local galaxies are determined by measuring the brightness of stars, called Cepheid variables, that can be observed individually and that have absolute brightnesses at a standard distance that are well known. Explain how the measured brightness would vary with distance as compared with the absolute brightness.

Exercise:

Problem:

Distances to very remote galaxies are estimated based on their apparent type, which indicate the number of stars in the galaxy, and their measured brightness. Explain how the measured brightness would vary with distance. Would there be any correction necessary to compensate for the red shift of the galaxy (all distant galaxies have significant red shifts)? Discuss possible causes of uncertainties in these measurements.

Exercise:**Problem:**

If the smallest meaningful time interval is greater than zero, will the lines in [\[link\]](#) ever meet?

Problems & Exercises**Exercise:****Problem:**

Find the approximate mass of the luminous matter in the Milky Way galaxy, given it has approximately 10^{11} stars of average mass 1.5 times that of our Sun.

Solution:

$$3 \times 10^{41} \text{ kg}$$

Exercise:

Problem:

Find the approximate mass of the dark and luminous matter in the Milky Way galaxy. Assume the luminous matter is due to approximately 10^{11} stars of average mass 1.5 times that of our Sun, and take the dark matter to be 10 times as massive as the luminous matter.

Exercise:**Problem:**

(a) Estimate the mass of the luminous matter in the known universe, given there are 10^{11} galaxies, each containing 10^{11} stars of average mass 1.5 times that of our Sun. (b) How many protons (the most abundant nuclide) are there in this mass? (c) Estimate the total number of particles in the observable universe by multiplying the answer to (b) by two, since there is an electron for each proton, and then by 10^9 , since there are far more particles (such as photons and neutrinos) in space than in luminous matter.

Solution:

(a) 3×10^{52} kg

(b) 2×10^{79}

(c) 4×10^{88}

Exercise:**Problem:**

If a galaxy is 500 Mly away from us, how fast do we expect it to be moving and in what direction?

Exercise:

Problem:

On average, how far away are galaxies that are moving away from us at 2.0% of the speed of light?

Solution:

0.30 Gly

Exercise:**Problem:**

Our solar system orbits the center of the Milky Way galaxy. Assuming a circular orbit 30,000 ly in radius and an orbital speed of 250 km/s, how many years does it take for one revolution? Note that this is approximate, assuming constant speed and circular orbit, but it is representative of the time for our system and local stars to make one revolution around the galaxy.

Exercise:**Problem:**

(a) What is the approximate speed relative to us of a galaxy near the edge of the known universe, some 10 Gly away? (b) What fraction of the speed of light is this? Note that we have observed galaxies moving away from us at greater than $0.9c$.

Solution:

(a) 2.0×10^5 km/s

(b) $0.67c$

Exercise:

Problem:

(a) Calculate the approximate age of the universe from the average value of the Hubble constant, $H_0 = 20 \text{ km/s} \cdot \text{Mly}$. To do this, calculate the time it would take to travel 1 Mly at a constant expansion rate of 20 km/s. (b) If deceleration is taken into account, would the actual age of the universe be greater or less than that found here? Explain.

Exercise:**Problem:**

Assuming a circular orbit for the Sun about the center of the Milky Way galaxy, calculate its orbital speed using the following information: The mass of the galaxy is equivalent to a single mass 1.5×10^{11} times that of the Sun (or $3 \times 10^{41} \text{ kg}$), located 30,000 ly away.

Solution:

$$2.7 \times 10^5 \text{ m/s}$$

Exercise:**Problem:**

(a) What is the approximate force of gravity on a 70-kg person due to the Andromeda galaxy, assuming its total mass is 10^{13} that of our Sun and acts like a single mass 2 Mly away? (b) What is the ratio of this force to the person's weight? Note that Andromeda is the closest large galaxy.

Exercise:**Problem:**

Andromeda galaxy is the closest large galaxy and is visible to the naked eye. Estimate its brightness relative to the Sun, assuming it has luminosity 10^{12} times that of the Sun and lies 2 Mly away.

Solution:

6×10^{-11} (an overestimate, since some of the light from Andromeda is blocked by gas and dust within that galaxy)

Exercise:**Problem:**

(a) A particle and its antiparticle are at rest relative to an observer and annihilate (completely destroying both masses), creating two γ rays of equal energy. What is the characteristic γ -ray energy you would look for if searching for evidence of proton-antiproton annihilation? (The fact that such radiation is rarely observed is evidence that there is very little antimatter in the universe.) (b) How does this compare with the 0.511-MeV energy associated with electron-positron annihilation?

Exercise:**Problem:**

The average particle energy needed to observe unification of forces is estimated to be 10^{19} GeV. (a) What is the rest mass in kilograms of a particle that has a rest mass of 10^{19} GeV/ c^2 ? (b) How many times the mass of a hydrogen atom is this?

Solution:

(a) 2×10^{-8} kg

(b) 1×10^{19}

Exercise:

Problem:

The peak intensity of the CMBR occurs at a wavelength of 1.1 mm. (a) What is the energy in eV of a 1.1-mm photon? (b) There are approximately 10^9 photons for each massive particle in deep space. Calculate the energy of 10^9 such photons. (c) If the average massive particle in space has a mass half that of a proton, what energy would be created by converting its mass to energy? (d) Does this imply that space is “matter dominated”? Explain briefly.

Exercise:**Problem:**

(a) What Hubble constant corresponds to an approximate age of the universe of 10^{10} y? To get an approximate value, assume the expansion rate is constant and calculate the speed at which two galaxies must move apart to be separated by 1 Mly (present average galactic separation) in a time of 10^{10} y. (b) Similarly, what Hubble constant corresponds to a universe approximately 2×10^{10} -y old?

Solution:

(a) $30 \text{ km/s} \cdot \text{Mly}$

(b) $15 \text{ km/s} \cdot \text{Mly}$

Exercise:**Problem:**

Show that the velocity of a star orbiting its galaxy in a circular orbit is inversely proportional to the square root of its orbital radius, assuming the mass of the stars inside its orbit acts like a single mass at the center of the galaxy. You may use an equation from a previous chapter to support your conclusion, but you must justify its use and define all terms used.

Exercise:

Problem:

The core of a star collapses during a supernova, forming a neutron star. Angular momentum of the core is conserved, and so the neutron star spins rapidly. If the initial core radius is 5.0×10^5 km and it collapses to 10.0 km, find the neutron star's angular velocity in revolutions per second, given the core's angular velocity was originally 1 revolution per 30.0 days.

Solution:

960 rev/s

Exercise:**Problem:**

Using data from the previous problem, find the increase in rotational kinetic energy, given the core's mass is 1.3 times that of our Sun. Where does this increase in kinetic energy come from?

Exercise:**Problem:**

Distances to the nearest stars (up to 500 ly away) can be measured by a technique called parallax, as shown in [\[link\]](#). What are the angles θ_1 and θ_2 relative to the plane of the Earth's orbit for a star 4.0 ly directly above the Sun?

Solution:

89.999773° (many digits are used to show the difference between 90°)

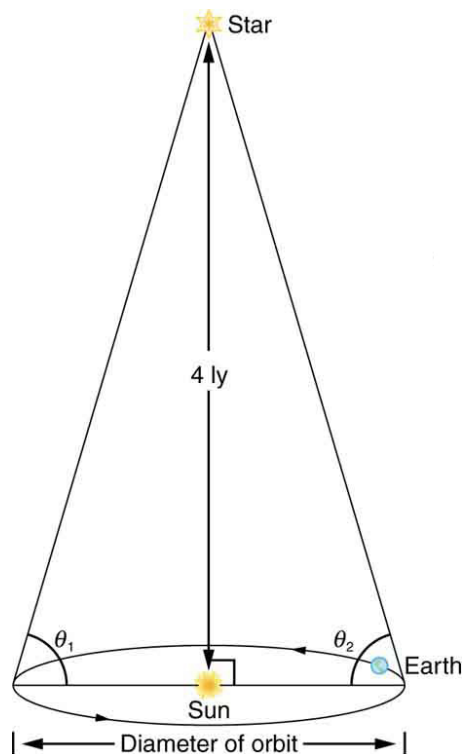
Exercise:

Problem:

(a) Use the Heisenberg uncertainty principle to calculate the uncertainty in energy for a corresponding time interval of 10^{-43} s. (b) Compare this energy with the 10^{19} GeV unification-of-forces energy and discuss why they are similar.

Exercise:**Problem: Construct Your Own Problem**

Consider a star moving in a circular orbit at the edge of a galaxy. Construct a problem in which you calculate the mass of that galaxy in kg and in multiples of the solar mass based on the velocity of the star and its distance from the center of the galaxy.



Distances to nearby stars are measured using triangulation, also called the parallax method. The angle of

line of sight to the star
is measured at
intervals six months
apart, and the distance
is calculated by using
the known diameter of
the Earth's orbit. This
can be done for stars
up to about 500 ly
away.

Glossary

Big Bang

a gigantic explosion that threw out matter a few billion years ago

cosmic microwave background

the spectrum of microwave radiation of cosmic origin

cosmological red shift

the photon wavelength is stretched in transit from the source to the observer because of the expansion of space itself

cosmology

the study of the character and evolution of the universe

electroweak epoch

the stage before 10^{-11} back to 10^{-34} after the Big Bang

GUT epoch

the time period from 10^{-43} to 10^{-34} after the Big Bang, when Grand Unification Theory, in which all forces except gravity are identical, governed the universe

Hubble constant

a central concept in cosmology whose value is determined by taking the slope of a graph of velocity versus distance, obtained from red shift measurements

inflationary scenario

the rapid expansion of the universe by an incredible factor of 10^{-50} for the brief time from 10^{-35} to about 10^{-32} s

spontaneous symmetry breaking

the transition from GUT to electroweak where the forces were no longer unified

superforce

hypothetical unified force in TOE epoch

TOE epoch

before 10^{-43} after the Big Bang

LEARNING OBJECTIVES

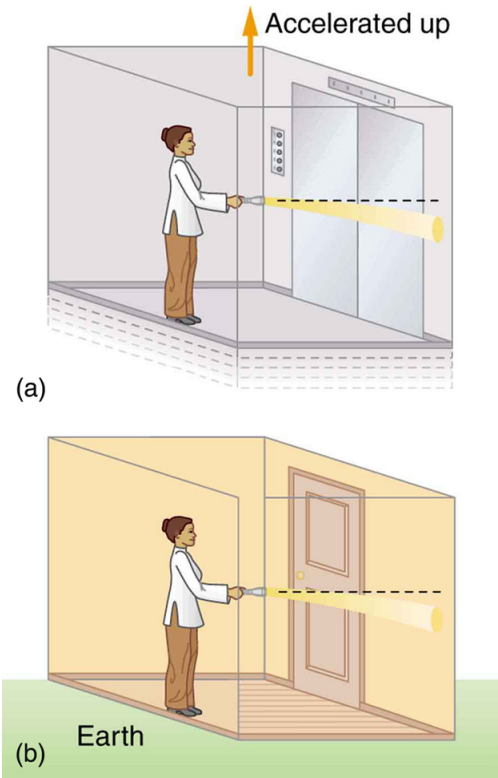
By the end of this section, you will be able to:

- Explain the effect of gravity on light.
- Discuss black holes.
- Explain quantum gravity.

When we talk of black holes or the unification of forces, we are actually discussing aspects of general relativity and quantum gravity. We know from [Special Relativity](#) that relativity is the study of how different observers measure the same event, particularly if they move relative to one another. Einstein's theory of **general relativity** describes all types of relative motion including accelerated motion and the effects of gravity. General relativity encompasses special relativity and classical relativity in situations where acceleration is zero and relative velocity is small compared with the speed of light. Many aspects of general relativity have been verified experimentally, some of which are better than science fiction in that they are bizarre but true. **Quantum gravity** is the theory that deals with particle exchange of gravitons as the mechanism for the force, and with extreme conditions where quantum mechanics and general relativity must both be used. A good theory of quantum gravity does not yet exist, but one will be needed to understand how all four forces may be unified. If we are successful, the theory of quantum gravity will encompass all others, from classical physics to relativity to quantum mechanics—truly a Theory of Everything (TOE).

General Relativity

Einstein first considered the case of no observer acceleration when he developed the revolutionary special theory of relativity, publishing his first work on it in 1905. By 1916, he had laid the foundation of general relativity, again almost on his own. Much of what Einstein did to develop his ideas was to mentally analyze certain carefully and clearly defined situations—doing this is to perform a **thought experiment**. [\[link\]](#) illustrates a thought experiment like the ones that convinced Einstein that light must fall in a gravitational field. Think about what a person feels in an elevator that is accelerated upward. It is identical to being in a stationary elevator in a gravitational field. The feet of a person are pressed against the floor, and objects released from hand fall with identical accelerations. In fact, it is not possible, without looking outside, to know what is happening—acceleration upward or gravity. This led Einstein to correctly postulate that acceleration and gravity will produce identical effects in all situations. So, if acceleration affects light, then gravity will, too. [\[link\]](#) shows the effect of acceleration on a beam of light shone horizontally at one wall. Since the accelerated elevator moves up during the time light travels across the elevator, the beam of light strikes low, seeming to the person to bend down. (Normally a tiny effect, since the speed of light is so great.) The same effect must occur due to gravity, Einstein reasoned, since there is no way to tell the effects of gravity acting downward from acceleration of the elevator upward. Thus gravity affects the path of light, even though we think of gravity as acting between masses and photons are massless.



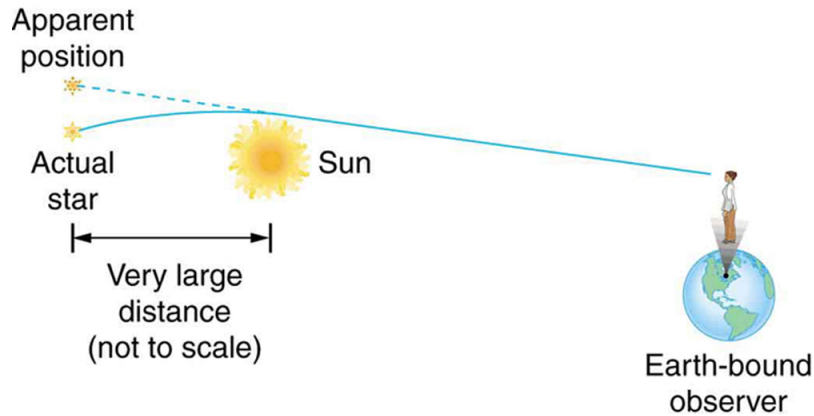
(a) A beam of light emerges from a flashlight in an upward-accelerating elevator. Since the elevator moves up during the time the light takes to reach the wall, the beam strikes lower than it would if the elevator were not accelerated. (b) Gravity has the same effect on light, since it is not possible to tell whether the elevator is accelerating upward or acted upon by gravity.

Einstein's theory of general relativity got its first verification in 1919 when starlight passing near the Sun was observed during a solar eclipse. (See [\[link\]](#).) During an eclipse, the sky is darkened and we can briefly see stars. Those in a line of sight nearest the Sun should have a shift in their apparent positions. Not only was this shift observed, but it agreed with Einstein's predictions well within experimental uncertainties. This discovery created a scientific and public sensation. Einstein was now a folk hero as well as a very great scientist. The bending of light by matter is equivalent to a bending of space itself, with light following the curve. This is another radical change in our concept of space and time. It is also another connection that any particle with mass or energy (massless photons) is affected by gravity.

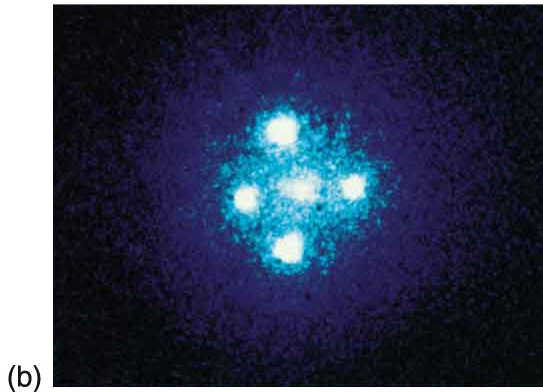
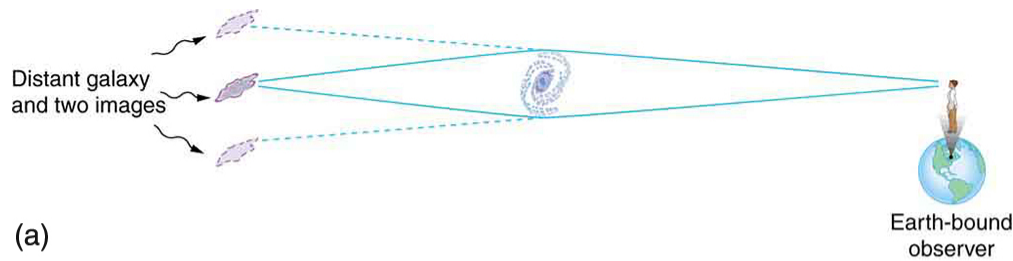
There are several current forefront efforts related to general relativity. One is the observation and analysis of gravitational lensing of light. Another is analysis of the definitive proof of the existence of black holes. Direct observation of gravitational waves or moving wrinkles in space is being searched for. Theoretical efforts are also being aimed at the possibility of time travel and wormholes into other parts of space due to black holes.

Gravitational lensing

As you can see in [\[link\]](#), light is bent toward a mass, producing an effect much like a converging lens (large masses are needed to produce observable effects). On a galactic scale, the light from a distant galaxy could be “lensed” into several images when passing close by another galaxy on its way to Earth. Einstein predicted this effect, but he considered it unlikely that we would ever observe it. A number of cases of this effect have now been observed; one is shown in [\[link\]](#). This effect is a much larger scale verification of general relativity. But such gravitational lensing is also useful in verifying that the red shift is proportional to distance. The red shift of the intervening galaxy is always less than that of the one being lensed, and each image of the lensed galaxy has the same red shift. This verification supplies more evidence that red shift is proportional to distance. Confidence that the multiple images are not different objects is bolstered by the observations that if one image varies in brightness over time, the others also vary in the same manner.



This schematic shows how light passing near a massive body like the Sun is curved toward it. The light that reaches the Earth then seems to be coming from different locations than the known positions of the originating stars. Not only was this effect observed, the amount of bending was precisely what Einstein predicted in his general theory of relativity.



(a) Light from a distant galaxy can travel different paths to the Earth because it is bent around an intermediary galaxy by gravity. This produces several images of the more distant galaxy. (b) The images around the central galaxy are produced by gravitational lensing. Each image has the same spectrum and a larger red shift than the intermediary. (credit: NASA, ESA, and STScI)

Black holes

Black holes are objects having such large gravitational fields that things can fall in, but nothing, not even light, can escape. Bodies, like the Earth or the Sun, have what is called an **escape velocity**. If an object moves straight up from the body, starting at the escape velocity, it will just be able to escape the gravity of the body. The greater the acceleration of gravity on the body, the greater is the escape velocity. As long ago as the late 1700s, it was proposed that if the escape velocity is greater than the speed of light, then light cannot escape. Simon Laplace (1749–1827), the French astronomer and mathematician, even incorporated this idea of a dark star into his writings. But the idea was dropped after Young's double slit experiment showed light to be a wave. For some time, light was thought not to have

particle characteristics and, thus, could not be acted upon by gravity. The idea of a black hole was very quickly reincarnated in 1916 after Einstein's theory of general relativity was published. It is now thought that black holes can form in the supernova collapse of a massive star, forming an object perhaps 10 km across and having a mass greater than that of our Sun. It is interesting that several prominent physicists who worked on the concept, including Einstein, firmly believed that nature would find a way to prohibit such objects.

Black holes are difficult to observe directly, because they are small and no light comes directly from them. In fact, no light comes from inside the **event horizon**, which is defined to be at a distance from the object at which the escape velocity is exactly the speed of light. The radius of the event horizon is known as the **Schwarzschild radius** R_S and is given by

Equation:

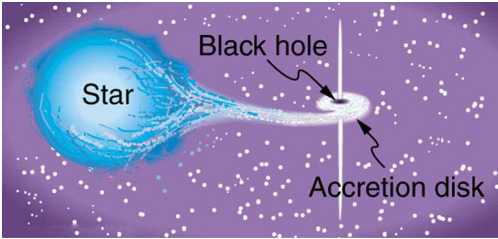
$$R_S = \frac{2GM}{c^2},$$

where G is the universal gravitational constant, M is the mass of the body, and c is the speed of light. The event horizon is the edge of the black hole and R_S is its radius (that is, the size of a black hole is twice R_S). Since G is small and c^2 is large, you can see that black holes are extremely small, only a few kilometers for masses a little greater than the Sun's. The object itself is inside the event horizon.

Physics near a black hole is fascinating. Gravity increases so rapidly that, as you approach a black hole, the tidal effects tear matter apart, with matter closer to the hole being pulled in with much more force than that only slightly farther away. This can pull a companion star apart and heat inflowing gases to the point of producing X rays. (See [\[link\]](#).) We have observed X rays from certain binary star systems that are consistent with such a picture. This is not quite proof of black holes, because the X rays could also be caused by matter falling onto a neutron star. These objects were first discovered in 1967 by the British astrophysicists, Jocelyn Bell and Anthony Hewish. **Neutron stars** are literally a star composed of neutrons. They are formed by the collapse of a star's core in a supernova,

during which electrons and protons are forced together to form neutrons (the reverse of neutron β decay). Neutron stars are slightly larger than a black hole of the same mass and will not collapse further because of resistance by the strong force. However, neutron stars cannot have a mass greater than about eight solar masses or they must collapse to a black hole. With recent improvements in our ability to resolve small details, such as with the orbiting Chandra X-ray Observatory, it has become possible to measure the masses of X-ray-emitting objects by observing the motion of companion stars and other matter in their vicinity. What has emerged is a plethora of X-ray-emitting objects too massive to be neutron stars. This evidence is considered conclusive and the existence of black holes is widely accepted. These black holes are concentrated near galactic centers.

We also have evidence that supermassive black holes may exist at the cores of many galaxies, including the Milky Way. Such a black hole might have a mass millions or even billions of times that of the Sun, and it would probably have formed when matter first coalesced into a galaxy billions of years ago. Supporting this is the fact that very distant galaxies are more likely to have abnormally energetic cores. Some of the moderately distant galaxies, and hence among the younger, are known as **quasars** and emit as much or more energy than a normal galaxy but from a region less than a light year across. Quasar energy outputs may vary in times less than a year, so that the energy-emitting region must be less than a light year across. The best explanation of quasars is that they are young galaxies with a supermassive black hole forming at their core, and that they become less energetic over billions of years. In closer superactive galaxies, we observe tremendous amounts of energy being emitted from very small regions of space, consistent with stars falling into a black hole at the rate of one or more a month. The Hubble Space Telescope (1994) observed an accretion disk in the galaxy M87 rotating rapidly around a region of extreme energy emission. (See [\[link\]](#).) A jet of material being ejected perpendicular to the plane of rotation gives further evidence of a supermassive black hole as the engine.



A black hole is shown pulling matter away from a companion star, forming a superheated accretion disk where X rays are emitted before the matter disappears forever into the hole. The in-fall energy also ejects some material, forming the two vertical spikes. (See also the photograph in [Introduction to Frontiers of Physics](#).) There are several X-ray-emitting objects in space that are consistent with this picture and are likely to be black holes.

Gravitational waves

If a massive object distorts the space around it, like the foot of a water bug on the surface of a pond, then movement of the massive object should create waves in space like those on a pond. **Gravitational waves** are mass-created distortions in space that propagate at the speed of light and are predicted by general relativity. Since gravity is by far the weakest force, extreme conditions are needed to generate significant gravitational waves. Gravity near binary neutron star systems is so great that significant gravitational wave energy is radiated as the two neutron stars orbit one

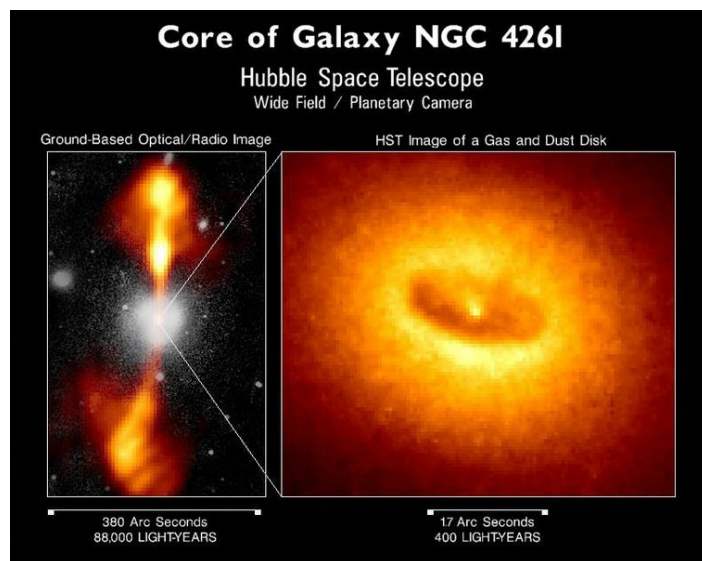
another. American astronomers, Joseph Taylor and Russell Hulse, measured changes in the orbit of such a binary neutron star system. They found its orbit to change precisely as predicted by general relativity, a strong indication of gravitational waves, and were awarded the 1993 Nobel Prize. But direct detection of gravitational waves on Earth would be conclusive. For many years, various attempts have been made to detect gravitational waves by observing vibrations induced in matter distorted by these waves. American physicist Joseph Weber pioneered this field in the 1960s, but no conclusive events have been observed. (No gravity wave detectors were in operation at the time of the 1987A supernova, unfortunately.) There are now several ambitious systems of gravitational wave detectors in use around the world. These include the LIGO (Laser Interferometer Gravitational Wave Observatory) system with two laser interferometer detectors, one in the state of Washington and another in Louisiana (See [\[link\]](#)) and the VIRGO (Variability of Irradiance and Gravitational Oscillations) facility in Italy with a single detector.

Quantum Gravity

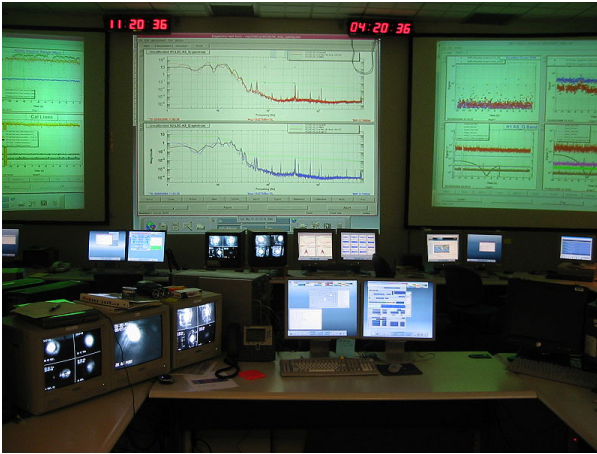
Black holes radiate

Quantum gravity is important in those situations where gravity is so extremely strong that it has effects on the quantum scale, where the other forces are ordinarily much stronger. The early universe was such a place, but black holes are another. The first significant connection between gravity and quantum effects was made by the Russian physicist Yakov Zel'dovich in 1971, and other significant advances followed from the British physicist Stephen Hawking. (See [\[link\]](#).) These two showed that black holes could radiate away energy by quantum effects just outside the event horizon (nothing can escape from inside the event horizon). Black holes are, thus, expected to radiate energy and shrink to nothing, although extremely slowly for most black holes. The mechanism is the creation of a particle-antiparticle pair from energy in the extremely strong gravitational field near the event horizon. One member of the pair falls into the hole and the other escapes, conserving momentum. (See [\[link\]](#).) When a black hole loses energy and, hence, rest mass, its event horizon shrinks, creating an even greater gravitational field. This increases the rate of pair production so that the process grows exponentially until the black hole is nuclear in size. A

final burst of particles and γ rays ensues. This is an extremely slow process for black holes about the mass of the Sun (produced by supernovas) or larger ones (like those thought to be at galactic centers), taking on the order of 10^{67} years or longer! Smaller black holes would evaporate faster, but they are only speculated to exist as remnants of the Big Bang. Searches for characteristic γ -ray bursts have produced events attributable to more mundane objects like neutron stars accreting matter.



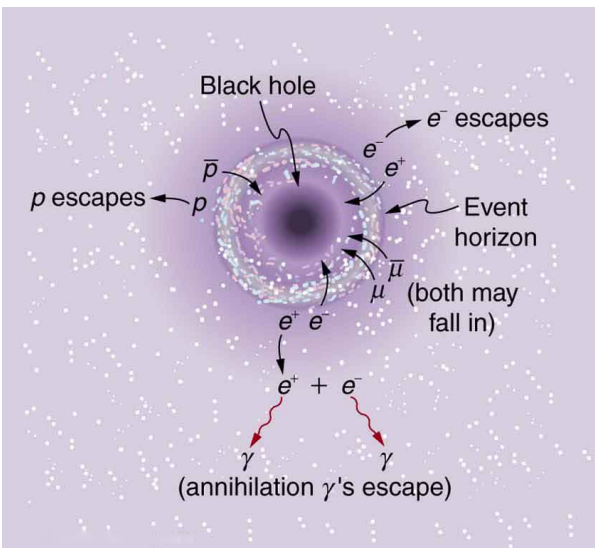
This Hubble Space Telescope photograph shows the extremely energetic core of the NGC 4261 galaxy. With the superior resolution of the orbiting telescope, it has been possible to observe the rotation of an accretion disk around the energy-producing object as well as to map jets of material being ejected from the object. A supermassive black hole is consistent with these observations, but other possibilities are not quite eliminated. (credit: NASA and ESA)



The control room of the LIGO gravitational wave detector. Gravitational waves will cause extremely small vibrations in a mass in this detector, which will be detected by laser interferometer techniques. Such detection in coincidence with other detectors and with astronomical events, such as supernovas, would provide direct evidence of gravitational waves. (credit: Tobin Fricke)



Stephen Hawking (b. 1942) has made many contributions to the theory of quantum gravity. Hawking is a long-time survivor of ALS and has produced popular books on general relativity, cosmology, and quantum gravity. (credit: Lwp Kommunikáció)



Gravity and quantum mechanics

come into play when a black hole creates a particle-antiparticle pair from the energy in its gravitational field. One member of the pair falls into the hole while the other escapes, removing energy and shrinking the black hole. The search is on for the characteristic energy.

Wormholes and time travel

The subject of time travel captures the imagination. Theoretical physicists, such as the American Kip Thorne, have treated the subject seriously, looking into the possibility that falling into a black hole could result in popping up in another time and place—a trip through a so-called wormhole. Time travel and wormholes appear in innumerable science fiction dramatizations, but the consensus is that time travel is not possible in theory. While still debated, it appears that quantum gravity effects inside a black hole prevent time travel due to the creation of particle pairs. Direct evidence is elusive.

The shortest time

Theoretical studies indicate that, at extremely high energies and correspondingly early in the universe, quantum fluctuations may make time intervals meaningful only down to some finite time limit. Early work indicated that this might be the case for times as long as 10^{-43} s, the time at which all forces were unified. If so, then it would be meaningless to consider the universe at times earlier than this. Subsequent studies indicate that the crucial time may be as short as 10^{-95} s. But the point remains—quantum gravity seems to imply that there is no such thing as a vanishingly short time. Time may, in fact, be grainy with no meaning to time intervals shorter than some tiny but finite size.

The future of quantum gravity

Not only is quantum gravity in its infancy, no one knows how to get started on a theory of gravitons and unification of forces. The energies at which TOE should be valid may be so high (at least 10^{19} GeV) and the necessary particle separation so small (less than 10^{-35} m) that only indirect evidence can provide clues. For some time, the common lament of theoretical physicists was one so familiar to struggling students—how do you even get started? But Hawking and others have made a start, and the approach many theorists have taken is called Superstring theory, the topic of the [Superstrings](#).

Section Summary

- Einstein's theory of general relativity includes accelerated frames and, thus, encompasses special relativity and gravity. Created by use of careful thought experiments, it has been repeatedly verified by real experiments.
- One direct result of this behavior of nature is the gravitational lensing of light by massive objects, such as galaxies, also seen in the microlensing of light by smaller bodies in our galaxy.
- Another prediction is the existence of black holes, objects for which the escape velocity is greater than the speed of light and from which nothing can escape.
- The event horizon is the distance from the object at which the escape velocity equals the speed of light c . It is called the Schwarzschild radius R_S and is given by

Equation:

$$R_S = \frac{2GM}{c^2},$$

where G is the universal gravitational constant, and M is the mass of the body.

- Physics is unknown inside the event horizon, and the possibility of wormholes and time travel are being studied.
- Candidates for black holes may power the extremely energetic emissions of quasars, distant objects that seem to be early stages of

galactic evolution.

- Neutron stars are stellar remnants, having the density of a nucleus, that hint that black holes could form from supernovas, too.
- Gravitational waves are wrinkles in space, predicted by general relativity but not yet observed, caused by changes in very massive objects.
- Quantum gravity is an incompletely developed theory that strives to include general relativity, quantum mechanics, and unification of forces (thus, a TOE).
- One unconfirmed connection between general relativity and quantum mechanics is the prediction of characteristic radiation from just outside black holes.

Conceptual Questions

Exercise:

Problem:

Quantum gravity, if developed, would be an improvement on both general relativity and quantum mechanics, but more mathematically difficult. Under what circumstances would it be necessary to use quantum gravity? Similarly, under what circumstances could general relativity be used? When could special relativity, quantum mechanics, or classical physics be used?

Exercise:

Problem:

Does observed gravitational lensing correspond to a converging or diverging lens? Explain briefly.

Exercise:

Problem:

Suppose you measure the red shifts of all the images produced by gravitational lensing, such as in [\[link\]](#). You find that the central image has a red shift less than the outer images, and those all have the same red shift. Discuss how this not only shows that the images are of the same object, but also implies that the red shift is not affected by taking different paths through space. Does it imply that cosmological red shifts are not caused by traveling through space (light getting tired, perhaps)?

Exercise:**Problem:**

What are gravitational waves, and have they yet been observed either directly or indirectly?

Exercise:**Problem:**

Is the event horizon of a black hole the actual physical surface of the object?

Exercise:**Problem:**

Suppose black holes radiate their mass away and the lifetime of a black hole created by a supernova is about 10^{67} years. How does this lifetime compare with the accepted age of the universe? Is it surprising that we do not observe the predicted characteristic radiation?

Problems & Exercises**Exercise:**

Problem:

What is the Schwarzschild radius of a black hole that has a mass eight times that of our Sun? Note that stars must be more massive than the Sun to form black holes as a result of a supernova.

Solution:

23.6 km

Exercise:**Problem:**

Black holes with masses smaller than those formed in supernovas may have been created in the Big Bang. Calculate the radius of one that has a mass equal to the Earth's.

Exercise:**Problem:**

Supermassive black holes are thought to exist at the center of many galaxies.

- (a) What is the radius of such an object if it has a mass of 10^9 Suns?
 - (b) What is this radius in light years?
-

Solution:

(a) 2.95×10^{12} m

(b) 3.12×10^{-4} ly

Exercise:**Problem: Construct Your Own Problem**

Consider a supermassive black hole near the center of a galaxy. Calculate the radius of such an object based on its mass. You must consider how much mass is reasonable for these large objects, and which is now nearly directly observed. (Information on black holes posted on the Web by NASA and other agencies is reliable, for example.)

Glossary

black holes

objects having such large gravitational fields that things can fall in, but nothing, not even light, can escape

general relativity

Einstein's theory that describes all types of relative motion including accelerated motion and the effects of gravity

gravitational waves

mass-created distortions in space that propagate at the speed of light and that are predicted by general relativity

escape velocity

takeoff velocity when kinetic energy just cancels gravitational potential energy

event horizon

the distance from the object at which the escape velocity is exactly the speed of light

neutron stars

literally a star composed of neutrons

Schwarzschild radius

the radius of the event horizon

thought experiment

mental analysis of certain carefully and clearly defined situations to develop an idea

quasars

the moderately distant galaxies that emit as much or more energy than a normal galaxy

Quantum gravity

the theory that deals with particle exchange of gravitons as the mechanism for the force

Superstrings

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define Superstring theory.
- Explain the relationship between Superstring theory and the Big Bang.

Introduced earlier in [GUTS: The Unification of Forces](#) **Superstring theory** is an attempt to unify gravity with the other three forces and, thus, must contain quantum gravity. The main tenet of Superstring theory is that fundamental particles, including the graviton that carries the gravitational force, act like one-dimensional vibrating strings. Since gravity affects the time and space in which all else exists, Superstring theory is an attempt at a Theory of Everything (TOE). Each independent quantum number is thought of as a separate dimension in some super space (analogous to the fact that the familiar dimensions of space are independent of one another) and is represented by a different type of Superstring. As the universe evolved after the Big Bang and forces became distinct (spontaneous symmetry breaking), some of the dimensions of superspace are imagined to have curled up and become unnoticed.

Forces are expected to be unified only at extremely high energies and at particle separations on the order of 10^{-35} m. This could mean that Superstrings must have dimensions or wavelengths of this size or smaller. Just as quantum gravity may imply that there are no time intervals shorter than some finite value, it also implies that there may be no sizes smaller

than some tiny but finite value. That may be about 10^{-35} m. If so, and if Superstring theory can explain all it strives to, then the structures of Superstrings are at the lower limit of the smallest possible size and can have no further substructure. This would be the ultimate answer to the question the ancient Greeks considered. There is a finite lower limit to space.

Not only is Superstring theory in its infancy, it deals with dimensions about 17 orders of magnitude smaller than the 10^{-18} m details that we have been able to observe directly. It is thus relatively unconstrained by experiment, and there are a host of theoretical possibilities to choose from. This has led theorists to make choices subjectively (as always) on what is the most elegant theory, with less hope than usual that experiment will guide them. It has also led to speculation of alternate universes, with their Big Bangs creating each new universe with a random set of rules. These speculations may not be tested even in principle, since an alternate universe is by definition unattainable. It is something like exploring a self-consistent field of mathematics, with its axioms and rules of logic that are not consistent with nature. Such endeavors have often given insight to mathematicians and scientists alike and occasionally have been directly related to the description of new discoveries.

Section Summary

- Superstring theory holds that fundamental particles are one-dimensional vibrations analogous to those on strings and is an attempt at a theory of quantum gravity.

Problems & Exercises

Exercise:

Problem:

The characteristic length of entities in Superstring theory is approximately 10^{-35} m.

(a) Find the energy in GeV of a photon of this wavelength.

(b) Compare this with the average particle energy of 10^{19} GeV needed for unification of forces.

Solution:

(a) 1×10^{20}

(b) 10 times greater

Glossary

Superstring theory

a theory to unify gravity with the other three forces in which the fundamental particles are considered to act like one-dimensional vibrating strings

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Discuss the evidence for the existence of dark matter.
- Explain neutrino oscillations and the consequences thereof.

One of the most exciting problems in physics today is the fact that there is far more matter in the universe than we can see. The motion of stars in galaxies and the motion of galaxies in clusters imply that there is about 10 times as much mass as in the luminous objects we can see. The indirectly observed non-luminous matter is called **dark matter**. Why is dark matter a problem? For one thing, we do not know what it is. It may well be 90% of all matter in the universe, yet there is a possibility that it is of a completely unknown form—a stunning discovery if verified. Dark matter has implications for particle physics. It may be possible that neutrinos actually have small masses or that there are completely unknown types of particles. Dark matter also has implications for cosmology, since there may be enough dark matter to stop the expansion of the universe. That is another problem related to dark matter—we do not know how much there is. We keep finding evidence for more matter in the universe, and we have an idea of how much it would take to eventually stop the expansion of the universe, but whether there is enough is still unknown.

Evidence

The first clues that there is more matter than meets the eye came from the Swiss-born American astronomer Fritz Zwicky in the 1930s; some initial work was also done by the American astronomer Vera Rubin. Zwicky measured the velocities of stars orbiting the galaxy, using the relativistic Doppler shift of their spectra (see [\[link\]](#)(a)). He found that velocity varied with distance from the center of the galaxy, as graphed in [\[link\]](#)(b). If the mass of the galaxy was concentrated in its center, as are its luminous stars, the velocities should decrease as the square root of the distance from the center. Instead, the velocity curve is almost flat, implying that there is a tremendous amount of matter in the galactic halo. Although not immediately recognized for its significance, such measurements have now been made for many galaxies, with similar results. Further, studies of galactic clusters have also indicated that galaxies have a mass distribution greater than that obtained from their brightness (proportional to the number of stars), which also extends into large halos surrounding the luminous parts of galaxies. Observations of other EM wavelengths, such as radio waves and X rays, have similarly confirmed the existence of dark matter. Take, for example, X rays in the relatively dark space between galaxies, which indicates the presence of previously unobserved hot, ionized gas (see [\[link\]](#) (c)).

Theoretical Yearnings for Closure

Is the universe open or closed? That is, will the universe expand forever or will it stop, perhaps to contract? This, until recently, was a question of whether there is enough gravitation to stop the expansion of the universe. In the past few years, it has become a question of the combination of gravitation and what is called the **cosmological constant**. The cosmological constant was invented by Einstein to prohibit the expansion or contraction of the universe. At the time he developed general relativity, Einstein considered that an illogical possibility. The cosmological constant was discarded after Hubble discovered the expansion, but has been re-invoked in recent years.

Gravitational attraction between galaxies is slowing the expansion of the universe, but the amount of slowing down is not known directly. In fact, the cosmological constant can counteract gravity's effect. As recent

measurements indicate, the universe is expanding *faster* now than in the past—perhaps a “modern inflationary era” in which the dark energy is thought to be causing the expansion of the present-day universe to accelerate. If the expansion rate were affected by gravity alone, we should be able to see that the expansion rate between distant galaxies was once greater than it is now. However, measurements show it was *less* than now. We can, however, calculate the amount of slowing based on the average density of matter we observe directly. Here we have a definite answer—there is far less visible matter than needed to stop expansion. The **critical density** ρ_c is defined to be the density needed to just halt universal expansion in a universe with no cosmological constant. It is estimated to be about

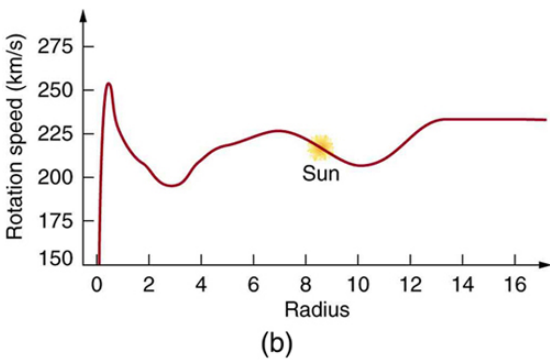
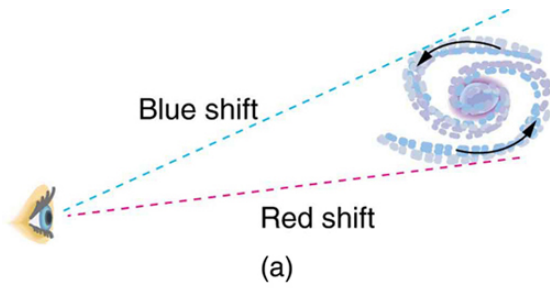
Equation:

$$\rho_c \approx 10^{-26} \text{ kg/m}^3.$$

However, this estimate of ρ_c is only good to about a factor of two, due to uncertainties in the expansion rate of the universe. The critical density is equivalent to an average of only a few nucleons per cubic meter, remarkably small and indicative of how truly empty intergalactic space is. Luminous matter seems to account for roughly 0.5% to 2% of the critical density, far less than that needed for closure. Taking into account the amount of dark matter we detect indirectly and all other types of indirectly observed normal matter, there is only 10% to 40% of what is needed for closure. If we are able to refine the measurements of expansion rates now and in the past, we will have our answer regarding the curvature of space and we will determine a value for the cosmological constant to justify this observation. Finally, the most recent measurements of the CMBR have implications for the cosmological constant, so it is not simply a device concocted for a single purpose.

After the recent experimental discovery of the cosmological constant, most researchers feel that the universe should be just barely open. Since matter can be thought to curve the space around it, we call an open universe **negatively curved**. This means that you can in principle travel an unlimited distance in any direction. A universe that is closed is called **positively**

curved. This means that if you travel far enough in any direction, you will return to your starting point, analogous to circumnavigating the Earth. In between these two is a **flat (zero curvature) universe.** The recent discovery of the cosmological constant has shown the universe is very close to flat, and will expand forever. Why do theorists feel the universe is flat? Flatness is a part of the inflationary scenario that helps explain the flatness of the microwave background. In fact, since general relativity implies that matter creates the space in which it exists, there is a special symmetry to a flat universe.



Evidence for dark matter: (a)

We can measure the velocities of stars relative to their galaxies by observing the Doppler shift in emitted light, usually using the hydrogen spectrum. These measurements indicate the

rotation of a spiral galaxy.
(b) A graph of velocity versus distance from the galactic center shows that the velocity does not decrease as it would if the matter were concentrated in luminous stars. The flatness of the curve implies a massive galactic halo of dark matter extending beyond the visible stars. (c) This is a computer-generated image of X rays from a galactic cluster. The X rays indicate the presence of otherwise unseen hot clouds of ionized gas in the regions of space previously considered more empty. (credit: NASA, ESA, CXC, M. Bradac (University of California, Santa Barbara), and S. Allen (Stanford University))

What Is the Dark Matter We See Indirectly?

There is no doubt that dark matter exists, but its form and the amount in existence are two facts that are still being studied vigorously. As always, we seek to explain new observations in terms of known principles. However, as more discoveries are made, it is becoming more and more difficult to explain dark matter as a known type of matter.

One of the possibilities for normal matter is being explored using the Hubble Space Telescope and employing the lensing effect of gravity on

light (see [\[link\]](#)). Stars glow because of nuclear fusion in them, but planets are visible primarily by reflected light. Jupiter, for example, is too small to ignite fusion in its core and become a star, but we can see sunlight reflected from it, since we are relatively close. If Jupiter orbited another star, we would not be able to see it directly. The question is open as to how many planets or other bodies smaller than about 1/1000 the mass of the Sun are there. If such bodies pass between us and a star, they will not block the star's light, being too small, but they will form a gravitational lens, as discussed in [General Relativity and Quantum Gravity](#).

In a process called **microlensing**, light from the star is focused and the star appears to brighten in a characteristic manner. Searches for dark matter in this form are particularly interested in galactic halos because of the huge amount of mass that seems to be there. Such microlensing objects are thus called **massive compact halo objects**, or **MACHOs**. To date, a few MACHOs have been observed, but not predominantly in galactic halos, nor in the numbers needed to explain dark matter.

MACHOs are among the most conventional of unseen objects proposed to explain dark matter. Others being actively pursued are red dwarfs, which are small dim stars, but too few have been seen so far, even with the Hubble Telescope, to be of significance. Old remnants of stars called white dwarfs are also under consideration, since they contain about a solar mass, but are small as the Earth and may dim to the point that we ordinarily do not observe them. While white dwarfs are known, old dim ones are not. Yet another possibility is the existence of large numbers of smaller than stellar mass black holes left from the Big Bang—here evidence is entirely absent.

There is a very real possibility that dark matter is composed of the known neutrinos, which may have small, but finite, masses. As discussed earlier, neutrinos are thought to be massless, but we only have upper limits on their masses, rather than knowing they are exactly zero. So far, these upper limits come from difficult measurements of total energy emitted in the decays and reactions in which neutrinos are involved. There is an amusing possibility of proving that neutrinos have mass in a completely different way.

We have noted in [Particles, Patterns, and Conservation Laws](#) that there are three flavors of neutrinos (ν_e , ν_μ , and ν_τ) and that the weak interaction could change quark flavor. It should also change neutrino flavor—that is, any type of neutrino could change spontaneously into any other, a process called **neutrino oscillations**. However, this can occur only if neutrinos have a mass. Why? Crudely, because if neutrinos are massless, they must travel at the speed of light and time will not pass for them, so that they cannot change without an interaction. In 1999, results began to be published containing convincing evidence that neutrino oscillations do occur. Using the Super-Kamiokande detector in Japan, the oscillations have been observed and are being verified and further explored at present at the same facility and others.

Neutrino oscillations may also explain the low number of observed solar neutrinos. Detectors for observing solar neutrinos are specifically designed to detect electron neutrinos ν_e produced in huge numbers by fusion in the Sun. A large fraction of electron neutrinos ν_e may be changing flavor to muon neutrinos ν_μ on their way out of the Sun, possibly enhanced by specific interactions, reducing the flux of electron neutrinos to observed levels. There is also a discrepancy in observations of neutrinos produced in cosmic ray showers. While these showers of radiation produced by extremely energetic cosmic rays should contain twice as many ν_μ s as ν_e s, their numbers are nearly equal. This may be explained by neutrino oscillations from muon flavor to electron flavor. Massive neutrinos are a particularly appealing possibility for explaining dark matter, since their existence is consistent with a large body of known information and explains more than dark matter. The question is not settled at this writing.

The most radical proposal to explain dark matter is that it consists of previously unknown leptons (sometimes obtusely referred to as non-baryonic matter). These are called **weakly interacting massive particles**, or **WIMPs**, and would also be chargeless, thus interacting negligibly with normal matter, except through gravitation. One proposed group of WIMPs would have masses several orders of magnitude greater than nucleons and are sometimes called **neutralinos**. Others are called **axions** and would have masses about 10^{-10} that of an electron mass. Both neutralinos and axions would be gravitationally attached to galaxies, but because they are

chargeless and only feel the weak force, they would be in a halo rather than interact and coalesce into spirals, and so on, like normal matter (see [\[link\]](#)).



The Hubble Space Telescope is producing exciting data with its corrected optics and with the absence of atmospheric distortion. It has observed some MACHOs, disks of material around stars thought to precede planet formation, black hole candidates, and collisions of comets with Jupiter. (credit: NASA (crew of STS-125))



Dark matter may shepherd normal matter gravitationally in space, as this stream moves the leaves. Dark matter may be invisible and even move through the normal matter, as neutrinos penetrate us without small-scale effect. (credit: Shinichi Sugiyama)

Some particle theorists have built WIMPs into their unified force theories and into the inflationary scenario of the evolution of the universe so popular today. These particles would have been produced in just the correct numbers to make the universe flat, shortly after the Big Bang. The proposal is radical in the sense that it invokes entirely new forms of matter, in fact *two* entirely new forms, in order to explain dark matter and other phenomena. WIMPs have the extra burden of automatically being very difficult to observe directly. This is somewhat analogous to quark confinement, which guarantees that quarks are there, but they can never be seen directly. One of the primary goals of the LHC at CERN, however, is to produce and detect WIMPs. At any rate, before WIMPs are accepted as the best explanation, all other possibilities utilizing known phenomena will have to be shown inferior. Should that occur, we will be in the unanticipated position of admitting that, to date, all we know is only 10% of what exists.

A far cry from the days when people firmly believed themselves to be not only the center of the universe, but also the reason for its existence.

Section Summary

- Dark matter is non-luminous matter detected in and around galaxies and galactic clusters.
- It may be 10 times the mass of the luminous matter in the universe, and its amount may determine whether the universe is open or closed (expands forever or eventually stops).
- The determining factor is the critical density of the universe and the cosmological constant, a theoretical construct intimately related to the expansion and closure of the universe.
- The critical density ρ_c is the density needed to just halt universal expansion. It is estimated to be approximately 10^{-26} kg/m^3 .
- An open universe is negatively curved, a closed universe is positively curved, whereas a universe with exactly the critical density is flat.
- Dark matter's composition is a major mystery, but it may be due to the suspected mass of neutrinos or a completely unknown type of leptonic matter.
- If neutrinos have mass, they will change families, a process known as neutrino oscillations, for which there is growing evidence.

Conceptual Questions

Exercise:

Problem:

Discuss the possibility that star velocities at the edges of galaxies being greater than expected is due to unknown properties of gravity rather than to the existence of dark matter. Would this mean, for example, that gravity is greater or smaller than expected at large distances? Are there other tests that could be made of gravity at large distances, such as observing the motions of neighboring galaxies?

Exercise:

Problem:

How does relativistic time dilation prohibit neutrino oscillations if they are massless?

Exercise:**Problem:**

If neutrino oscillations do occur, will they violate conservation of the various lepton family numbers (L_e , L_μ , and L_τ)? Will neutrino oscillations violate conservation of the total number of leptons?

Exercise:**Problem:**

Lacking direct evidence of WIMPs as dark matter, why must we eliminate all other possible explanations based on the known forms of matter before we invoke their existence?

Problems Exercises**Exercise:****Problem:**

If the dark matter in the Milky Way were composed entirely of MACHOs (evidence shows it is not), approximately how many would there have to be? Assume the average mass of a MACHO is 1/1000 that of the Sun, and that dark matter has a mass 10 times that of the luminous Milky Way galaxy with its 10^{11} stars of average mass 1.5 times the Sun's mass.

Solution:

$$1.5 \times 10^{15}$$

Exercise:

Problem:

The critical mass density needed to just halt the expansion of the universe is approximately 10^{-26} kg/m^3 .

(a) Convert this to $\text{eV}/c^2 \cdot \text{m}^3$.

(b) Find the number of neutrinos per cubic meter needed to close the universe if their average mass is $7 \text{ eV}/c^2$ and they have negligible kinetic energies.

Exercise:**Problem:**

Assume the average density of the universe is 0.1 of the critical density needed for closure. What is the average number of protons per cubic meter, assuming the universe is composed mostly of hydrogen?

Solution:

$$0.6 \text{ m}^{-3}$$

Exercise:**Problem:**

To get an idea of how empty deep space is on the average, perform the following calculations:

(a) Find the volume our Sun would occupy if it had an average density equal to the critical density of 10^{-26} kg/m^3 thought necessary to halt the expansion of the universe.

(b) Find the radius of a sphere of this volume in light years.

(c) What would this radius be if the density were that of luminous matter, which is approximately 5% that of the critical density?

(d) Compare the radius found in part (c) with the 4-ly average separation of stars in the arms of the Milky Way.

Glossary

axions

a type of WIMPs having masses about 10^{-10} of an electron mass

cosmological constant

a theoretical construct intimately related to the expansion and closure of the universe

critical density

the density of matter needed to just halt universal expansion

dark matter

indirectly observed non-luminous matter

flat (zero curvature) universe

a universe that is infinite but not curved

microlensing

a process in which light from a distant star is focused and the star appears to brighten in a characteristic manner, when a small body (smaller than about 1/1000 the mass of the Sun) passes between us and the star

MACHOs

massive compact halo objects; microlensing objects of huge mass

neutrino oscillations

a process in which any type of neutrino could change spontaneously into any other

neutralinos

a type of WIMPs having masses several orders of magnitude greater than nucleon masses

negatively curved

an open universe that expands forever

positively curved

a universe that is closed and eventually contracts

WIMPs

weakly interacting massive particles; chargeless leptons (non-baryonic matter) interacting negligibly with normal matter

Complexity and Chaos

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain complex systems.
- Discuss chaotic behavior of different systems.

Much of what impresses us about physics is related to the underlying connections and basic simplicity of the laws we have discovered. The language of physics is precise and well defined because many basic systems we study are simple enough that we can perform controlled experiments and discover unambiguous relationships. Our most spectacular successes, such as the prediction of previously unobserved particles, come from the simple underlying patterns we have been able to recognize. But there are systems of interest to physicists that are inherently complex. The simple laws of physics apply, of course, but complex systems may reveal patterns that simple systems do not. The emerging field of **complexity** is devoted to the study of complex systems, including those outside the traditional bounds of physics. Of particular interest is the ability of complex systems to adapt and evolve.

What are some examples of complex adaptive systems? One is the primordial ocean. When the oceans first formed, they were a random mix of elements and compounds that obeyed the laws of physics and chemistry. In a relatively short geological time (about 500 million years), life had emerged. Laboratory simulations indicate that the emergence of life was far too fast to have come from random combinations of compounds, even if

driven by lightning and heat. There must be an underlying ability of the complex system to organize itself, resulting in the self-replication we recognize as life. Living entities, even at the unicellular level, are highly organized and systematic. Systems of living organisms are themselves complex adaptive systems. The grandest of these evolved into the biological system we have today, leaving traces in the geological record of steps taken along the way.

Complexity as a discipline examines complex systems, how they adapt and evolve, looking for similarities with other complex adaptive systems. Can, for example, parallels be drawn between biological evolution and the evolution of *economic systems*? Economic systems do emerge quickly, they show tendencies for self-organization, they are complex (in the number and types of transactions), and they adapt and evolve. Biological systems do all the same types of things. There are other examples of complex adaptive systems being studied for fundamental similarities. *Cultures* show signs of adaptation and evolution. The comparison of different cultural evolutions may bear fruit as well as comparisons to biological evolution. *Science* also is a complex system of human interactions, like culture and economics, that adapts to new information and political pressure, and evolves, usually becoming more organized rather than less. Those who study *creative thinking* also see parallels with complex systems. Humans sometimes organize almost random pieces of information, often subconsciously while doing other things, and come up with brilliant creative insights. The development of *language* is another complex adaptive system that may show similar tendencies. *Artificial intelligence* is an overt attempt to devise an adaptive system that will self-organize and evolve in the same manner as an intelligent living being learns. These are a few of the broad range of topics being studied by those who investigate complexity. There are now institutes, journals, and meetings, as well as popularizations of the emerging topic of complexity.

In traditional physics, the discipline of complexity may yield insights in certain areas. Thermodynamics treats systems on the average, while statistical mechanics deals in some detail with complex systems of atoms and molecules in random thermal motion. Yet there is organization, adaptation, and evolution in those complex systems. Non-equilibrium

phenomena, such as heat transfer and phase changes, are characteristically complex in detail, and new approaches to them may evolve from complexity as a discipline. Crystal growth is another example of self-organization spontaneously emerging in a complex system. Alloys are also inherently complex mixtures that show certain simple characteristics implying some self-organization. The organization of iron atoms into magnetic domains as they cool is another. Perhaps insights into these difficult areas will emerge from complexity. But at the minimum, the discipline of complexity is another example of human effort to understand and organize the universe around us, partly rooted in the discipline of physics.

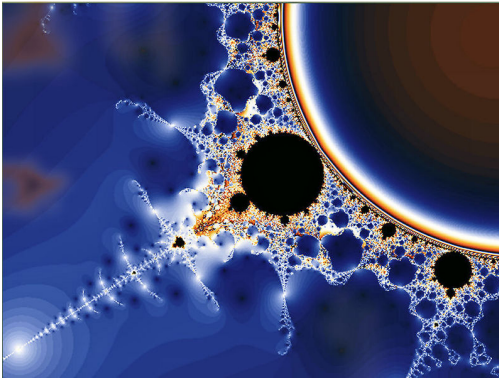
A predecessor to complexity is the topic of chaos, which has been widely publicized and has become a discipline of its own. It is also based partly in physics and treats broad classes of phenomena from many disciplines.

Chaos is a word used to describe systems whose outcomes are extremely sensitive to initial conditions. The orbit of the planet Pluto, for example, may be chaotic in that it can change tremendously due to small interactions with other planets. This makes its long-term behavior impossible to predict with precision, just as we cannot tell precisely where a decaying Earth satellite will land or how many pieces it will break into. But the discipline of chaos has found ways to deal with such systems and has been applied to apparently unrelated systems. For example, the heartbeat of people with certain types of potentially lethal arrhythmias seems to be chaotic, and this knowledge may allow more sophisticated monitoring and recognition of the need for intervention.

Chaos is related to complexity. Some chaotic systems are also inherently complex; for example, vortices in a fluid as opposed to a double pendulum. Both are chaotic and not predictable in the same sense as other systems. But there can be organization in chaos and it can also be quantified. Examples of chaotic systems are beautiful fractal patterns such as in [\[link\]](#). Some chaotic systems exhibit self-organization, a type of stable chaos. The orbits of the planets in our solar system, for example, may be chaotic (we are not certain yet). But they are definitely organized and systematic, with a simple formula describing the orbital radii of the first eight planets *and* the asteroid belt. Large-scale vortices in Jupiter's atmosphere are chaotic, but the Great

Red Spot is a stable self-organization of rotational energy. (See [\[link\]](#).) The Great Red Spot has been in existence for at least 400 years and is a complex self-adaptive system.

The emerging field of complexity, like the now almost traditional field of chaos, is partly rooted in physics. Both attempt to see similar systematics in a very broad range of phenomena and, hence, generate a better understanding of them. Time will tell what impact these fields have on more traditional areas of physics as well as on the other disciplines they relate to.



This image is related to the Mandelbrot set, a complex mathematical form that is chaotic. The patterns are infinitely fine as you look closer and closer, and they indicate order in the presence of chaos. (credit: Gilberto Santa Rosa)



The Great Red Spot on Jupiter is an example of self-organization in a complex and chaotic system. Smaller vortices in Jupiter's atmosphere behave chaotically, but the triple-Earth-size spot is self-organized and stable for at least hundreds of years. (credit: NASA)

Section Summary

- Complexity is an emerging field, rooted primarily in physics, that considers complex adaptive systems and their evolution, including self-organization.
- Complexity has applications in physics and many other disciplines, such as biological evolution.
- Chaos is a field that studies systems whose properties depend extremely sensitively on some variables and whose evolution is impossible to predict.
- Chaotic systems may be simple or complex.

- Studies of chaos have led to methods for understanding and predicting certain chaotic behaviors.

Conceptual Questions

Exercise:

Problem:

Must a complex system be adaptive to be of interest in the field of complexity? Give an example to support your answer.

Exercise:

Problem: State a necessary condition for a system to be chaotic.

Glossary

complexity

an emerging field devoted to the study of complex systems

chaos

word used to describe systems the outcomes of which are extremely sensitive to initial conditions

High-Temperature Superconductors

LEARNING OBJECTIVES

By the end of this section, you will be able to:

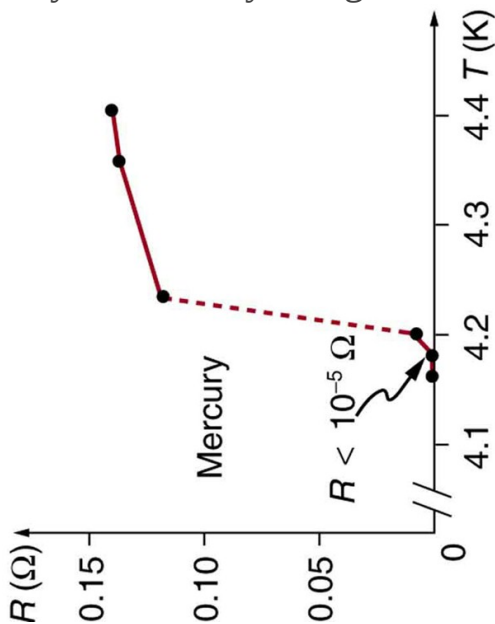
- Identify superconductors and their uses.
- Discuss the need for a high- T_c superconductor.

Superconductors are materials with a resistivity of zero. They are familiar to the general public because of their practical applications and have been mentioned at a number of points in the text. Because the resistance of a piece of superconductor is zero, there are no heat losses for currents through them; they are used in magnets needing high currents, such as in MRI machines, and could cut energy losses in power transmission. But most superconductors must be cooled to temperatures only a few kelvin above absolute zero, a costly procedure limiting their practical applications. In the past decade, tremendous advances have been made in producing materials that become superconductors at relatively high temperatures. There is hope that room temperature superconductors may someday be manufactured.

Superconductivity was discovered accidentally in 1911 by the Dutch physicist H. Kamerlingh Onnes (1853–1926) when he used liquid helium to cool mercury. Onnes had been the first person to liquefy helium a few years earlier and was surprised to observe the resistivity of a mediocre conductor like mercury drop to zero at a temperature of 4.2 K. We define the temperature at which and below which a material becomes a superconductor to be its **critical temperature**, denoted by T_c . (See [\[link\]](#).) Progress in understanding how and why a material became a

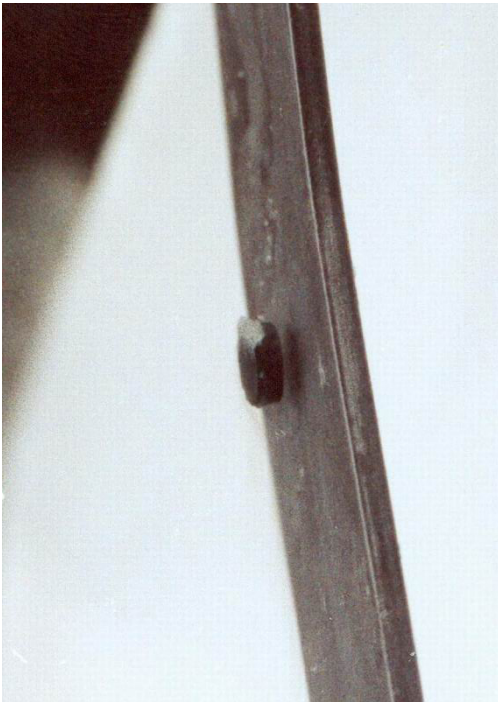
superconductor was relatively slow, with the first workable theory coming in 1957. Certain other elements were also found to become superconductors, but all had T_c s less than 10 K, which are expensive to maintain. Although Onnes received a Nobel prize in 1913, it was primarily for his work with liquid helium.

In 1986, a breakthrough was announced—a ceramic compound was found to have an unprecedented T_c of 35 K. It looked as if much higher critical temperatures could be possible, and by early 1988 another ceramic (this of thallium, calcium, barium, copper, and oxygen) had been found to have $T_c = 125$ K (see [link](#).) The economic potential of perfect conductors saving electric energy is immense for T_c s above 77 K, since that is the temperature of liquid nitrogen. Although liquid helium has a boiling point of 4 K and can be used to make materials superconducting, it costs about \$5 per liter. Liquid nitrogen boils at 77 K, but only costs about \$0.30 per liter. There was general euphoria at the discovery of these complex ceramic superconductors, but this soon subsided with the sobering difficulty of forming them into usable wires. The first commercial use of a high temperature superconductor is in an electronic filter for cellular phones. High-temperature superconductors are used in experimental apparatus, and they are actively being researched, particularly in thin film applications.



A graph of resistivity versus temperature for a

superconductor shows a sharp transition to zero at the critical temperature T_c . High temperature superconductors have verifiable T_c s greater than 125 K, well above the easily achieved 77-K temperature of liquid nitrogen.



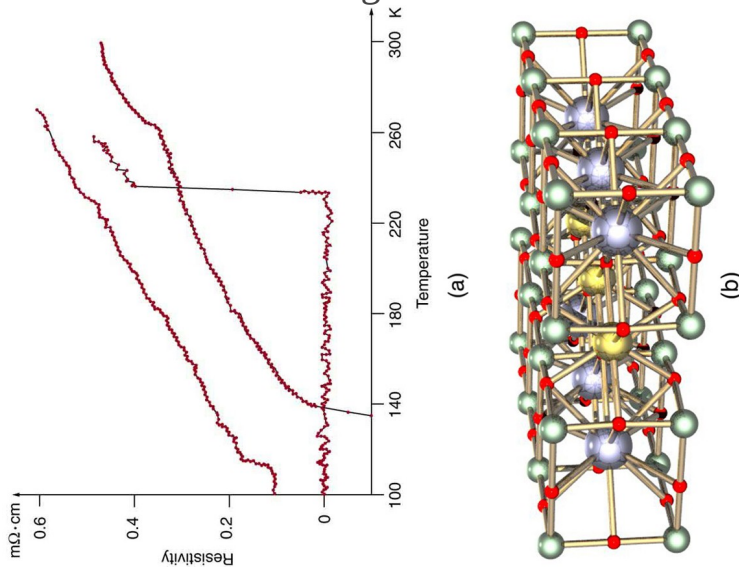
One characteristic of a superconductor is that it excludes magnetic flux and, thus, repels other magnets. The small magnet levitated above a high-temperature superconductor, which is cooled by liquid nitrogen,

gives evidence that the
material is
superconducting. When
the material warms and
becomes conducting,
magnetic flux can
penetrate it, and the
magnet will rest upon it.
(credit: Saperaud)

The search is on for even higher T_c superconductors, many of complex and exotic copper oxide ceramics, sometimes including strontium, mercury, or yttrium as well as barium, calcium, and other elements. Room temperature (about 293 K) would be ideal, but any temperature close to room temperature is relatively cheap to produce and maintain. There are persistent reports of T_c s over 200 K and some in the vicinity of 270 K. Unfortunately, these observations are not routinely reproducible, with samples losing their superconducting nature once heated and re-cooled (cycled) a few times (see [\[link\]](#).) They are now called USOs or unidentified superconducting objects, out of frustration and the refusal of some samples to show high T_c even though produced in the same manner as others. Reproducibility is crucial to discovery, and researchers are justifiably reluctant to claim the breakthrough they all seek. Time will tell whether USOs are real or an experimental quirk.

The theory of ordinary superconductors is difficult, involving quantum effects for widely separated electrons traveling through a material. Electrons couple in a manner that allows them to get through the material without losing energy to it, making it a superconductor. High- T_c superconductors are more difficult to understand theoretically, but theorists seem to be closing in on a workable theory. The difficulty of understanding how electrons can sneak through materials without losing energy in collisions is even greater at higher temperatures, where vibrating atoms should get in the way. Discoverers of high T_c may feel something analogous

to what a politician once said upon an unexpected election victory—“I wonder what we did right?”



(a) This graph, adapted from an article in *Physics Today*, shows the behavior of a single sample of a high-temperature superconductor in three different trials. In one case the sample exhibited a T_c of about 230 K, whereas in the others it did not become superconducting at all. The lack of reproducibility is typical of forefront experiments and prohibits definitive conclusions. (b) This colorful diagram shows the complex but systematic nature of the lattice structure of a high-temperature superconducting ceramic. (credit: en:Cadmium, Wikimedia Commons)

Section Summary

- High-temperature superconductors are materials that become superconducting at temperatures well above a few kelvin.
- The critical temperature T_c is the temperature below which a material is superconducting.
- Some high-temperature superconductors have verified T_c s above 125 K, and there are reports of T_c s as high as 250 K.

Conceptual Questions

Exercise:

Problem:

What is critical temperature T_c ? Do all materials have a critical temperature? Explain why or why not.

Exercise:

Problem:

Explain how good thermal contact with liquid nitrogen can keep objects at a temperature of 77 K (liquid nitrogen's boiling point at atmospheric pressure).

Exercise:

Problem:

Not only is liquid nitrogen a cheaper coolant than liquid helium, its boiling point is higher (77 K vs. 4.2 K). How does higher temperature help lower the cost of cooling a material? Explain in terms of the rate of heat transfer being related to the temperature difference between the sample and its surroundings.

Problem Exercises

Exercise:

Problem:

A section of superconducting wire carries a current of 100 A and requires 1.00 L of liquid nitrogen per hour to keep it below its critical temperature. For it to be economically advantageous to use a superconducting wire, the cost of cooling the wire must be less than the cost of energy lost to heat in the wire. Assume that the cost of liquid nitrogen is \$0.30 per liter, and that electric energy costs \$0.10 per kW·h. What is the resistance of a normal wire that costs as much in wasted electric energy as the cost of liquid nitrogen for the superconductor?

Solution:

0.30 Ω

Glossary

Superconductors

materials with resistivity of zero

critical temperature

the temperature at which and below which a material becomes a superconductor

Some Questions We Know to Ask

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Identify sample questions to be asked on the largest scales.
- Identify sample questions to be asked on the intermediate scale.
- Identify sample questions to be asked on the smallest scales.

Throughout the text we have noted how essential it is to be curious and to ask questions in order to first understand what is known, and then to go a little farther. Some questions may go unanswered for centuries; others may not have answers, but some bear delicious fruit. Part of discovery is knowing which questions to ask. You have to know something before you can even phrase a decent question. As you may have noticed, the mere act of asking a question can give you the answer. The following questions are a sample of those physicists now know to ask and are representative of the forefronts of physics. Although these questions are important, they will be replaced by others if answers are found to them. The fun continues.

On the Largest Scale

1. *Is the universe open or closed?* Theorists would like it to be just barely closed and evidence is building toward that conclusion. Recent measurements in the expansion rate of the universe and in CMBR support a flat universe. There is a connection to small-scale physics in

the type and number of particles that may contribute to closing the universe.

2. *What is dark matter?* It is definitely there, but we really do not know what it is. Conventional possibilities are being ruled out, but one of them still may explain it. The answer could reveal whole new realms of physics and the disturbing possibility that most of what is out there is unknown to us, a completely different form of matter.
3. *How do galaxies form?* They exist since very early in the evolution of the universe and it remains difficult to understand how they evolved so quickly. The recent finer measurements of fluctuations in the CMBR may yet allow us to explain galaxy formation.
4. *What is the nature of various-mass black holes?* Only recently have we become confident that many black hole candidates cannot be explained by other, less exotic possibilities. But we still do not know much about how they form, what their role in the history of galactic evolution has been, and the nature of space in their vicinity. However, so many black holes are now known that correlations between black hole mass and galactic nuclei characteristics are being studied.
5. *What is the mechanism for the energy output of quasars?* These distant and extraordinarily energetic objects now seem to be early stages of galactic evolution with a supermassive black-hole-devouring material. Connections are now being made with galaxies having energetic cores, and there is evidence consistent with less consuming, supermassive black holes at the center of older galaxies. New instruments are allowing us to see deeper into our own galaxy for evidence of our own massive black hole.
6. *Where do the γ bursts come from?* We see bursts of γ rays coming from all directions in space, indicating the sources are very distant objects rather than something associated with our own galaxy. Some γ bursts finally are being correlated with known sources so that the possibility they may originate in binary neutron star interactions or black holes eating a companion neutron star can be explored.

On the Intermediate Scale

1. *How do phase transitions take place on the microscopic scale?* We know a lot about phase transitions, such as water freezing, but the

details of how they occur molecule by molecule are not well understood. Similar questions about specific heat a century ago led to early quantum mechanics. It is also an example of a complex adaptive system that may yield insights into other self-organizing systems.

2. *Is there a way to deal with nonlinear phenomena that reveals underlying connections?* Nonlinear phenomena lack a direct or linear proportionality that makes analysis and understanding a little easier. There are implications for nonlinear optics and broader topics such as chaos.
3. *How do high- T_c superconductors become resistanceless at such high temperatures?* Understanding how they work may help make them more practical or may result in surprises as unexpected as the discovery of superconductivity itself.
4. *There are magnetic effects in materials we do not understand—how do they work?* Although beyond the scope of this text, there is a great deal to learn in condensed matter physics (the physics of solids and liquids). We may find surprises analogous to lasing, the quantum Hall effect, and the quantization of magnetic flux. Complexity may play a role here, too.

On the Smallest Scale

1. *Are quarks and leptons fundamental, or do they have a substructure?* The higher energy accelerators that are just completed or being constructed may supply some answers, but there will also be input from cosmology and other systematics.
2. *Why do leptons have integral charge while quarks have fractional charge?* If both are fundamental and analogous as thought, this question deserves an answer. It is obviously related to the previous question.
3. *Why are there three families of quarks and leptons?* First, does this imply some relationship? Second, why three and only three families?
4. *Are all forces truly equal (unified) under certain circumstances?* They don't have to be equal just because we want them to be. The answer may have to be indirectly obtained because of the extreme energy at which we think they are unified.

5. *Are there other fundamental forces?* There was a flurry of activity with claims of a fifth and even a sixth force a few years ago. Interest has subsided, since those forces have not been detected consistently. Moreover, the proposed forces have strengths similar to gravity, making them extraordinarily difficult to detect in the presence of stronger forces. But the question remains; and if there are no other forces, we need to ask why only four and why these four.
6. *Is the proton stable?* We have discussed this in some detail, but the question is related to fundamental aspects of the unification of forces. We may never know from experiment that the proton is stable, only that it is very long lived.
7. *Are there magnetic monopoles?* Many particle theories call for very massive individual north- and south-pole particles—magnetic monopoles. If they exist, why are they so different in mass and elusiveness from electric charges, and if they do not exist, why not?
8. *Do neutrinos have mass?* Definitive evidence has emerged for neutrinos having mass. The implications are significant, as discussed in this chapter. There are effects on the closure of the universe and on the patterns in particle physics.
9. *What are the systematic characteristics of high- Z nuclei?* All elements with $Z = 118$ or less (with the exception of 115 and 117) have now been discovered. It has long been conjectured that there may be an island of relative stability near $Z = 114$, and the study of the most recently discovered nuclei will contribute to our understanding of nuclear forces.

These lists of questions are not meant to be complete or consistently important—you can no doubt add to it yourself. There are also important questions in topics not broached in this text, such as certain particle symmetries, that are of current interest to physicists. Hopefully, the point is clear that no matter how much we learn, there always seems to be more to know. Although we are fortunate to have the hard-won wisdom of those who preceded us, we can look forward to new enlightenment, undoubtedly sprinkled with surprise.

Section Summary

- On the largest scale, the questions which can be asked may be about dark matter, dark energy, black holes, quasars, and other aspects of the universe.
- On the intermediate scale, we can query about gravity, phase transitions, nonlinear phenomena, high- T_c superconductors, and magnetic effects on materials.
- On the smallest scale, questions may be about quarks and leptons, fundamental forces, stability of protons, and existence of monopoles.

Conceptual Questions

Exercise:

Problem:

For experimental evidence, particularly of previously unobserved phenomena, to be taken seriously it must be reproducible or of sufficiently high quality that a single observation is meaningful. Supernova 1987A is not reproducible. How do we know observations of it were valid? The fifth force is not broadly accepted. Is this due to lack of reproducibility or poor-quality experiments (or both)? Discuss why forefront experiments are more subject to observational problems than those involving established phenomena.

Exercise:

Problem:

Discuss whether you think there are limits to what humans can understand about the laws of physics. Support your arguments.

Atomic Masses

Atomic Number, Z	Name	Atomic Mass Number, A	Symbol	Atomic Mass (u)	Percent Abundance or Decay Mode	Half-life, $t_{1/2}$
0	neutron	1	n	1.008 665	β^-	10.37 min
1	Hydrogen	1	^1H	1.007 825	99.985%	
	Deuterium	2	^2H or D	2.014 102	0.015%	
	Tritium	3	^3H or T	3.016 050	β^-	12.33 y
2	Helium	3	^3He	3.016 030	$1.38 \times 10^{-4}\%$	
		4	^4He	4.002 603	$\approx 100\%$	
3	Lithium	6	^6Li	6.015 121	7.5%	
		7	^7Li	7.016 003	92.5%	
4	Beryllium	7	^7Be	7.016 928	EC	53.29 d
		9	^9Be	9.012 182	100%	
5	Boron	10	^{10}B	10.012 937	19.9%	
		11	^{11}B	11.009 305	80.1%	
6	Carbon	11	^{11}C	11.011 432	EC, β^+	
		12	^{12}C	12.000 000	98.90%	
		13	^{13}C	13.003 355	1.10%	

Atomic Number, Z	Name	Atomic Mass Number, A	Symbol	Atomic Mass (u)	Percent Abundance or Decay Mode	Half-life, $t_{1/2}$
		14	^{14}C	14.003 241	β^-	5730 y
7	Nitrogen	13	^{13}N	13.005 738	β^+	9.96 min
		14	^{14}N	14.003 074	99.63%	
		15	^{15}N	15.000 108	0.37%	
8	Oxygen	15	^{15}O	15.003 065	EC, β^+	122 s
		16	^{16}O	15.994 915	99.76%	
		18	^{18}O	17.999 160	0.200%	
9	Fluorine	18	^{18}F	18.000 937	EC, β^+	1.83 h
		19	^{19}F	18.998 403	100%	
10	Neon	20	^{20}Ne	19.992 435	90.51%	
		22	^{22}Ne	21.991 383	9.22%	
11	Sodium	22	^{22}Na	21.994 434	β^+	2.602 y
		23	^{23}Na	22.989 767	100%	
		24	^{24}Na	23.990 961	β^-	14.96 h
12	Magnesium	24	^{24}Mg	23.985 042	78.99%	
13	Aluminum	27	^{27}Al	26.981 539	100%	
14	Silicon	28	^{28}Si	27.976 927	92.23%	2.62h

Atomic Number, Z	Name	Atomic Mass Number, A	Symbol	Atomic Mass (u)	Percent Abundance or Decay Mode	Half-life, $t_{1/2}$
		31	^{31}Si	30.975 362	β^-	
15	Phosphorus	31	^{31}P	30.973 762	100%	
		32	^{32}P	31.973 907	β^-	14.28 d
16	Sulfur	32	^{32}S	31.972 070	95.02%	
		35	^{35}S	34.969 031	β^-	87.4 d
17	Chlorine	35	^{35}Cl	34.968 852	75.77%	
		37	^{37}Cl	36.965 903	24.23%	
18	Argon	40	^{40}Ar	39.962 384	99.60%	
19	Potassium	39	^{39}K	38.963 707	93.26%	
		40	^{40}K	39.963 999	0.0117%, EC, β^-	$1.28 \times 10^9 \text{ y}$
20	Calcium	40	^{40}Ca	39.962 591	96.94%	
21	Scandium	45	^{45}Sc	44.955 910	100%	
22	Titanium	48	^{48}Ti	47.947 947	73.8%	
23	Vanadium	51	^{51}V	50.943 962	99.75%	
24	Chromium	52	^{52}Cr	51.940 509	83.79%	
25	Manganese	55	^{55}Mn	54.938 047	100%	
26	Iron	56	^{56}Fe	55.934 939	91.72%	

Atomic Number, Z	Name	Atomic Mass Number, A	Symbol	Atomic Mass (u)	Percent Abundance or Decay Mode	Half-life, $t_{1/2}$
27	Cobalt	59	^{59}Co	58.933 198	100%	
		60	^{60}Co	59.933 819	β^-	5.271 y
28	Nickel	58	^{58}Ni	57.935 346	68.27%	
		60	^{60}Ni	59.930 788	26.10%	
29	Copper	63	^{63}Cu	62.939 598	69.17%	
		65	^{65}Cu	64.927 793	30.83%	
30	Zinc	64	^{64}Zn	63.929 145	48.6%	
		66	^{66}Zn	65.926 034	27.9%	
31	Gallium	69	^{69}Ga	68.925 580	60.1%	
32	Germanium	72	^{72}Ge	71.922 079	27.4%	
		74	^{74}Ge	73.921 177	36.5%	
33	Arsenic	75	^{75}As	74.921 594	100%	
34	Selenium	80	^{80}Se	79.916 520	49.7%	
35	Bromine	79	^{79}Br	78.918 336	50.69%	
36	Krypton	84	^{84}Kr	83.911 507	57.0%	
37	Rubidium	85	^{85}Rb	84.911 794	72.17%	
38	Strontium	86	^{86}Sr	85.909 267	9.86%	

Atomic Number, Z	Name	Atomic Mass Number, A	Symbol	Atomic Mass (u)	Percent Abundance or Decay Mode	Half-life, $t_{1/2}$
		88	^{88}Sr	87.905 619	82.58%	
		90	^{90}Sr	89.907 738	β^-	28.8 y
39	Yttrium	89	^{89}Y	88.905 849	100%	
		90	^{90}Y	89.907 152	β^-	64.1 h
40	Zirconium	90	^{90}Zr	89.904 703	51.45%	
41	Niobium	93	^{93}Nb	92.906 377	100%	
42	Molybdenum	98	^{98}Mo	97.905 406	24.13%	
43	Technetium	98	^{98}Tc	97.907 215	β^-	$4.2 \times 10^6 \text{ y}$
44	Ruthenium	102	^{102}Ru	101.904 348	31.6%	
45	Rhodium	103	^{103}Rh	102.905 500	100%	
46	Palladium	106	^{106}Pd	105.903 478	27.33%	
47	Silver	107	^{107}Ag	106.905 092	51.84%	
		109	^{109}Ag	108.904 757	48.16%	
48	Cadmium	114	^{114}Cd	113.903 357	28.73%	
49	Indium	115	^{115}In	114.903 880	95.7%, β^-	$4.4 \times 10^{14} \text{ y}$
50	Tin	120	^{120}Sn	119.902 200	32.59%	
51	Antimony	121	^{121}Sb	120.903 821	57.3%	

Atomic Number, Z	Name	Atomic Mass Number, A	Symbol	Atomic Mass (u)	Percent Abundance or Decay Mode	Half-life, $t_{1/2}$
52	Tellurium	130	^{130}Te	129.906 229	33.8%, β^-	$2.5 \times 10^{21}\text{y}$
53	Iodine	127	^{127}I	126.904 473	100%	
		131	^{131}I	130.906 114	β^-	8.040 d
54	Xenon	132	^{132}Xe	131.904 144	26.9%	
		136	^{136}Xe	135.907 214	8.9%	
55	Cesium	133	^{133}Cs	132.905 429	100%	
		134	^{134}Cs	133.906 696	EC, β^-	2.06 y
56	Barium	137	^{137}Ba	136.905 812	11.23%	
		138	^{138}Ba	137.905 232	71.70%	
57	Lanthanum	139	^{139}La	138.906 346	99.91%	
58	Cerium	140	^{140}Ce	139.905 433	88.48%	
59	Praseodymium	141	^{141}Pr	140.907 647	100%	
60	Neodymium	142	^{142}Nd	141.907 719	27.13%	
61	Promethium	145	^{145}Pm	144.912 743	EC, α	17.7 y
62	Samarium	152	^{152}Sm	151.919 729	26.7%	
63	Europium	153	^{153}Eu	152.921 225	52.2%	
64	Gadolinium	158	^{158}Gd	157.924 099	24.84%	

Atomic Number, Z	Name	Atomic Mass Number, A	Symbol	Atomic Mass (u)	Percent Abundance or Decay Mode	Half-life, $t_{1/2}$
65	Terbium	159	^{159}Tb	158.925 342	100%	
66	Dysprosium	164	^{164}Dy	163.929 171	28.2%	
67	Holmium	165	^{165}Ho	164.930 319	100%	
68	Erbium	166	^{166}Er	165.930 290	33.6%	
69	Thulium	169	^{169}Tm	168.934 212	100%	
70	Ytterbium	174	^{174}Yb	173.938 859	31.8%	
71	Lutecium	175	^{175}Lu	174.940 770	97.41%	
72	Hafnium	180	^{180}Hf	179.946 545	35.10%	
73	Tantalum	181	^{181}Ta	180.947 992	99.98%	
74	Tungsten	184	^{184}W	183.950 928	30.67%	
75	Rhenium	187	^{187}Re	186.955 744	62.6%, β^-	$4.6 \times 10^{10}\text{y}$
76	Osmium	191	^{191}Os	190.960 920	β^-	15.4 d
		192	^{192}Os	191.961 467	41.0%	
77	Iridium	191	^{191}Ir	190.960 584	37.3%	
		193	^{193}Ir	192.962 917	62.7%	
78	Platinum	195	^{195}Pt	194.964 766	33.8%	
79	Gold	197	^{197}Au	196.966 543	100%	

Atomic Number, Z	Name	Atomic Mass Number, A	Symbol	Atomic Mass (u)	Percent Abundance or Decay Mode	Half-life, $t_{1/2}$
		198	^{198}Au	197.968 217	β^-	2.696 d
80	Mercury	199	^{199}Hg	198.968 253	16.87%	
		202	^{202}Hg	201.970 617	29.86%	
81	Thallium	205	^{205}Tl	204.974 401	70.48%	
82	Lead	206	^{206}Pb	205.974 440	24.1%	
		207	^{207}Pb	206.975 872	22.1%	
		208	^{208}Pb	207.976 627	52.4%	
		210	^{210}Pb	209.984 163	α, β^-	22.3 y
		211	^{211}Pb	210.988 735	β^-	36.1 min
		212	^{212}Pb	211.991 871	β^-	10.64 h
83	Bismuth	209	^{209}Bi	208.980 374	100%	
		211	^{211}Bi	210.987 255	α, β^-	2.14 min
84	Polonium	210	^{210}Po	209.982 848	α	138.38 d
85	Astatine	218	^{218}At	218.008 684	α, β^-	1.6 s
86	Radon	222	^{222}Rn	222.017 570	α	3.82 d
87	Francium	223	^{223}Fr	223.019 733	α, β^-	21.8 min
88	Radium	226	^{226}Ra	226.025 402	α	$1.60 \times 10^3 \text{ y}$

Atomic Number, Z	Name	Atomic Mass Number, A	Symbol	Atomic Mass (u)	Percent Abundance or Decay Mode	Half-life, $t_{1/2}$
89	Actinium	227	^{227}Ac	227.027 750	α, β^-	21.8 y
90	Thorium	228	^{228}Th	228.028 715	α	1.91 y
		232	^{232}Th	232.038 054	100%, α	$1.41 \times 10^{10}\text{y}$
91	Protactinium	231	^{231}Pa	231.035 880	α	$3.28 \times 10^4\text{y}$
92	Uranium	233	^{233}U	233.039 628	α	$1.59 \times 10^3\text{y}$
		235	^{235}U	235.043 924	0.720%, α	$7.04 \times 10^8\text{y}$
		236	^{236}U	236.045 562	α	$2.34 \times 10^7\text{y}$
		238	^{238}U	238.050 784	99.2745%, α	$4.47 \times 10^9\text{y}$
		239	^{239}U	239.054 289	β^-	23.5 min
93	Neptunium	239	^{239}Np	239.052 933	β^-	2.355 d
94	Plutonium	239	^{239}Pu	239.052 157	α	$2.41 \times 10^4\text{y}$
95	Americium	243	^{243}Am	243.061 375	α , fission	$7.37 \times 10^3\text{y}$
96	Curium	245	^{245}Cm	245.065 483	α	$8.50 \times 10^3\text{y}$
97	Berkelium	247	^{247}Bk	247.070 300	α	$1.38 \times 10^3\text{y}$
98	Californium	249	^{249}Cf	249.074 844	α	351 y
99	Einsteinium	254	^{254}Es	254.088 019	α, β^-	276 d
100	Fermium	253	^{253}Fm	253.085 173	EC, α	3.00 d

Atomic Number, Z	Name	Atomic Mass Number, A	Symbol	Atomic Mass (u)	Percent Abundance or Decay Mode	Half-life, $t_{1/2}$
101	Mendelevium	255	^{255}Md	255.091 081	EC, α	27 min
102	Nobelium	255	^{255}No	255.093 260	EC, α	3.1 min
103	Lawrencium	257	^{257}Lr	257.099 480	EC, α	0.646 s
104	Rutherfordium	261	^{261}Rf	261.108 690	α	1.08 min
105	Dubnium	262	^{262}Db	262.113 760	α , fission	34 s
106	Seaborgium	263	^{263}Sg	263.11 86	α , fission	0.8 s
107	Bohrium	262	^{262}Bh	262.123 1	α	0.102 s
108	Hassium	264	^{264}Hs	264.128 5	α	0.08 ms
109	Meitnerium	266	^{266}Mt	266.137 8	α	3.4 ms

Atomic Masses

Selected Radioactive Isotopes

Decay modes are α , β^- , β^+ , electron capture (EC) and isomeric transition (IT). EC results in the same daughter nucleus as would β^+ decay. IT is a transition from a metastable excited state. Energies for β^\pm decays are the maxima; average energies are roughly one-half the maxima.

Isotope	$t_{1/2}$	DecayMode(s)	Energy(MeV)	Percent		γ -Ray Energy(MeV)
^3H	12.33 y	β^-	0.0186	100%		
^{14}C	5730 y	β^-	0.156	100%		
^{13}N	9.96 min	β^+	1.20	100%		
^{22}Na	2.602 y	β^+	0.55	90%	γ	1.27
^{32}P	14.28 d	β^-	1.71	100%		
^{35}S	87.4 d	β^-	0.167	100%		
^{36}Cl	$3.00 \times 10^5 \text{y}$	β^-	0.710	100%		
^{40}K	$1.28 \times 10^9 \text{y}$	β^-	1.31	89%		
^{43}K	22.3 h	β^-	0.827	87%	γs	0.373
						0.618
^{45}Ca	165 d	β^-	0.257	100%		
^{51}Cr	27.70 d	EC			γ	0.320
^{52}Mn	5.59d	β^+	3.69	28%	γs	1.33
						1.43
^{52}Fe	8.27 h	β^+	1.80	43%		0.169
						0.378
^{59}Fe	44.6 d	$\beta^- \text{s}$	0.273	45%	γs	1.10
			0.466	55%		1.29
^{60}Co	5.271 y	β^-	0.318	100%	γs	1.17
						1.33
^{65}Zn	244.1 d	EC			γ	1.12

Isotope	$t_{1/2}$	DecayMode(s)	Energy(MeV)	Percent		γ -Ray Energy(MeV)
^{67}Ga	78.3 h	EC			γ s	0.0933
						0.185
						0.300
						others
^{75}Se	118.5 d	EC			γ s	0.121
						0.136
						0.265
						0.280
						others
^{86}Rb	18.8 d	β^- s	0.69	9%	γ	1.08
			1.77	91%		
^{85}Sr	64.8 d	EC			γ	0.514
^{90}Sr	28.8 y	β^-	0.546	100%		
^{90}Y	64.1 h	β^-	2.28	100%		
$^{99\text{m}}\text{Tc}$	6.02 h	IT			γ	0.142
$^{113\text{m}}\text{In}$	99.5 min	IT			γ	0.392
^{123}I	13.0 h	EC			γ	0.159
^{131}I	8.040 d	β^- s	0.248	7%	γ s	0.364
			0.607	93%		others
			others			
^{129}Cs	32.3 h	EC			γ s	0.0400
						0.372
						0.411
						others
^{137}Cs	30.17 y	β^- s	0.511	95%	γ	0.662
			1.17	5%		

Isotope	$t_{1/2}$	DecayMode(s)	Energy(MeV)	Percent		γ -Ray Energy(MeV)
^{140}Ba	12.79 d	β^-	1.035	$\approx 100\%$	γ s	0.030
						0.044
						0.537
						others
^{198}Au	2.696 d	β^-	1.161	$\approx 100\%$	γ	0.412
^{197}Hg	64.1 h	EC			γ	0.0733
^{210}Po	138.38 d	α	5.41	100%		
^{226}Ra	$1.60 \times 10^3 \text{ y}$	α s	4.68	5%	γ	0.186
			4.87	95%		
^{235}U	$7.038 \times 10^8 \text{ y}$	α	4.68	$\approx 100\%$	γ s	numerous
^{238}U	$4.468 \times 10^9 \text{ y}$	α s	4.22	23%	γ	0.050
			4.27	77%		
^{237}Np	$2.14 \times 10^6 \text{ y}$	α s	numerous		γ s	numerous
			4.96 (max.)			
^{239}Pu	$2.41 \times 10^4 \text{ y}$	α s	5.19	11%	γ s	7.5×10^{-5}
			5.23	15%		0.013
			5.24	73%		0.052
						others
^{243}Am	$7.37 \times 10^3 \text{ y}$	α s	Max. 5.44		γ s	0.075
			5.37	88%		others
			5.32	11%		
			others			

Selected Radioactive Isotopes

Useful Information

This appendix is broken into several tables.

- [\[link\]](#), Important Constants
- [\[link\]](#), Submicroscopic Masses
- [\[link\]](#), Solar System Data
- [\[link\]](#), Metric Prefixes for Powers of Ten and Their Symbols
- [\[link\]](#), The Greek Alphabet
- [\[link\]](#), SI units
- [\[link\]](#), Selected British Units
- [\[link\]](#), Other Units
- [\[link\]](#), Useful Formulae

Symbol	Meaning	Best Value	Approximate Value
c	Speed of light in vacuum	$2.99792458 \times 10^8 \text{ m/s}$	$3.00 \times 10^8 \text{ m/s}$
G	Gravitational constant	$6.67408(31) \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2$	$6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2$
N_A	Avogadro's number	$6.02214129(27) \times 10^{23}$	6.02×10^{23}
k	Boltzmann's constant	$1.3806488(13) \times 10^{-23} \text{ J/K}$	$1.38 \times 10^{-23} \text{ J/K}$
R	Gas constant	$8.3144621(75) \text{ J/mol} \cdot \text{K}$	$8.31 \text{ J/mol} \cdot \text{K} = 1.99 \text{ cal/mol} \cdot \text{K} =$
σ	Stefan-Boltzmann constant	$5.670373(21) \times 10^{-8} \text{ W/m}^2 \cdot \text{K}$	$5.67 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}$
k	Coulomb force constant	$8.987551788... \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2$	$8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2$
q_e	Charge on electron	$-1.602176565(35) \times 10^{-19} \text{ C}$	$-1.60 \times 10^{-19} \text{ C}$
ϵ_0	Permittivity of free space	$8.854187817... \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2$	$8.85 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2$
μ_0	Permeability of free space	$4\pi \times 10^{-7} \text{ T} \cdot \text{m/A}$	$1.26 \times 10^{-6} \text{ T} \cdot \text{m/A}$
h	Planck's constant	$6.62606957(29) \times 10^{-34} \text{ J} \cdot \text{s}$	$6.63 \times 10^{-34} \text{ J} \cdot \text{s}$

Important Constants^[footnote]

Stated values are according to the National Institute of Standards and Technology Reference on Constants, Units, and Uncertainty, www.physics.nist.gov/cuu (accessed May 18, 2012). Values in parentheses are the uncertainties in the last digits. Numbers without uncertainties are exact as defined.

Symbol	Meaning	Best Value	Approximate Value
m_e	Electron mass	$9.10938291(40) \times 10^{-31}\text{kg}$	$9.11 \times 10^{-31}\text{kg}$
m_p	Proton mass	$1.672621777(74) \times 10^{-27}\text{kg}$	$1.6726 \times 10^{-27}\text{kg}$
m_n	Neutron mass	$1.674927351(74) \times 10^{-27}\text{kg}$	$1.6749 \times 10^{-27}\text{kg}$
u	Atomic mass unit	$1.660538921(73) \times 10^{-27}\text{kg}$	$1.6605 \times 10^{-27}\text{kg}$

Submicroscopic Masses^[footnote]

Stated values are according to the National Institute of Standards and Technology Reference on Constants, Units, and Uncertainty, www.physics.nist.gov/cuu (accessed May 18, 2012). Values in parentheses are the uncertainties in the last digits. Numbers without uncertainties are exact as defined.

Sun	mass	$1.99 \times 10^{30}\text{kg}$
	average radius	$6.96 \times 10^8\text{m}$
	Earth-sun distance (average)	$1.496 \times 10^{11}\text{m}$
Earth	mass	$5.9736 \times 10^{24}\text{kg}$
	average radius	$6.376 \times 10^6\text{m}$
	orbital period	$3.16 \times 10^7\text{s}$

Epsilon	Ε	ε	Lambda	Λ	λ	Rho	Ρ	ρ	Psi	Ψ	ψ
Zeta	Ζ	ζ	Mu	Μ	μ	Sigma	Σ	σ	Omega	Ω	ω

The Greek Alphabet

	Entity	Abbreviation	Name
Fundamental units	Length	m	meter
	Mass	kg	kilogram
	Time	s	second
	Current	A	ampere
Supplementary unit	Angle	rad	radian
Derived units	Force	$N = kg \cdot m/s^2$	newton
	Energy	$J = kg \cdot m^2/s^2$	joule
	Power	$W = J/s$	watt
	Pressure	$Pa = N/m^2$	pascal
	Frequency	$Hz = 1/s$	hertz
	Electronic potential	$V = J/C$	volt
	Capacitance	$F = C/V$	farad
	Charge	$C = s \cdot A$	coulomb
	Resistance	$\Omega = V/A$	ohm

	Entity	Abbreviation	Name
	Magnetic field	$T = N/(A \cdot m)$	tesla
	Nuclear decay rate	$Bq = 1/s$	becquerel

SI Units

Length	1 inch (in.) = 2.54 cm (exactly)
	1 foot (ft) = 0.3048 m
	1 mile (mi) = 1.609 km
Force	1 pound (lb) = 4.448 N
Energy	1 British thermal unit (Btu) = 1.055×10^3 J
Power	1 horsepower (hp) = 746 W
Pressure	$1 \text{ lb/in}^2 = 6.895 \times 10^3 \text{ Pa}$

Selected British Units

Length	1 light year (ly) = 9.46×10^{15} m
	1 astronomical unit (au) = 1.50×10^{11} m
	1 nautical mile = 1.852 km
	1 angstrom(\AA) = 10^{-10} m
Area	1 acre (ac) = 4.05×10^3 m ²
	1 square foot (ft ²) = 9.29×10^{-2} m ²
	1 barn (<i>b</i>) = 10^{-28} m ²
Volume	1 liter (<i>L</i>) = 10^{-3} m ³

	1 U.S. gallon (gal) = $3.785 \times 10^{-3} \text{ m}^3$
Mass	1 solar mass = $1.99 \times 10^{30} \text{ kg}$
	1 metric ton = 10^3 kg
	1 atomic mass unit (u) = $1.6605 \times 10^{-27} \text{ kg}$
Time	1 year (y) = $3.16 \times 10^7 \text{ s}$
	1 day (d) = 86,400 s
Speed	1 mile per hour (mph) = 1.609 km/h
	1 nautical mile per hour (naut) = 1.852 km/h
Angle	1 degree ($^\circ$) = $1.745 \times 10^{-2} \text{ rad}$
	1 minute of arc ($'$) = 1/60 degree
	1 second of arc ($''$) = 1/60 minute of arc
	1 grad = $1.571 \times 10^{-2} \text{ rad}$
Energy	1 kiloton TNT (kT) = $4.2 \times 10^{12} \text{ J}$
	1 kilowatt hour ($\text{kW} \cdot \text{h}$) = $3.60 \times 10^6 \text{ J}$
	1 food calorie (kcal) = 4186 J
	1 calorie (cal) = 4.186 J
	1 electron volt (eV) = $1.60 \times 10^{-19} \text{ J}$
Pressure	1 atmosphere (atm) = $1.013 \times 10^5 \text{ Pa}$
	1 millimeter of mercury (mm Hg) = 133.3 Pa
	1 torricelli (torr) = 1 mm Hg = 133.3 Pa
Nuclear decay rate	1 curie (Ci) = $3.70 \times 10^{10} \text{ Bq}$

Other Units

Circumference of a circle with radius r or diameter d	$C = 2\pi r = \pi d$
Area of a circle with radius r or diameter d	$A = \pi r^2 = \pi d^2/4$
Area of a sphere with radius r	$A = 4\pi r^2$

Volume of a sphere with radius r

$$V = (4/3)(\pi r^3)$$

Useful Formulae

Glossary of Key Symbols and Notation

In this glossary, key symbols and notation are briefly defined.

Symbol	Definition
any symbol	average (indicated by a bar over a symbol— e.g., \bar{v} is average velocity)
$^{\circ}\text{C}$	Celsius degree
$^{\circ}\text{F}$	Fahrenheit degree
//	parallel
\perp	perpendicular
\propto	proportional to
\pm	plus or minus

Symbol	Definition
0	zero as a subscript denotes an initial value
α	alpha rays
α	angular acceleration
α	temperature coefficient(s) of resistivity
β	beta rays
β	sound level
β	volume coefficient of expansion
β^{-}	electron emitted in nuclear beta decay
β^{+}	positron decay
γ	gamma rays

Symbol	Definition
γ	surface tension
$\gamma = 1/\sqrt{1 - v^2/c^2}$	a constant used in relativity
Δ	change in whatever quantity follows
δ	uncertainty in whatever quantity follows
ΔE	change in energy between the initial and final orbits of an electron in an atom
ΔE	uncertainty in energy
Δm	difference in mass between initial and final products
ΔN	number of decays that occur
Δp	change in momentum

Symbol	Definition
Δp	uncertainty in momentum
ΔPE_g	change in gravitational potential energy
$\Delta\theta$	rotation angle
Δs	distance traveled along a circular path
Δt	uncertainty in time
Δt_0	proper time as measured by an observer at rest relative to the process
ΔV	potential difference
Δx	uncertainty in position
ϵ_0	permittivity of free space
η	viscosity

Symbol	Definition
θ	angle between the force vector and the displacement vector
θ	angle between two lines
θ	contact angle
θ	direction of the resultant
θ_b	Brewster's angle
θ_c	critical angle
κ	dielectric constant
λ	decay constant of a nuclide
λ	wavelength
λ_n	wavelength in a medium

Symbol	Definition
μ_0	permeability of free space
μ_k	coefficient of kinetic friction
μ_s	coefficient of static friction
ν_e	electron neutrino
π^+	positive pion
π^-	negative pion
π^0	neutral pion
ρ	density
ρ_c	critical density, the density needed to just halt universal expansion
ρ_{fl}	fluid density

Symbol	Definition
ρ_{obj}	average density of an object
ρ/ρ_{w}	specific gravity
τ	characteristic time constant for a resistance and inductance (RL) or resistance and capacitance (RC) circuit
τ	characteristic time for a resistor and capacitor (RC) circuit
τ	torque
Υ	upsilon meson
Φ	magnetic flux
ϕ	phase angle
Ω	ohm (unit)
ω	angular velocity

Symbol	Definition
A	ampere (current unit)
A	area
A	cross-sectional area
A	total number of nucleons
a	acceleration
a_B	Bohr radius
a_c	centripetal acceleration
a_t	tangential acceleration
AC	alternating current
AM	amplitude modulation

Symbol	Definition
atm	atmosphere
B	baryon number
B	blue quark color
B	antiblue (yellow) antiquark color
b	quark flavor bottom or beauty
B	bulk modulus
B	magnetic field strength
B_{int}	electron's intrinsic magnetic field
B_{orb}	orbital magnetic field
BE	binding energy of a nucleus—it is the energy required to completely disassemble it into separate protons and neutrons

Symbol	Definition
BE/A	binding energy per nucleon
Bq	becquerel—one decay per second
C	capacitance (amount of charge stored per volt)
C	coulomb (a fundamental SI unit of charge)
C_p	total capacitance in parallel
C_s	total capacitance in series
CG	center of gravity
CM	center of mass
c	quark flavor charm
c	specific heat

Symbol	Definition
c	speed of light
Cal	kilocalorie
cal	calorie
COP_{hp}	heat pump's coefficient of performance
COP_{ref}	coefficient of performance for refrigerators and air conditioners
$\cos \theta$	cosine
$\cot \theta$	cotangent
$\csc \theta$	cosecant
D	diffusion constant
d	displacement

Symbol	Definition
d	quark flavor down
dB	decibel
d_i	distance of an image from the center of a lens
d_o	distance of an object from the center of a lens
DC	direct current
E	electric field strength
ε	emf (voltage) or Hall electromotive force
emf	electromotive force
E	energy of a single photon
E	nuclear reaction energy

Symbol	Definition
E	relativistic total energy
E	total energy
E_0	ground state energy for hydrogen
E_0	rest energy
EC	electron capture
E_{cap}	energy stored in a capacitor
Eff	efficiency—the useful work output divided by the energy input
Eff _C	Carnot efficiency
E_{in}	energy consumed (food digested in humans)
E_{ind}	energy stored in an inductor

Symbol	Definition
E_{out}	energy output
e	emissivity of an object
e^+	antielectron or positron
eV	electron volt
F	farad (unit of capacitance, a coulomb per volt)
F	focal point of a lens
F	force
F	magnitude of a force
F	restoring force
F_{B}	buoyant force

Symbol	Definition
F_c	centripetal force
F_i	force input
\mathbf{F}_{net}	net force
F_o	force output
FM	frequency modulation
f	focal length
f	frequency
f_0	resonant frequency of a resistance, inductance, and capacitance (RLC) series circuit
f_0	threshold frequency for a particular material (photoelectric effect)

Symbol	Definition
f_1	fundamental
f_2	first overtone
f_3	second overtone
f_B	beat frequency
f_k	magnitude of kinetic friction
f_s	magnitude of static friction
G	gravitational constant
G	green quark color
\bar{G}	antigreen (magenta) antiquark color

Symbol	Definition
g	acceleration due to gravity
g	gluons (carrier particles for strong nuclear force)
h	change in vertical position
h	height above some reference point
h	maximum height of a projectile
h	Planck's constant
hf	photon energy
h_i	height of the image
h_o	height of the object
I	electric current

Symbol	Definition
I	intensity
I	intensity of a transmitted wave
I	moment of inertia (also called rotational inertia)
I_0	intensity of a polarized wave before passing through a filter
I_{ave}	average intensity for a continuous sinusoidal electromagnetic wave
I_{rms}	average current
J	joule
J/Ψ	Joules/psi meson
K	kelvin
k	Boltzmann constant

Symbol	Definition
k	force constant of a spring
K_{α}	x rays created when an electron falls into an $n = 1$ shell vacancy from the $n = 3$ shell
K_{β}	x rays created when an electron falls into an $n = 2$ shell vacancy from the $n = 3$ shell
kcal	kilocalorie
KE	translational kinetic energy
KE + PE	mechanical energy
KE_e	kinetic energy of an ejected electron
KE_{rel}	relativistic kinetic energy
KE_{rot}	rotational kinetic energy
KE	thermal energy

Symbol	Definition
kg	kilogram (a fundamental SI unit of mass)
L	angular momentum
L	liter
L	magnitude of angular momentum
L	self-inductance
ℓ	angular momentum quantum number
L_{α}	x rays created when an electron falls into an $n = 2$ shell from the $n = 3$ shell
L_e	electron total family number
L_{μ}	muon family total number
L_{τ}	tau family total number

Symbol	Definition
L_f	heat of fusion
L_f and L_v	latent heat coefficients
L_{orb}	orbital angular momentum
L_s	heat of sublimation
L_v	heat of vaporization
L_z	z - component of the angular momentum
M	angular magnification
M	mutual inductance
m	indicates metastable state
m	magnification

Symbol	Definition
m	mass
m	mass of an object as measured by a person at rest relative to the object
m	meter (a fundamental SI unit of length)
m	order of interference
m	overall magnification (product of the individual magnifications)
$m(^AX)$	atomic mass of a nuclide
MA	mechanical advantage
m_e	magnification of the eyepiece
m_e	mass of the electron
m_ℓ	angular momentum projection quantum number

Symbol	Definition
m_n	mass of a neutron
m_o	magnification of the objective lens
mol	mole
m_p	mass of a proton
m_s	spin projection quantum number
N	magnitude of the normal force
N	newton
N	normal force
N	number of neutrons
n	index of refraction

Symbol	Definition
n	number of free charges per unit volume
N_A	Avogadro's number
N_r	Reynolds number
$N \cdot m$	newton-meter (work-energy unit)
$N \cdot m$	newtons times meters (SI unit of torque)
OE	other energy
P	power
P	power of a lens
P	pressure
p	momentum

Symbol	Definition
p	momentum magnitude
p	relativistic momentum
\mathbf{p}_{tot}	total momentum
\mathbf{p}'_{tot}	total momentum some time later
P_{abs}	absolute pressure
P_{atm}	atmospheric pressure
P_{atm}	standard atmospheric pressure
PE	potential energy
PE _{el}	elastic potential energy
PE _{elec}	electric potential energy

Symbol	Definition
PE_s	potential energy of a spring
P_g	gauge pressure
P_{in}	power consumption or input
P_{out}	useful power output going into useful work or a desired, form of energy
Q	latent heat
Q	net heat transferred into a system
Q	flow rate—volume per unit time flowing past a point
$+Q$	positive charge
$-Q$	negative charge

Symbol	Definition
q	electron charge
q_p	charge of a proton
q	test charge
QF	quality factor
R	activity, the rate of decay
R	radius of curvature of a spherical mirror
R	red quark color
R	antired (cyan) quark color
R	resistance
R	resultant or total displacement

Symbol	Definition
R	Rydberg constant
R	universal gas constant
r	distance from pivot point to the point where a force is applied
r	internal resistance
r_{\perp}	perpendicular lever arm
r	radius of a nucleus
r	radius of curvature
r	resistivity
r or rad	radiation dose unit
rem	roentgen equivalent man

Symbol	Definition
rad	radian
RBE	relative biological effectiveness
RC	resistor and capacitor circuit
rms	root mean square
r_n	radius of the n th H-atom orbit
R_p	total resistance of a parallel connection
R_s	total resistance of a series connection
R_s	Schwarzschild radius
S	entropy
S	intrinsic spin (intrinsic angular momentum)

Symbol	Definition
S	magnitude of the intrinsic (internal) spin angular momentum
S	shear modulus
S	strangeness quantum number
s	quark flavor strange
s	second (fundamental SI unit of time)
s	spin quantum number
s	total displacement
$\sec \theta$	secant
$\sin \theta$	sine
s_z	z-component of spin angular momentum

Symbol	Definition
T	period—time to complete one oscillation
T	temperature
T_c	critical temperature—temperature below which a material becomes a superconductor
T	tension
T	tesla (magnetic field strength B)
t	quark flavor top or truth
t	time
$t_{1/2}$	half-life—the time in which half of the original nuclei decay
$\tan \theta$	tangent
U	internal energy

Symbol	Definition
u	quark flavor up
u	unified atomic mass unit
u	velocity of an object relative to an observer
u'	velocity relative to another observer
V	electric potential
V	terminal voltage
V	volt (unit)
V	volume
v	relative velocity between two observers
v	speed of light in a material

Symbol	Definition
\mathbf{v}	velocity
\mathbf{v}	average fluid velocity
$V_B - V_A$	change in potential
\mathbf{v}_d	drift velocity
V_p	transformer input voltage
V_{rms}	rms voltage
V_s	transformer output voltage
\mathbf{v}_{tot}	total velocity
v_w	propagation speed of sound or other wave
\mathbf{v}_w	wave velocity

Symbol	Definition
W	work
W	net work done by a system
W	watt
w	weight
w_{fl}	weight of the fluid displaced by an object
W_{c}	total work done by all conservative forces
W_{nc}	total work done by all nonconservative forces
W_{out}	useful work output
X	amplitude
X	symbol for an element

Symbol	Definition
${}^Z_A X_N$	notation for a particular nuclide
x	deformation or displacement from equilibrium
x	displacement of a spring from its undeformed position
x	horizontal axis
X_C	capacitive reactance
X_L	inductive reactance
x_{rms}	root mean square diffusion distance
y	vertical axis
Y	elastic modulus or Young's modulus
Z	atomic number (number of protons in a nucleus)

Symbol	Definition
Z	impedance

Selected Answers